Instabilities in Anisotropic Chiral Plasmas

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arXiv:1405.2865
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XXI DAE-BRNS High Energy Physics Symposium
IITG, 8-12 December, 2014

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5. Summary
Recently it has been proposed that P and CP violation should manifest in heavyion collisions through the electric charge separation with respect to reaction plane in noncentral collisions.

Motivation

- First preliminary result of such a study has been presented recently by STAR Collaboration at RHIC.
- Recently, based on Berry curvature corrections, a modified kinetic theory has been developed which allows one to study the CP violating or chiral effects in non-equilibrium conditions.
- It has been found that for modified kinetic theory the presence of CP-violating effects can lead to the instabilities in the transverse branch of dispersion relation in the quasi-stationary limit.
- However in many realistic situations in plasma physics it is important to consider initial distribution function to be anisotropic in momentum space. It is well known that momentum anisotropy can lead to so called Weibel instability. Therefore it is important to consider effect of anisotropy in modified kinetic theory and to see how the two instabilities compete with each other.

B. I. Abelev et. al.[STAR Collaboration], Phys. Rev. Lett. 103, 251601 (2009),
B. I. Abelev et. al.[STAR Collaboration], Phys. Rev. C. 81, 054908 (2010),
Chiral kinetic theory

- In this description we have considered the weak gauge field limit, where there is no essential difference between Abelian and non-Abelian gauge fields up to color and flavor degrees of freedom.

\[
\dot{n}_p + \frac{1}{1 + eB \cdot \Omega_p} \left[ \left( e\tilde{E} + e\tilde{v} \times B + e^2 (\tilde{E} \cdot B) \Omega_p \right) \cdot \frac{\partial n_p}{\partial p} \right. \\
\left. + \left( \tilde{v} + e\tilde{E} \times \Omega_p + e(\tilde{v} \cdot \Omega_p)B \right) \cdot \frac{\partial n_p}{\partial x} \right] = 0,
\]

where \( \tilde{v} = \frac{\partial \epsilon_p}{\partial p} \), \( e\tilde{E} = eE - \frac{\partial \epsilon_p}{\partial x} \), \( \epsilon_p = p(1 - eB \cdot \Omega_p) \) and \( \Omega_p = \pm p/2p^3 \). Here \( \pm \) sign corresponds to right and left handed fermions respectively.

- If \( \Omega_p = 0 \) and \( \frac{\partial \epsilon_p}{\partial x} = 0 \) above equation reduces to Vlasov equation.

- From above equation it is easy to get,

\[
\partial_t n + \nabla \cdot j = e^2 \int \frac{d^3 p}{(2\pi)^3} \left( \Omega_p \cdot \frac{\partial n_p}{\partial p} \right) E \cdot B,
\]

where,

\[
n = \int \frac{d^3 p}{(2\pi)^3} (1 + eB \cdot \Omega_p) n_p,
\]

\[
j = -e \int \frac{d^3 p}{(2\pi)^3} \left[ \epsilon_p \frac{\partial n_p}{\partial p} + e \left( \Omega_p \cdot \frac{\partial n_p}{\partial p} \right) \epsilon_p B + \epsilon_p \Omega_p \times \frac{\partial n_p}{\partial x} \right] + E \times \sigma.
\]

\[
\sigma = \int \frac{d^3 p}{(2\pi)^3} \Omega_p n_p.
\]
Chiral kinetic theory

- We follow the power counting scheme $A_\mu = O(\epsilon)$ and $\partial x = O(\delta)$ where $\epsilon$ and $\delta$ are small parameters.

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) n_p + \left( eE + ev \times B - \frac{\partial e_p}{\partial x} \right) \cdot \frac{\partial n_p}{\partial p} = 0$$

- Where $\mathbf{v} = \frac{\mathbf{p}}{p}$.

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Linear response analysis of anisotropic chiral plasma

- We consider the equilibrium distribution of the form
  \[ n_p^0 = \frac{1}{e^{(\epsilon_p - \mu)/T} + 1} \]
  \[ n_p^0 = n_p^{0(0)} + e n_p^{0(\epsilon \delta)} , \]

- where, \( n_p^{0(0)} = \frac{1}{[e^{(\bar{\epsilon} - \mu)/T} + 1]} \) and \( n_p^{0(\epsilon \delta)} = \left( \frac{B \cdot v}{2 \bar{p} T} \right) \frac{e^{(\bar{\epsilon} - \mu)/T}}{[e^{(\bar{\epsilon} - \mu)/T} + 1]^2} \).

- \( \bar{p} = \sqrt{p^2 + \xi (p \cdot \hat{n})^2} \).

