NEUTRINO MASSES AND MIXING

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This is a review of the present status of neutrino masses and mixing and the theoretical ideas proposed to understand them. The talk will include some general comments on the theory of flavours.

Content

- The neutrino parameters and their status (a brief history)
- Two neutrino puzzles
  - Why are the masses so tiny?
  - Why are the angles so large?
- Answers to the puzzles
- Models?
- Theory of flavours :- does not exist so far.

An angle to tackle the neutrinos

- A brief history of neutrino oscillations is presented, highlighting the breakthrough in the determination of a crucial neutrino parameter by the Daya Bay and RENO reactor experiments that occurred two years ago . The importance of this parameter in the context of one of the goals of the India-based Neutrino Observatory (INO) project and also in advancing the frontier of neutrino physics is explained.
- The two neutrino experiments, one in China, called Daya Bay and the other in Korea, called RENO measured the flux of the antineutrinos at some distance from a complex of powerful nuclear reactors in their respective countries.
- The measurements showed that the antineutrinos oscillated, thus allowing the determination of a fundamental parameter of neutrino physics, the reactor angle.

An angle to tackle the neutrinos

- One must also mention that, a little earlier, the Double CHOOZ experiment in France also had measured this angle, but with larger errors.

  F P An et al (The Daya Bay Collaboration), arXiv:1203.1669 [hep-ex]
  J K Ahn et al (RENO Collaboration), arXiv:1204.0626 [hep-ex]
  Y Abe et al (Double CHOOZ Collaboration), arXiv:1112.6353 [hep-ex]

- To appreciate the importance of this experimental discovery, one must go back in time, a little bit.

- India was a pioneer in neutrino physics. The very first detection of atmospheric neutrinos was made in the Kolar Gold Field (KGF) mines in South India in 1965.
- INO Laboratory and INO Centre will come up in Theni District and near Madurai city respectively, both in the southern part of India. In the underground (more correctly, under-mountain) laboratory a gigantic 50-Kton magnetised particle detector will be erected to study atmospheric neutrinos.
- The INO Centre, named as the Inter-institutional Centre for High Energy Physics (ICHEP) will function as the nerve centre for the INO Collaboration.

- Neutrino oscillations occur among three types of neutrinos that are known to exist. The mixing among these three types is governed by a 3 x 3 unitary matrix which can be specified by three angles and a certain number of phases.
- The three angles can be called atmospheric, solar and reactor angles since they respectively control the oscillations of atmospheric, solar and reactor neutrinos.
- The atmospheric and solar neutrino studies had determined the atmospheric and solar angles as about 45 degrees and 30 degrees respectively more than 10 years ago.
- Now the Daya Bay and RENO experiments have determined the reactor angle to be about 9 degrees.

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31} \\ 0 & 1 & 0 \\ -s_{31} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_{12} = sin\theta_{12}, \quad c_{12} = cos\theta_{12} \text{ etc.}$$

$$\theta_{12} = \text{solar angle} \approx 30^\circ$$
$$\theta_{23} = \text{atm. angle} \approx 45^\circ$$
$$\theta_{31} = \text{reactor angle} \approx 9^\circ$$

An angle to tackle the neutrinos

The three angles

- Again let us go back in history a little bit. During the exciting period in the 90's when neutrino oscillations were discovered, our group at IMSc, Chennai was one of the earliest to initiate a comprehensive study of both solar and atmospheric neutrino oscillations using the full mixing among the three types of neutrinos.

  M Narayan, M V N Murthy, G Rajasekaran and S Uma Sankar, Phys Rev D 53, 2809 (1996)
  M Narayan, G Rajasekaran and S Uma Sankar, Phys Rev D 56, 437 (1997)

- Others were using toy-models of mixing among two types of neutrinos to describe solar and atmospheric neutrinos separately.

An angle to tackle the neutrinos

The three angles

- Since we were working with the complete three-neutrino framework, we became the first to analyze the reactor neutrino data that came in 1997 from the CHOOZ experiment in France. Analyzing the data within this framework, we showed that the reactor angle was smaller than 12 degrees and also showed that as a consequence the solar and atmospheric oscillations became approximately decoupled.

  M Narayan, G Rajasekaran and S Uma Sankar, Phys Rev D 58, 031301 (1998)

- This decoupling played a major role in all the subsequent analyses of atmospheric and solar data helping to pin down the parameters in these two sectors more easily.