- Anomalous Hall current depends on electric field, it can be of order \( O(\epsilon \delta) \) or higher. We are interested in finding deviations in current and distribution function up to order \( O(\epsilon \delta) \), only \( n_p^{0(0)} \) will contribute to the Hall current term.

\[
\sigma = \frac{1}{2} \int d\Omega d\bar{p} \frac{v}{[1 + \xi (v \cdot \hat{n})]^{1/2}} \frac{1}{(1 + e^{(\bar{\epsilon} - \mu)/T})} = 0.
\]

- Anomalous Hall current vanishes.

The distribution function can be decomposed into separate scales as follows,

\[ n_p = n_p^0 + e(n_p^{(\epsilon)} + n_p^{(\epsilon \delta)}). \]

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Linear response analysis of anisotropic chiral plasma

\[ \Pi^i j_+(K) = m_D^2 \int \frac{d\Omega}{4\pi} \frac{\nu^i (\nu^l + \xi (\nu \cdot \hat{n}) \hat{n}^l)}{(1 + \xi (\nu \cdot \hat{n})^2)^2} \left( \delta^{jl} + \frac{\nu^j k^l}{\nu. k + i\epsilon} \right), \]

\[ \Pi^{im}_-(K) = C_E \int \frac{d\Omega}{4\pi} \left[ i\epsilon^{ilm} k^l \nu^j (\omega + \xi (\nu \cdot \hat{n})(k \cdot \hat{n})) \right. \]

\[ + \left. \left( \frac{\nu^j + \xi (\nu \cdot \hat{n}) \hat{n}^j}{(1 + \xi (\nu \cdot \hat{n})^2)^{3/2}} \right) i\epsilon^{ilm} k^l \nu^j \right] \left( \delta^{mn} + \frac{\nu^m k^n}{\nu. k + i\epsilon} \right) \left( \frac{\nu^n + \xi (\nu \cdot \hat{n}) \hat{n}^n}{(1 + \xi (\nu \cdot \hat{n})^2)^{3/2}} \right) \]

where,

\[ m_D^2 = -\frac{1}{2\pi^2} \int_0^\infty d\tilde{p} \tilde{p}^2 \left[ \frac{\partial n_{p(0)}^{0(0)} (\tilde{p} - \mu)}{\partial \tilde{p}} + \frac{\partial n_{p(0)}^{0(0)} (\tilde{p} + \mu)}{\partial \tilde{p}} \right] \]

\[ C_E = -\frac{1}{4\pi^2} \int_0^\infty d\tilde{p} \tilde{p} \left[ \frac{\partial n_{p(0)}^{0(0)} (\tilde{p} - \mu)}{\partial \tilde{p}} - \frac{\partial n_{p(0)}^{0(0)} (\tilde{p} + \mu)}{\partial \tilde{p}} \right] \]

- After performing above integrations one can get \( m_D^2 = \frac{\mu^2}{2\pi^2} + \frac{T^2}{6} \) and \( C_E = \frac{\mu}{4\pi^2} \). It can be noticed that the terms with anisotropy parameter \( \xi \) are contributing in both parity-even and odd part of the self-energy or polarization tensor.

\[ j_{\text{ind}}^{\mu} = \Pi^{\mu \nu}(K) A_{\nu}(K), \]
Linear response analysis of anisotropic chiral plasma

- Maxwell equation,

\[ \partial_{\nu} F^{\nu\mu} = j_{\text{ind}}^{\mu} + j_{\text{ext}}^{\mu}. \]

\[ j_{\text{ind}}^{\mu} = \Pi^{\mu\nu}(K)A_{\nu}(K), \]

- \( \Pi^{\mu\nu}(K) \) is the retarded self energy in Fourier space. Here we denote the Fourier transform as \( F(K) = \int d^4xe^{-i(\omega t - k \cdot x)}F(x, t). \)

- Choosing temporal gauge \( A_0 = 0 \)

\[ \left[(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)\right]E^j = i\omega j_{\text{ext}}^i(k). \]

- From this one can define,

\[ [\Delta^{-1}(K)]^{ij} = (k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K). \]

- The poles of \( [\Delta(K)]^{ij} \) will give us the dispersion relation.
In the quasi stationary limit $|\omega| \ll k$ one can get the final form of dispersion relation as $\omega = i\rho(k)$, where $\rho(k)$ is given by.

$$\rho(k) = \left( \frac{4\alpha^3 \mu^3}{\pi^4 m_D^2} \right) k_N^2 \left[ 1 - k_N + \frac{\xi}{12} (1 + 5 \cos 2\theta_n) + \frac{\xi}{12} (1 + 3 \cos 2\theta_n) \frac{\pi^2 m_D^2}{\mu^2 \alpha^2 k_N} \right].$$

Where $k_N = \frac{\pi k}{\mu \alpha}$, and $\alpha = \frac{e^2}{4\pi}$ is the electromagnetic coupling.

In the limit $\xi \rightarrow 0$ we will get,

$$\rho(k) = \left( \frac{4\alpha^3 \mu^3}{\pi^4 m_D^2} \right) k_N^2 [1 - k_N]$$

From here it is easy to determine that the two instabilities have comparable growth rate at a critical angle $\theta_c = \frac{1}{2} \cos^{-1} \left[ \left( \frac{2}{27} \right)^{2/3} \frac{12\mu^2 \alpha^2}{\xi \pi^2 m_D^2} - \frac{1}{3} \right].$
Figure: Shows plots of real and imaginary part of the dispersion relation. Here $\theta_n$ is the angle between the wave vector $k$ and the anisotropy vector. Real part of dispersion relation is zero. Fig. (1a-1b) show plots for three cases: (i) Pure chiral (no anisotropy), (ii) Pure Weibel (chiral chemical potential=0) and (iii) When both chiral and Weibel instabilities are present.
Figure: Shows plots of real and imaginary part of the dispersion relation. Here $\theta_n$ is the angle between the wave vector $k$ and the anisotropy vector. Real part of dispersion relation is zero. Fig. (2a-2b) represent the case when both the instabilities are present but the anisotropy parameter varies at different values of $\theta_n$. Fig. (2c) represents the case when for a particular value of $\theta_n \sim \theta_C$ two instabilities have equal growth rates at different $\xi$ values. Here frequency is normalized in unit of $\omega / \left( \frac{4 \alpha^3 \mu^3}{\pi^4 m_d^2} \right)$ and wave-number $k$ by $k_N = \frac{\pi}{\mu \alpha} k$. 

(a) 

(b) 

(c)
We have studied collective modes in anisotropic chiral plasmas. We have considered two cases of the instabilities together namely chiral imbalance instability and Weibel instability.

We found that even for small value of anisotropy parameter ($\xi << 1$), the range and the magnitude of chiral imbalance instability is strongly modified.

For $\xi > 0$, the growth rate and range increases significantly when the wave vector $k$ is in the direction parallel to anisotropy vector $\mathbf{n}$.

Instability become weaker when $k$ is in the direction perpendicular to $\mathbf{n}$.

Growth rates for the two instabilities become comparable at a critical angle

$$\theta_c = \frac{1}{2} \cos^{-1} \left( \left( \frac{2}{27} \right)^{2/3} \frac{12\mu^2 \alpha^2}{\xi \pi^2 m_D^2} - \frac{1}{3} \right).$$

THANK YOU
We consider a stream of particles passing through an isotropic thermalized background plasma with four velocity $u^\mu = (u^0, u^0 v_{st})$ where $u^0 = \gamma$.

$$n_p^0 = \tilde{n} u^0 \delta^3(p - \Lambda u)(1 + eB \cdot \Omega_p)$$

where, $\tilde{n}$ describes the density of the jet particles and it is considered to be constant. Here $\Lambda$ represents the scale of energy of the jet. The term with $eB \cdot \Omega_p$ regarded as $O(\epsilon \delta)$ corrections in the equilibrium distribution function arising due to the Berry-curvature.

We take the following isotropic thermalized distribution function as,

$$n_p^0 = n_p^{0(0)} + e n_p^{0(\epsilon \delta)} ,$$

where, $n_p^{0(0)} = \frac{1}{[e^{(p-\mu)/T} + 1]}$ and $n_p^{0(\epsilon \delta)} = \left( \frac{B \cdot v}{2pT} \right) \frac{e^{p-\mu}/T}{[e^{(p-\mu)/T} + 1]^2}.$

Considering streaming velocity $v_{st}$ in $z$-direction and $k$ parallel to $v_{st}$ i.e $k_z$. In the limit $|\omega| \ll k$ imaginary part of dispersion relation can be written as,

$$B_1(k) = \frac{4\alpha \mu_1 k_1^2}{3\pi^2(1-b)} \left[ 1 - \frac{\pi k_1}{\mu_1 \alpha} - \frac{\pi b}{\alpha \mu_1 k_1} \left( 1 - \frac{3(1-v_{3st}^2)^{1/2}}{2} \frac{k_1}{\Lambda_1} + v_{3st}^2(1 - v_{3st}^2)^{1/2} \frac{k_1}{\Lambda_1} \right) \right] \left( 1 + \left( \frac{2k_1 v_{3st}(1-v_{3st}^2)^{1/2} b}{3\pi(1-b)\Lambda_1} \right)^2 \right).$$

$k_1 = k/\omega_t$, $b = \omega_{st}^2/\omega_t$, $\mu_1 = \mu/\omega_t$, $\Lambda_1 = \Lambda/\omega_t$, $\omega_t = (m_D^2/3 + \omega_{st}^2)^{1/2}$ and $\omega_{st}^2 = \frac{\epsilon e^2}{2\Lambda}.$