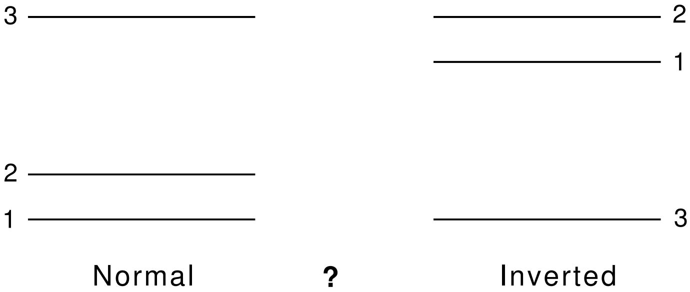
- The upper limit remained as our only information on the crucial reactor angle for the last 15 years until it was determined 2 years ago to be 9 degrees, not far away from the upper limit.

- Now we are ready to explain the importance of this measurement. There are two points: one in the context of INO and the other in the context of matter-antimatter asymmetry.

- Although oscillations establish that the neutrinos are massive, their actual mass cannot be determined by oscillation experiments; only the mass differences (actually differences of squares of masses) are determined. Calling the three neutrinos as 1, 2 and 3, the 2-1 mass difference is determined by the solar neutrino oscillations while the 3-2 mass difference is determined by the atmospheric sector.

- The mass-square differences so determined turn out to be very tiny. The sign of the 2-1 mass difference is determined to be positive but the sign of the other is not determined. So although neutrino 2 is heavier than 1, we do not know whether 3 is heavier than the 2-1 doublet or lighter.

$$\delta m^2_{21} = m^2_2 - m^2_1 \simeq 7 \times 10^{-5} eV^2$$
$$\delta m^2_{32} = m^2_3 - m^2_2; \;\; |\delta m^2_{32}| \simeq 2 \times 10^{-3} eV^2$$



- A major discovery item in the agenda of the big magnetised particle detector at INO is to resolve this ambiguity in the sign of the 3-2 mass difference and thus determine the actual mass-ordering of the neutrinos.
- A non-zero value for the reactor angle is crucial for this discovery and that is the importance of the reactor angle for INO.
- The rather large measured value of this angle has enhanced the optimism of the INO Collaboration. However in order to achieve this discovery, all the components of the project have to be executed according to strict time schedules. There is no time to lose.

An angle to tackle the neutrinos

Matter-antimatter asymmetry

- This is about the phases of the 3 x 3 unitary mixing matrix for the neutrinos. Earlier we mentioned the three angles of this unitary matrix.
- Now we come to the phases. Phases lead to matter-antimatter asymmetry (CP violation).
- And it is this asymmetry which is presumed to be responsible for the evolution of a original matter-antimatter symmetric universe into the present-day asymmetric universe that contains only matter and no antimatter. So that is the cosmological importance of the phases in the unitary mixing matrix.
- The question is: apart from the three angles, how many phases exist?
- It was the simple remark of Kobayashi and Maskawa that the dimension of the unitary matrix has to be atleast 3 for a phase to exist that won a Nobel Prize for them in the year 2008.
- That was in the context of quarks and so for the 3 x 3 unitary mixing matrix for quarks there exists precisely one phase.

An angle to tackle the neutrinos

Matter-antimatter asymmetry

- For the neutrino case, there is some difference. Ignoring this difference for the moment, there is one phase in the case of neutrino mixing too.

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31} \\ 0 & 1 & 0 \\ -s_{31} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An angle to tackle the neutrinos

- However if the reactor angle were zero, then as already mentioned the 3-neutrino problem would be reduced to two uncoupled 2-neutrino problems described by two 2 x 2 matrices which will not have any matter-antimatter-symmetry-violating phase.

- Hence the importance of the non-zero reactor angle that couples the solar and atmospheric sector into one 3-neutrino problem. Now that the angle has been measured and found to be 9 degrees, the door is open for measuring the CP-violating phase.

- That will however require long-base-line neutrino experiments. In the second phase of the INO experiment, the magnetised detector may play the role of the end-detector in such an experiment.

An angle to tackle the neutrinos

Towards a complete picture

- 1. Although neutrinos are now known to be massive from the existence of neutrino oscillations, we do not know the value of the masses since only the differences in neutrino-mass-squares can be determined from the oscillation phenomena. Nuclear beta decay experiments (in particular decay of tritium) can give the absolute masses. So far it has only led to an upper limit of 2.2 eV. KATRIN will improve this limit by an order of magnitude. Since the mass differences are very tiny as already mentioned above, we see that all the three neutrino masses are clustered around a mass level below 2.2 eV.