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$$det[\Delta^{-1}(K)]^{ij} = det[(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)] = 0.$$
Figure: show plots of dispersion relation of the instability in a chiral-plasma background with a stream for the situation when the wave-vector $k_1$ propagating in the direction parallel to the stream velocity $v_{st}$. The $b = 0$ corresponds to the situation when there is no stream in the background plasma. Fig.(2a) shows how the instability varies for different values of $b$ while the stream velocity $v_{3st} = 0.9$ and $\Lambda_1 = 30$. Fig.(2b) shows that for a given values of $b$ and $v_{3st}$, the parity-odd terms in the jet self-energy can enhance the instability. Fig.(2c) shows the dependence of the instability on the stream velocity for given values of $b$ and $\Lambda_1$. Inset Figures in Fig. (2b,2c) shows the instability for $b=0$ case with better resolution.
Choosing $k$ to be in $x$-direction and $v_{st}$ in $z$-direction one can get the following dispersion relation (in the limit $|\omega| << k$).

\[
\left( \lambda_1^2 - (k_1^2 + \alpha_1 + b)^2 \right) (-\omega_1^2 + \beta_1 + b) - (k_1^2 + \alpha_1 + b) \left( \left( -1 + \frac{\beta_1}{\omega_1^2} \right) b k_1^2 v_3^2 \right) \\
+ 6 \left( \frac{k_1(1 - v_3^2)^{1/2}}{2\Lambda_1} \right) \lambda_1 b \left( (\omega_1^2 - \beta_1) \left( 1 - \frac{k_1^2 v_3^2}{6\omega_1^2} \right) + \left(1 - \frac{v_3^2}{2} \right) b \right) \\
+ \left( \frac{k_1(1 - v_3^2)^{1/2}}{2\Lambda_1} \right)^2 b^2 \left( 9 \left( \omega_1^2 \left( 1 - \frac{v_3^2}{2} \right) - \beta_1 \left( 1 - \frac{k_1^2 v_3^2}{3\omega_1^2} \left( 1 - \frac{v_3^2}{3} \right) + \frac{k_1^2 v_3^4}{9} - b \right) \right) \\
+ v_3^2 \left( -2k_1^2 + \alpha_1 + 7b \right) \right) = 0
\]

where, $\lambda_1 = \frac{\mu_1 k_1 e^2}{4\pi^2} \left( 1 + i \frac{\pi}{2} \frac{\omega_1}{k_1} \right)$, $\alpha_1 = -i \frac{3\pi(1-b)}{4} \frac{\omega_1}{k_1}$, $\beta_1 = -3(1-b) \frac{\omega_1^2}{k_1^2}$.

\[
det[\Delta^{-1}(K)]^{ij} = det[(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)] = 0.
\]

Figure: show plots of dispersion relation of instabilities in a chiral plasma with a stream passing through it when \( k_1 \) perpendicular to \( v_{st} \). Fig. 3(a) shows a comparison in the instabilities when a stream with parameters \( b = 0.01 \) and \( v_{3st} = 0.06 \), \( \Lambda_1 = 30 \) passing through chiral plasma [red (solid) curve] to the cases, when there is no streaming i.e. \( b = 0 \) [blue (dotted) curve] and when there is stream with \( b = 0.01, v_{3st} = 0.06 \) passing through parity even plasma [green (dashed) curve]. Fig. 3(b) shows the effect on instability by changing the parameter \( \Lambda_1 \) keeping parameters \( b = 0.02 \) and \( v_{3st} = 0.1 \) fixed. Fig. 3(c) shows the effect on instability by changing parameter \( b \) keeping parameters \( v_{3st} = 0.4 \) and \( \Lambda_1 = 30 \) fixed.
Berry’s Phase

- Consider the Hamiltonian of the system with an external time dependent parameter $R(t)$ denoting it as $H(R(t))$.

- The ket $|n(R(t))\rangle$ of the $n^{th}$ energy eigenstate corresponding to $R(t)$ satisfies the eigenvalue equation at time $t$.

$$H(R(t))|n(R(t))\rangle = E_n(R(t))|n(R(t))\rangle$$

$$\langle n(R(t))|n'(R(t))\rangle = \delta_{n,n'}$$

- Let $R$ evolve in time from $R(0) = R_0$.

- Let at time $t$ the state ket is $|n(R_0), t_0; t\rangle$

- Time dependent Schrödinger equation that the state ket obeys is.

$$H(R(t))|n(R_0), t_0; t\rangle = i\hbar \frac{\partial}{\partial t} |n(R_0), t_0; t\rangle \quad (1)$$

- where $t_0 = 0$.

J. J. Sakurai, Modern Quantum Mechanics: (Pearson Education, 1994), 18
Berry’s Phase

When \( R(t) \) is slow enough, we expect from the adiabatic theorem that 
\( |n(R_0), t_0; t_i\rangle \) would be proportional to the \( n^{th} \) energy eigenket \( |n(R(t))\rangle \) of 
\( H(R(t)) \) at time \( t \).

\[
|n(R_0), t_0; t_i\rangle = A_n(t) \exp \left\{ -\frac{i}{\hbar} \int_0^t E_n(R(t')) dt' \right\} |n(R(t))\rangle. \tag{2}
\]

Using Eq. (2) in Eq. (1), one can get;

\[
\frac{dA_n(t)}{dt} = -A_n(t) \langle n(R(t)) | \frac{d}{dt} | n(R(t)) \rangle \\
A_n(t) = A_n(0) \exp \left\{ -\int_0^t dt' \langle n(R(t')) | \frac{d}{dt'} | n(R(t')) \rangle \right\} \underbrace{i\gamma_n(t)}_{i\gamma_n(t)}
\]

Let us call it phase factor \( \gamma_n(t) \)

\[
\gamma_n(t) = -i \int_0^t dt' \langle n(R(t')) | \frac{d}{dt'} | n(R(t')) \rangle
\]

\[
|n(R_0), t_0; t_i\rangle = A_n(0) \exp \left\{ -i\gamma_n(t) - \frac{i}{\hbar} \int_0^t E_n(R(t')) dt' \right\} |n(R(t))\rangle. \tag{Berry Phase}
\]
Berry's Phase

- $\gamma_n(t)$ can also be represented by a path-integral in parameter $R(t)$ space as,

\[
\gamma_n(t) = -i \int_{R_0}^{R_f} dR \langle n(R(t')) | \nabla_R | n(R(t')) \rangle
\]

- If $R$ describes a closed loop in parameter space i.e. $R_f = R_0$

\[
\gamma_n(t) = -i \oint_c dR \langle n(R(t')) | \nabla_R | n(R(t')) \rangle = \int \int_{S(c)} dS \cdot \nabla_R \times Q(R)
\]

- Where, $Q(R) = -i \langle n(R(t')) | \nabla_R | n(R(t')) \rangle$. $\Rightarrow$ Berry connection

- While $\nabla_R \times Q(R) = \Omega(R) \Rightarrow$ Berry Curvature.
Berry Curvature for Chiral Fermions

- Consider a chiral fermion expressed by the two-component spinor $u_p$ satisfying the Weyl equation.

\[(\sigma \cdot p)u_p = \pm |p|u_p\]

- Two component spinor described above has a nonzero Berry connection

\[Q_p \equiv -i u_p^\dagger \nabla_p u_p\]

- Nonzero Berry curvature,

\[\Omega(p) \equiv \nabla_p \times Q_p = \pm \frac{\hat{p}}{2|p|^2}\]

- where $\hat{p} = \frac{\hat{p}}{|p|}$ is a unit vector.

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Chiral Kinetic Theory

- Considering a charged fermion in electromagnetic fields and Berry curvature, the action:

\[
S(x, p) = \int dt \left[ (p^i + eA^i(x)) \dot{x}^i - Q^i(p) \dot{p}^i - \epsilon_p(p) - A^0(x) \right]
\]

\[
S(\xi) = \int dt \left[ \Sigma_a(\xi) \dot{\xi}^a - H(\xi) \right]
\]

- Where, \( \Sigma_a(\xi) = (p^i + eA_i(x), -Q^i(p)) \) and \( \xi^a = (x^i, p^i) \)

- Equations of motion of the action read.

\[
\Sigma_{ab} \dot{\xi}^b = - \frac{\partial H(\xi)}{\partial \xi^a}
\]

- Where \( \Sigma_{ab} = \frac{\partial \Sigma_a(\xi)}{\partial \xi^b} - \frac{\partial \Sigma_b(\xi)}{\partial \xi^a} \).

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Further we rewrite above equation as,

\[ \dot{\xi}^a = - (\Sigma^{-1})^{ab} \frac{\partial H(\xi)}{\partial \xi^b} \]

Hamilton’s equation of motion is,

\[ \dot{\xi}^a = -\{\xi^a, H(\xi)\} = -\{\xi^a, \xi^b\} \frac{\partial H(\xi)}{\partial \xi^b} \]

\[ \implies \{\xi^a, \xi^b\} = (\Sigma^{-1})^{ab} \]

Explicit form of poision brackets with berry curvature,

\[ \{x^i, x^j\} = \frac{\epsilon_{ijk} \Omega_k}{1 + eB \cdot \Omega}, \quad \{x^i, p^j\} = -\frac{\delta_{ij} + e \Omega_i B_j}{1 + eB \cdot \Omega}, \quad \{p^i, p^j\} = -\frac{\epsilon \epsilon_{ijk} B_k}{1 + eB \cdot \Omega}, \]

Where \( B^i = \epsilon^{ijk} \frac{\partial A^k}{\partial x^j} \)
Kinetic Equation

- Invariant Phase space gets modified, \( d\Gamma = \sqrt{\det \Sigma_{ab}} d\xi = (1 + e \mathbf{B} \cdot \Omega) \frac{dpdx}{2\pi^3} \).