- 2. It must be pointed out that the fundamental nature of the neutrino is still not known, namely whether neutrino is a Dirac or Majorana particle. If neutrino is a Majorana particle, then the Kobayashi-Maskawa counting of the number of phases is not valid for the neutrino sector. It has to be augmented by two more phases. The question whether neutrino is a Majorana particle can be answered only by the neutrinoless double beta decay experiment which is therefore the most important experiment in all of neutrino physics. This experiment also will be a part of the INO project.

- For Majorana neutrinos, multiply U on the right by

$$\begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$m_\beta \equiv \left[\sum_i |U_{ei}|^2 \; m_i^2 \right]^{1/2} < \; 2.2 \; eV \; \text{(Tritium } \beta \text{ decay)}$$

$$m_{\beta\beta} \equiv \left[\sum_i U_{ei}^2 \; m_i \right] < \; 0.2 \; eV \; (0\nu 2\beta \text{ decay)}$$

$$\sum_i m_i < 0.5 \; eV \; \text{(CMBR, WMAP, PLANCK)}$$

- The reactor angle whose upper bound was found 15 years ago in Chennai has now been determined recently by Daya Bay and RENO.

- The rather large value of this angle gives strong impetus to INO to pursue without delay its original goal of determining the neutrino mass ordering and also to participate in the long-base-line neutrino programmes aiming to fix the matter-antimatter-symmetry-violating phase which is of cosmological importance.

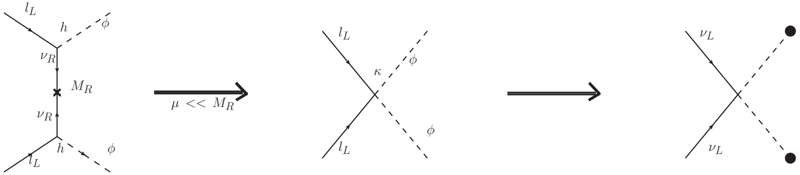
Two Neutrino Puzzles

- Two Neutrino puzzles
  - Why are the masses so tiny?
  - Why are the angles so large?
- Answers to the puzzles
  - See-saw mechanism, with Majorana ν's
  - Magnification of the angles by RG

Extension of SM with RH neutrinos and the Seesaw

- $$\mathcal{L} = h_{ij}\bar{l}_{Li}\nu_{Rj}\phi + \frac{1}{2}M^R_{ij}\bar{\nu}^c_{Ri}\nu_{Rj} + \text{h.c}$$

  where $l_i = (\nu_e, e^-)^T$, h_{ij} are the Yukawa couplings and M_{ij} are Majorana mass terms.
- After SSB $\langle\phi\rangle = v$ and $m_D = hv$.



- $\kappa = h\,\frac{1}{M_R}\,h^T$

Extension of SM with RH neutrinos and the Seesaw

- $$\mathcal{L} \xrightarrow[\mu<<M_R]{} \mathcal{L}_{\text{eff}} \sim \kappa \bar{l}_L \phi l_L \phi \rightarrow \nu_L^T M_\nu \nu_L$$

  where $M_\nu = m_D \frac{1}{M_R} m_D^T$

- This is the famous Seesaw mechanism

How RG evolution solves the large angle problem

- At high scales, both CKM and PMNS are assumed to be of the Wolfenstein form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda & \lambda^3 \\ \lambda & 0 & \lambda^2 \\ \lambda^3 & \lambda^2 & 0 \end{pmatrix}$$

- More correctly, write U in terms of θ_{12}, θ_{23} and θ_{31} with $\sin\theta_{12} \sim \lambda$, $\sin\theta_{23} \sim \lambda^2$ and $\sin\theta_{31} \sim \lambda^3$.
- Use RG to evolve U to low scales.
- CKM does not change much but PMNS changes dramatically because of the quasi-degenerate nature of the neutrino masses.

High Scale Mixing Unification

- High Scale Mixing Unification :

$$U_{\mathrm{PMNS}} = U_{\mathrm{CKM}} \text{ at high scales.}$$

- All our papers use this.
- But now I am taking the point of view that this may not be necessary.
- What is needed is only the Wolfenstein structure

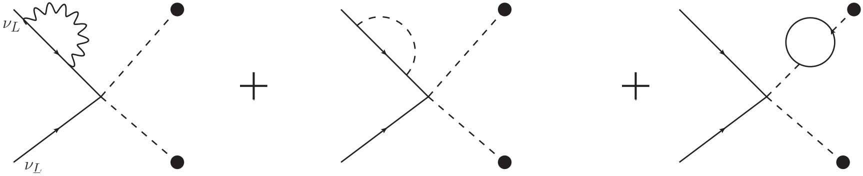
$$U_{\mathrm{PMNS}} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix},$$

  with λ small (may be ≈ 0.2)
- RG evolution then magnifies the angles.

HSMU and RG

- Before θ_{13}:
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 69 (2004)
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 71 (2005)
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 72 (2005)
  - S. K. Agarwalla, R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 75 (2007)
- After θ_{13}:
  - G. Abbas, S. Gupta, R. Srivastava abd GR, Phys. Rev. D 89 (2014)
  - G. Abbas, S. Gupta, R. Srivastava abd GR, arXiv: 1312.7384
- Works for a large range of SUSY scales and GUT scales
- Works even for Dirac neutrinos. (Rahul's talk in this symposium)

Radiative Correction and RG Evolution



- $$16\pi^2 \frac{dM_\nu}{dt} = \left\{ -\left(\frac{6}{5} g_1^2 + 6 g_2^2 \right) + \mathrm{Tr}\left(6 \mathrm{Y_U} \mathrm{Y_U^\dagger} \right) \right\} M_\nu + \frac{1}{2} \left\{ \left(Y_E Y_E^\dagger \right) M_\nu + M_\nu \left(Y_E Y_E^\dagger \right)^T \right\}$$

Radiative Correction and RG Evolution

- $Y_U Y_U^\dagger = 3 \times 3$ up-quark Yukawa matrix $\simeq \begin{pmatrix} 0 & & \\ & 0 & \\ & & h_t^2 \end{pmatrix}$

  $Y_E Y_E^\dagger = 3 \times 3$ charged lepton Yukawa matrix $\simeq \begin{pmatrix} 0 & & \\ & 0 & \\ & & h_\tau^2 \end{pmatrix}$

- Divide these by $\sin^2\beta$ and $\cos^2\beta$ respectively for MSSM, where $\tan\beta = \frac{\langle\phi_u^0\rangle}{\langle\phi_d^0\rangle}$

- Chankowski, Krolikowski and Pokorski

- Casas, Espinosa and Navaroo

Radiative Correction and RG Evolution

- Diagonalize and Run

$$
\begin{aligned}
\frac{dm_i}{dt} &= -2F_\tau m_i U_{\tau i}^2 - m_i F_U \qquad (i = 1,2,3) \\
\frac{ds_{23}}{dt} &= -F_\tau c_{23}^2(-s_{12}U_{\tau 1}D_{31} + c_{12}U_{\tau 2}D_{32}) \\
\frac{ds_{13}}{dt} &= -F_\tau c_{23}c_{13}^2(c_{12}U_{\tau 1}D_{31} + s_{12}U_{\tau 2}D_{32}) \\
\frac{ds_{12}}{dt} &= -F_\tau c_{12}(c_{23}s_{13}s_{12}U_{\tau 1}D_{31} - c_{23}s_{13}c_{12}U_{\tau 2}D_{32} + U_{\tau 1}U_{\tau 2}D_{21})
\end{aligned}
$$

where $D_{ij} = \frac{m_i+m_j}{m_i-m_j}$; $i \neq j$ and

	F_τ	F_U
MSSM	$-\frac{h_\tau^2}{16\pi^2\cos^2\beta}$	$\frac{1}{16\pi^2}\left(\frac{6}{5}g_1^2+6g_2^2-\frac{6h_t^2}{\sin^2\beta}\right)$
SM	$\frac{3h_\tau^2}{32\pi^2}$	$\frac{1}{16\pi^2}\left(3g_2^2-2\lambda-6h_t^2-2h_\tau^2\right)$

$$U=\begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12}-c_{12}s_{13}s_{23} & c_{12}c_{23}-s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23} & -c_{12}s_{23}-c_{23}s_{13}s_{12} & c_{13}c_{23} \end{pmatrix}$$

also

$$U^T M_\nu U = \mathrm{diag(m_1, m_2, m_3)}$$

Radiative Correction and RG Evolution

Some simplifications for understanding the RG evolution

- In MSSM, F_τ is enhanced by a factor $\sim 10^3$, for $\tan\beta \simeq 50$, as compared to its value in SM. So, the rapid evolution can be attributed to SUSY.
- For quasi-degenerate neutrino masses, $D_{ij} \to \infty$. Where $D_{ij} = \frac{m_i+m_j}{m_i-m_j}$; $i \neq j$ and $|D_{31}| \simeq |D_{32}| << |D_{21}|$. This contributes to quite rapid evolution.
- At high scale,

$$\begin{aligned} s_{12} &\sim \lambda \sim 0.2; \quad s_{23} \sim O(\lambda^2) \sim 0.035; \quad s_{31} \sim O(\lambda^3) \sim 0.0025 \\ &\Rightarrow U_{\tau 1} \sim O(\lambda^3); \quad U_{\tau 2} \sim O(\lambda^2) \end{aligned}$$