- Equivalent Liouville’s theorem,

\[
\dot{n}_p - (\Sigma)^{-1}_{ab} \frac{\partial H(\xi)}{\partial \xi^b} \frac{\partial n_p}{\partial \xi^a} = 0
\]

- Taking \( H = \epsilon_p + A_0 \), One can explicitly write down the kinetic equation as,
Therefore,
\[ j = - \int \frac{d^3 p}{(2\pi)^3} \left[ \epsilon_p \frac{\partial n_p}{\partial p} + e \left( \Omega_p \cdot \frac{\partial n_p}{\partial p} \right) \right. \]
\[ \left. \epsilon_p B + \epsilon_p \Omega_p \times \frac{\partial n_p}{\partial x} \right]. \]

The distribution function can be decomposed into separate scales as follows,
\[ n_p = n_p^0 + e\left(n_p^{(e)} + n_p^{(e\delta)}\right). \]

\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} \right) n_p^{(e)} = -\left( E + v \times B \right) \cdot \frac{\partial n_p^{(0)}}{\partial p}
\]
\[
\left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} \right) \left( n_p^{0(e\delta)} + n_p^{(e\delta)} \right) = -\frac{1}{e} \frac{\partial \epsilon_p}{\partial x} \cdot \frac{\partial n_p^{0(0)}}{\partial p}
\]
\[ j^{\mu(e)} = e^2 \int \frac{d^3 p}{(2\pi)^3} \nu^\mu n_p^{(e)} \]
\[ j^{i(e\delta)} = e^2 \int \frac{d^3 p}{(2\pi)^3} \left[ \nu^i n_p^{(e\delta)} - \left( \frac{\nu^j}{2p} \frac{\partial n_p^{0(0)}}{\partial p^j} \right) B^i - \epsilon^{ijk} \frac{\nu^j}{2p} \frac{\partial n_p^{(e)}}{\partial x^k} \right] \]
\[ \left( \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} \right) n_p + \left( eE + ev \times B - \frac{\partial \epsilon_p}{\partial x} \right) \cdot \frac{\partial n_p}{\partial p} = 0. \]
Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

- We decompose first $\Pi^{ij}(K)$ in following six tensorial basis,

$$\Pi^{ij} = \alpha P_T^{ij} + \beta P_L^{ij} + \gamma P_n^{ij} + \delta P_{kn}^{ij} + \lambda P_A^{ij} + \chi P_{An}^{ij}.$$  

- Where,

$$P_T^{ij} = \delta^{ij} - k^i k^j / k^2$$
$$P_L^{ij} = k^i k^j / k^2$$
$$P_n^{ij} = \hat{n}^i \hat{n}^j / \hat{n}^2$$
$$P_{kn}^{ij} = k^i \hat{n}^j + k^j \hat{n}^i$$
$$P_A^{ij} = i \epsilon^{ijk} \hat{k}^k$$
$$P_{An}^{ij} = i \epsilon^{ijk} \hat{n}^k.$$

- $\alpha, \beta, \gamma, \delta, \lambda$ and $\chi$ are some scalar functions of $k$ and $\omega$ which can be determined by $\alpha = (P_T^{ij} - P_n^{ij})\Pi^{ij}$, $\beta = P_L^{ij}\Pi^{ij}$, $\gamma = (2P_n^{ij} - P_T^{ij})\Pi^{ij}$, $\delta = \frac{1}{2k^2 \hat{n}^2} P_{kn}^{ij} \Pi^{ij}$, $\lambda = -\frac{1}{2} P_A^{ij} \Pi^{ij}$ and $\chi = -\frac{1}{2\hat{n}^2} P_{An}^{ij} \Pi^{ij}$. 


Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

- We shall do the analysis in the small $\xi$ limit (Very weak anisotropy),

\[
\alpha = \Pi_T + \xi \left[ \frac{z^2}{12} (3 + 5 \cos 2 \theta_n) m_D^2 - \frac{1}{6} (1 + \cos 2 \theta_n) m_D^2 + \frac{1}{4} \Pi_T \left( (1 + 3 \cos 2 \theta_n) - \frac{z^2}{2} (3 + 5 \cos 2 \theta_n) \right) \right];
\]

\[
z^{-2} \beta = \Pi_L + \xi \left[ \frac{1}{6} (1 + 3 \cos 2 \theta_n) m_D^2 + \Pi_L \left( \cos 2 \theta_n - \frac{z^2}{2} (1 + 3 \cos 2 \theta_n) \right) \right];
\]

\[
\gamma = \frac{\xi}{3} (3 \Pi_T - m_D^2) (z^2 - 1) \sin^2 \theta_n;
\]

\[
\delta = \frac{\xi}{3k} (4 z^2 m_D^2 + 3 \Pi_T (1 - 4 z^2)) \cos \theta_n;
\]