Radiative Correction and RG Evolution

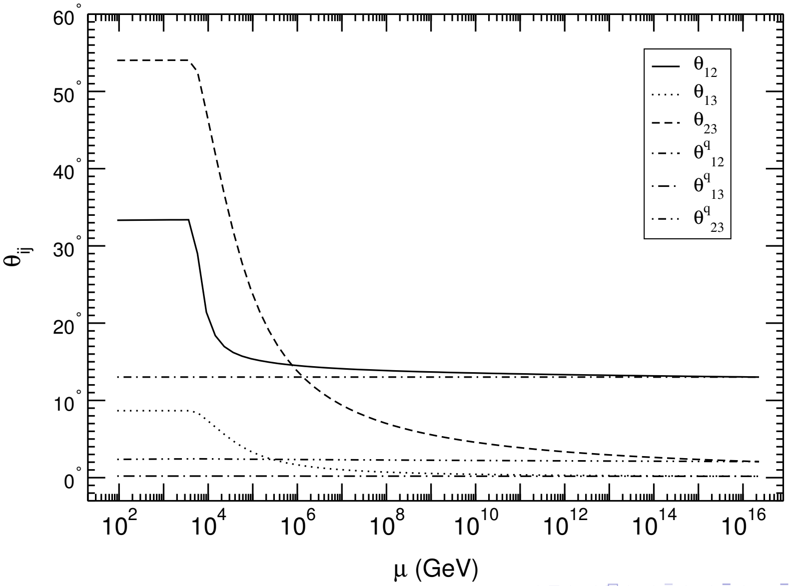
Approximate evolution eqs:

$$\begin{aligned}
\frac{ds_{23}}{dt} &\sim \lambda^2 F_\tau D_{32} \quad \text{fast; faster than } \frac{\mathrm{ds}_{12}}{\mathrm{dt}} \\
\frac{ds_{13}}{dt} &\sim \lambda^3 F_\tau (D_{32} + D_{31}) \quad \text{remains small} \\
\frac{ds_{12}}{dt} &\sim \lambda^5 F_\tau D_{21} \quad \text{smallness of } \lambda^5 \text{ compensated by largeness of } \mathrm{D}_{21}
\end{aligned}$$

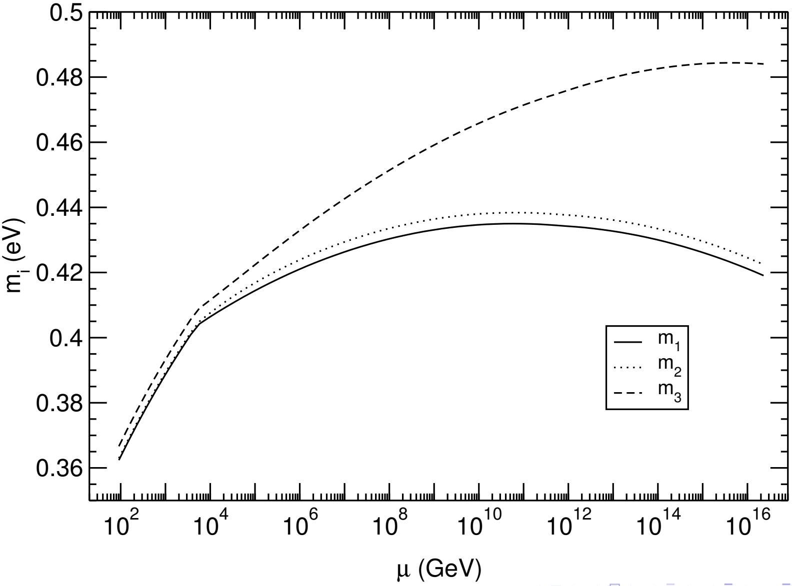
Remember

$$\begin{aligned}
D_{ij} &= \frac{m_i + m_j}{m_i - m_j}; \quad i \neq j \\
|D_{31}| &\simeq |D_{32}| << |D_{21}|.
\end{aligned}$$

RG Evolution of Mixing Angles



RG Evolution of Neutrino Masses



Models

Models (a dime a dozen!)

- Models have not solved the problem.
- Will describe one model, to illustrate the negative point I am making about the models in general.

TRIBIMAXIMAL MIXING

- $$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

- $$s_1 = \sqrt{\frac{1}{2}}, \quad s_3 = \sqrt{\frac{1}{3}}, \quad s_2 = 0$$

- Harrison, Perkins and Scott (2002).

The Truly Maximal Mixing Matrix

- The most beautiful mixing matrix

$$U = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$, $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

  - Maximal mixing
  - Equal mixtures
  - On par with $U = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
  - Maximal CP violation

The Truly Maximal Mixing Matrix

- Proposed by Cabibbo and Wolfenstein in 1978 !
  - Cabibbo, PLB, 72,333 (1978)
  - Wolfenstein, PRD 18, 958 (1978)
- Theoretically derived in A_4 symmetry in 2001 by Ernest Ma and GR (PRD 64)

COMPARISON

- $$U = \begin{pmatrix} c_2c_3 & c_2s_3 & s_2 \\ -c_1s_3 - s_1s_2s_3e^{i\delta} & c_1c_3 - s_1s_2s_3e^{i\delta} & s_1c_2e^{i\delta} \\ s_1s_3 - c_1s_2s_3e^{i\delta} & -s_1c_3 - c_1s_2s_3e^{i\delta} & c_1c_2e^{i\delta} \end{pmatrix}$$
  $$\rightarrow \begin{pmatrix} c_2c_3 & c_2s_3 & s_2 \\ -c_1s_3e^{-i\delta} - s_1s_2s_3 & c_1c_3e^{-i\delta} - s_1s_2s_3 & s_1c_2 \\ s_1s_3e^{-i\delta} - c_1s_2s_3 & -s_1c_3e^{-i\delta} - c_1s_2s_3 & c_1c_2 \end{pmatrix}$$
  $$= \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}$$
- Truly Maximal Mixing: $s_1 = s_3 = \frac{1}{\sqrt{2}},\ s_2 = \frac{1}{\sqrt{3}},\ \delta = \frac{\pi}{2}$
- Tribimaximal Mixing: $s_1 = \frac{1}{\sqrt{2}},\ s_3 = \frac{1}{\sqrt{3}},\ s_2 = 0$

CONNECTION

- Connection between Truly Maximal Mixing and Tribimaximal Mixing

$$U = \sqrt{\frac{1}{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

- This massaging can be achieved in a A_4 model.
- E. Ma, PRD 70 (2004)

A_4 Group

- Symmetry group of a tetrahedron
- 12 elements
- S_4 (24 elements) $\supset A_4$
- IRREPs: 1, $1'$, $1''$, 3
- Product rule: $3 \times 3 = 1 + 1' + 1'' + 3 + 3$
- If $(a_1, a_2, a_3) \sim 3$ and $(b_1, b_2, b_3) \sim 3$ then

$$
\begin{aligned}
1 &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
1' &= a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \\
1'' &= a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \\
3 &= (a_2 b_3, a_3 b_1, a_1 b_2) \\
3 &= (a_3 b_2, a_1 b_3, a_2 b_1)
\end{aligned}
$$

A_4 Model

- Higgs Doublets and Triplets: Many of Them !
- The leptons and scalars with their A_4 representations are:

$$(\nu_i, l_i)_L \sim 3, \quad l_{1R} \sim 1, \quad l_{2R} \sim 1', \quad l_{3R} \sim 1''$$
$$\Phi_i \equiv (\phi_i^+, \phi_i^0) \sim 3, \quad \xi_i \equiv (\xi_i^{++}, \xi_i^+, \xi_i^0)$$
$$\xi_1 \sim 1, \quad \xi_2 \sim 1', \quad \xi_3 \sim 1'', \quad \xi_i \sim 3; i = 4, 5, 6$$