\[
\lambda = - \frac{\mu k}{4 \pi^2} \left[ (1 - z^2) \frac{\Pi_L}{m_D^2} \right] - \xi \frac{\mu k}{8 \pi^2} \left[ (1 - z^2) \frac{\Pi_L}{m_D^2} \left( 3 \cos 2 \theta_n - 1 \right) \right]
\]

\[
- 2 z^2 (1 + 3 \cos 2 \theta_n) + \frac{2 z^2}{3} (1 - 3 \cos 2 \theta_n) - \frac{43}{15} + \frac{22}{10} (1 + \cos 2 \theta_n) ;
\]

\[
\chi = \xi \left[ f(\omega, k) \right],
\]

- Expressions for $\Pi_T, \Pi_L$ are given as,

\[
\Pi_T = m_D^2 \frac{\omega^2}{2k^2} \left[ 1 + \frac{k^2 - \omega^2}{2\omega k} \ln \frac{\omega + k}{\omega - k} \right],
\]

\[
\Pi_L = m_D^2 \left[ \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} - 1 \right],
\]
Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

- Similarly we can write $[\Delta^{-1}(k)]^{ij}$ as

$$[\Delta^{-1}(K)]^{ij} = C_T P_T^{ij} + C_L P_L^{ij} + C_n P_n^{ij} + C_{kn} P_{kn}^{ij} + C_A P_A^{ij} + C_{An} P_{An}^{ij}. $$

- Coefficients C’s and $\alpha$’s have the following relationship.

$$
C_T = k^2 - \omega^2 + \alpha
$$

$$
C_L = -\omega^2 + \beta
$$

$$
C_n = \gamma
$$

$$
C_{kn} = \delta
$$

$$
C_A = \lambda
$$

$$
C_{An} = \chi.
$$

- So once we know $\alpha, \beta, \gamma, \delta, \lambda$ and $\chi$ we can determine coefficient C’s.

- But in order to get dispersion relation we have to find poles of $[\Delta(K)]^{ij}$ not of $[\Delta^{-1}(K)]^{ij}$. We can use the following formula,

$$
[\Delta^{-1}(K)]^{ij} [\Delta(K)]^{jl} = \delta^{il}
$$
Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

- we can obtain following formula for $[\Delta(K)]^{ij}$.

$$[\Delta(K)]^{ij} = aP_L^{ij} + bP_T^{ij} + cP_n^{ij} + dP_{kn}^{ij} + eP_A^{ij} + fP_{An}^{ij}$$

- where,

$$a = \frac{C_A^2 - C_T(C_n + C_T)}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$

$$b = \frac{k^2\tilde{n}^2 C_{kn}^2 - C_L(C_n + C_T)}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$

$$c = \frac{(C_A C_A + kC_{kn} C_T)/k}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$

$$d = \frac{k\tilde{n}^2 C_A C_{kn} + C_A C_L}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$

$$e = \frac{\tilde{n}^2(C_A^2 - k^2 C_{kn}^2) + C_L C_n}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$

$$f = \frac{kC_A C_{kn} + C_A(C_n + C_T)}{2k\tilde{n}^2 C_A C_A C_{kn} + C_A^2 C_L + \tilde{n}^2 C_A^2 (C_n + C_T) - C_T(-k^2\tilde{n}^2 C_{kn}^2 + C_L(C_n + C_T))}$$
Finding the Poles of $[\Delta(K)]^{ij}$ or Dispersion relation

Therefore the dispersion relation is,

$$2k\bar{n}^2 C_A C_{An} C_{kn} + C_A^2 C_L + \bar{n}^2 C_{An}^2 (C_n + C_T) - C_T (-k^2 \bar{n}^2 C_{kn}^2 + C_L (C_n + C_T)) = 0.$$ 

In the weak anisotropy limit, one can write the dispersion relation as,

$$C_A^2 C_L - C_T C_L (C_n + C_T) = 0,$$

Which give following two branches of Dispersion relation,

$$C_A^2 - C_T^2 - C_n C_T = 0.$$

$$C_L = 0.$$ 

When $C_A = 0$, above equations reduces to exactly the same dispersion relation discussed in Ref. given below for an anisotropic plasma where there is no parity violating effect.

Equation for transverse modes give the following solution,

$$(k^2 - \omega^2) = \frac{-(2\alpha + \gamma) \pm 2\lambda}{2}.$$
Expressions for the self energy of background plasma can be obtained simply by taking limit $\xi \to 0$ as,

$$\Pi^{ij}_+(K) = \alpha P_L^{ij} + \beta P_T^{ij}$$
$$\Pi^{ij}_-(K) = \lambda P_A^{ij}$$

where $\alpha = m^2 D \frac{\omega^2}{2k^2} \left[ 1 + \frac{k^2 - \omega^2}{2\omega k} \ln \frac{\omega + k}{\omega - k} \right]$, $\beta = m^2 D \frac{\omega^2}{k^2} \left[ \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} - 1 \right]$, $\lambda = -\frac{\mu ke^2}{4\pi^2} \left( 1 - \frac{\omega^2}{k^2} \right) \left[ \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} - 1 \right]$. $\Pi_T = \delta^{ij} - k^i k^j / k^2$, $\Pi_L = k^i k^j$ and $\Pi_A = i \epsilon^{ijk} \hat{k}^k$.