-

$$\mathcal{L} \supset f_{ijk}(\bar{l}_{iL}\phi_j^0 - \bar{\nu}_{iL}\bar{\phi}_j)l_{kR} + \mathrm{h.c} + \mathrm{h_{ijk}}\left[\xi_\mathrm{i}^0\nu_\mathrm{j}\nu_\mathrm{k} - \frac{\xi_\mathrm{i}^+}{\sqrt{2}}(\nu_\mathrm{j}\mathrm{l_k} + \mathrm{l_j}\nu_\mathrm{k})\right.$$
$$\left. + \quad \xi_i^{++}l_j l_k\right] + \mathrm{h.c.}$$
$$\langle\phi_j^0\rangle \to \text{masses for l} \quad \langle\xi_\mathrm{j}^0\rangle \to \text{masses for } \nu$$

The Triplet See-saw

- The Higgs sector:

$$
\begin{aligned}
V &= m^2\Phi^\dagger\Phi + M^2\xi^\dagger\xi + \frac{1}{2}\lambda_1(\Phi^\dagger\Phi)^2 + \frac{1}{2}\lambda_2(\xi^\dagger\xi)^2 + \lambda_3\Phi^\dagger\Phi\xi^\dagger\xi \\
&- \mu(\bar{\xi}^0\phi^0\phi^0 + \sqrt{2}\xi^-\phi^+\phi^0 + \xi^{--}\phi^+\phi^+) + \text{h.c.}
\end{aligned}
$$

- Put $\langle\phi_j^0\rangle = v$, $\langle\xi_j^0\rangle = u$
- Minimum of V :

$$
\begin{aligned}
\frac{\partial V}{\partial v} = 0 \quad &\Rightarrow \quad m^2 + \lambda_1 v^2 + \lambda_3 u^2 - 2\mu v = 0 \\
\frac{\partial V}{\partial u} = 0 \quad &\Rightarrow \quad u(M^2 + \lambda_2 u^2 + \lambda_3 v^2) - \mu v^2 = 0
\end{aligned}
$$

The Triplet See-saw

- Take M to be very large. Then,

$$m^2 + \lambda_1 v^2 = 0$$
$$uM^2 + \mu v^2 = 0$$
$$\Rightarrow u = \frac{\mu v^2}{M^2}$$

- Also

$$m_\nu = hu = hv\frac{\mu v}{M^2}$$

Charged Lepton Masses

- $$\begin{aligned}\mathcal{L} &= f_1(\bar{l}_{1L}\phi_1^0 + \bar{l}_{2L}\phi_2^0 + \bar{l}_{3L}\phi_3^0)l_{1R} \\ &+ f_2(\bar{l}_{1L}\phi_1^0 + \omega\bar{l}_{2L}\phi_2^0 + \omega^2\bar{l}_{3L}\phi_3^0)l_{2R} \\ &+ f_3(\bar{l}_{1L}\phi_1^0 + \omega^2\bar{l}_{2L}\phi_2^0 + \omega\bar{l}_{3L}\phi_3^0)l_{3R}\end{aligned}$$
- Put $\langle\phi_i^0\rangle = v_i$. The masses are given by

$$M_l = \begin{pmatrix} f_1 v_1 & f_2 v_1 & f_3 v_1 \\ f_1 v_2 & \omega f_2 v_2 & \omega^2 f_3 v_2 \\ f_1 v_3 & \omega^2 f_2 v_3 & \omega f_3 v_3 \end{pmatrix}$$

- Assuming $v_1 = v_2 = v_3 = v$

$$M_l = v\begin{pmatrix} f_1 & f_2 & f_3 v_1 \\ f_1 & \omega f_2 & \omega^2 f_3 \\ f_1 & \omega^2 f_2 & \omega f_3 \end{pmatrix} = U_{lL}\begin{pmatrix} \sqrt{3}vf_1 & 0 & 0 \\ 0 & \sqrt{3}vf_2 & 0 \\ 0 & 0 & \sqrt{3}vf_3 \end{pmatrix}U_{lR}^\dagger$$

Charged Lepton Masses

- Where U_{lL} is the truely maximal mixing matrix given by

$$U_{lL} \quad = \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

- And

$$U_{lR} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Neutrino Masses

- $$\begin{aligned}\mathcal{L} &= h_1(\nu_1\nu_1 + \nu_2\nu_2 + \nu_3\nu_3)\langle\xi_1^0\rangle \\ &+ h_2(\nu_1\nu_1 + \omega\nu_2\nu_2 + \omega^2\nu_3\nu_3)\langle\xi_2^0\rangle \\ &+ h_3(\nu_1\nu_1 + \omega^2\nu_2\nu_2 + \omega\nu_3\nu_3)\langle\xi_3^0\rangle \\ &+ h_4(\nu_2\nu_3\langle\xi_4^0\rangle + \nu_3\nu_1\langle\xi_5^0\rangle + \nu_1\nu_1\langle\xi_6^0\rangle)\end{aligned}$$
- Denoting the VEVs of ξ_i's as $\langle\xi_1^0\rangle = a$, $\langle\xi_2^0\rangle = b$, $\langle\xi_3^0\rangle = c$, $\langle\xi_4^0\rangle = d$, $\langle\xi_5^0\rangle = \langle\xi_6^0\rangle = 0$. These are natural minima of Higgs potential for a continuous range of parameter values.
- The neutrino mass matrix becomes