Expressions for the self energy for the stream of particles,

$$\Pi^i_{st} (K) = \omega_{st}^2 \left[ \delta^{ij} + \frac{k^i v^j_{st} + k^j v^i_{st}}{\omega - k \cdot v_{st}} - \frac{(\omega^2 - k^2) v^i_{st} v^j_{st}}{(\omega - k \cdot v_{st})^2} \right] .$$

$$\Pi^{im}_{st} (K) = \frac{i \epsilon^{iml} k^l v^i_{st} v^j_{st} \omega_{st}^2}{2\Lambda u^0 (\omega - k \cdot v_{st})} - \frac{i \epsilon^{iml} k^l (1 - 2 v^2_{st}) \omega_{st}^2}{2\Lambda u^0} - \frac{i \omega \epsilon^{iml} v^l_{st} \omega_{st}^2}{2\Lambda u^0}$$

$$+ \frac{i \epsilon^{ijl} k^l \omega_{st}^2}{2\Lambda u^0} \left[ \delta^{jm} + \frac{k^j v^m_{st} + k^m v^j_{st}}{\omega - k \cdot v_{st}} - \frac{(\omega^2 - k^2) v^j_{st} v^m_{st}}{(\omega - k \cdot v_{st})^2} - \frac{v^j_{st} v^m_{st}}{(\omega - k \cdot v_{st})} \right] .$$

Note that third term on the right hand side of above equation is due to the anomalous Hall-current.

Where $\omega_{st}^2 = \frac{\tilde{n} e^2}{2\Lambda}$.
Total self energy of the jet plasma system is,

\[ \Pi^{ij}(K) = \Pi^{ij}_+(K) + \Pi^{ij}_-(K) + \Pi^{ij}_{+st}(K) + \Pi^{ij}_{-st}(K) \]

In order to analyze the collective mode one can evaluate determinant of \([\Delta^{-1}(K)]^{ij}\).

\[
\det[[\Delta^{-1}(K)]^{ij}] = \det[(k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K)] = 0.
\]

We choose the streaming velocity \(v_{st}\) in z-direction only and \(k\) parallel to \(v_{st}\) i.e \(k_z\). In this case we shall get the following dispersion relation using above equation.

\[
\left(\beta + \omega^2 \left(-1 - \left(-1 + \frac{v_{3st}^2}{(\omega - kv_{3st})^2}\right)\right)\right) = 0,
\]

\[
(4\bar{\Lambda}^2(k^2 - \omega^2 + \alpha + \omega_{st}^2)^2 - (2\bar{\Lambda}\lambda + (3k - v_{3st}(\omega + 2kv_{3st}))\omega_{st}^2)^2) = 0.
\]

In the Quasi-stationary limit \(|\omega| \ll k\) one can write,

\[
\alpha|\omega|<<k = -i \frac{\pi}{4} \frac{\omega}{k} m_D^2,
\]

\[
\beta|\omega|<<k \approx -m_D^2 \frac{\omega^2}{k^2},
\]

\[
\lambda|\omega|<<k \approx \frac{\mu k}{4\pi^2}.
\]

In this limit the last equation will give, \(\omega = A + iB\) where \(A\) and \(B\) are real and imaginary part of the frequency one can write the imaginary part of the dispersion relation as follows:

\[
B(k) = + \frac{\mu k^2}{\pi^3 m_D^2} \left[ 1 - \frac{4\pi^2 k}{\mu} - \frac{4\pi^2 \omega_{st}^2}{\mu k} \left( 1 - \frac{3(1 - v_{3st}^2)^{1/2}}{2} \frac{k}{\Lambda} + v_{3st}^2 (1 - v_{3st}^2)^{1/2} \frac{k}{\Lambda} \right) \right]
\]

\[
\left( 1 + \left( \frac{2k v_{3st}(1 - v_{3st}^2)^{1/2} \omega_{st}^2}{\pi m_D^2 \Lambda} \right)^2 \right)^{-1/2}
\]

We normalize above equation by \(\omega_t = (m_D^2/3 + \omega_{st}^2)^{1/2}\), so we will have following parameters: \(k_1 = k/\omega_t\) \(b = \omega_{st}^2/\omega_t^2\), \(\mu_1 = \mu/\omega_t\), \(\Lambda_1 = \Lambda/\omega_t\).