$$M_\nu = \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+\omega b+\omega^2 c & d \\ 0 & d & a+\omega^2 b+\omega c1 \end{pmatrix}$$

Neutrino Masses

- If $b = c$

$$
\begin{aligned}
M_\nu &= \begin{pmatrix} a+2b & 0 & 0 \\ 0 & a-b & d \\ 0 & d & a-b \end{pmatrix} \\
&= U_{\nu L} \begin{pmatrix} \frac{1}{\sqrt{2}}(d+a-b) & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}(d-a+b) & 0 \\ 0 & 0 & a+2b \end{pmatrix} U_{\nu R}^T
\end{aligned}
$$

where

$$
U_{\nu L} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix}
$$

PMNS Mixing Matrix

- $$\begin{aligned} U_{\text{PMNS}} &= U_{lL}^{\dagger} U_{\nu L} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \end{aligned}$$
- The desired tribimaximal matrix has been generated from the truly maximal mixing matrix occuring naturally in A_4.

Other A_4 models and their particle content

Φ	ξ	χ^0	N	SUSY	Ref
$1, 3$	$-$	1	3	No	1
3	$1, 1', 1'', 3$	$-$	$-$	No	2
$1, 1$	$-$	3	3	Yes	3
$1, 1$				Ex Dim	4
3	$1', 1'', 3$			No	5
$1, 1$		$1, 1, 3, 3$	3	Yes	6
3	$-$	$-$	$-$	No	7
$1, 1$	$-$	$1, 1, 1, 3, 3, 3, 3$	$-$	No	8
$1, 1', 1''$	1	3	3		9
1	3	$1, 1', 1'', 3$	3		10
$1, 1', 1''$	$1, 1', 1'', 3$	$-$	$-$		11
$1, 1', 1'', 3$	$-$	$-$	$1, 1', 1''$		12

More than 30! (upto a few years ago)

Other A_4 models and their particle content

References:

- 1 : Ma and GR, PRD 64 (01)
- 2 : Ma, PRD 70 (04)
- 3 : Babu, Ma, Valle, PLB 552 (03)
- 4 : Altarelli, Feruglio, NPB 720 (05)
- 5 : Ma, PRD 72 (05)
- 6 : Babu, He, hep-ph/0507217 (05)
- 7 : Zee, PLB 630 (05)
- 8 : Altarelli, Feruglio, NPB 741 (05)
- 9 : Chen, Frigerio, Ma, NPB 724 (05)
- 10 : Ma, MPLA 20 (05)
- 11 : Hirsch, Ma, Moreli,Valle, PRD, 72 (05)
- 12 : Ma, PLB, 632 (06)

Other Discrete Symmetries

- Discrete subgroups of $SU(3)$ and $U(3)$:
- $SU(3)$ contains two series of discrete groups: $\Delta(3n^2)$ and $\Delta(3n^2-3)$.

n	1	2	3	4	5
$\Delta(3n^2)$	Z_3	A_4	$\Delta(27)$	$\Delta(48)$	$\Delta(75)$
$\Delta(3n^2-3)$		$Z_3 \times Z_3$	S_4		

- $U(3)$ has the series $\Sigma(3n^3)$ which is:
  - $\Sigma(24) = A_4 \times Z_2$ for $n = 2$.
  - $\Sigma(81)$ for $n = 3$

Other Discrete Symmetries

- $Q(24)$: Quaternion group, which is a subgroup of $SU(2)$. It is also the double cover of A_4 (which is a subgroup of $SO(3)$).
- S_4 : Permutation of 4 objects
- D_4 : Symmetry group of the square
- Q_4, D_5, D_6, Q_6, Q_7
- B_4 : Coexeter group (symmetry group of the hyperoctahedron with 384 elements)
- $B_3 \times Z_2^3$: 384 elements

A Theorem on Higgs

- Scalar bosons have become the theoretician's tool in building models. Whether one wants to build models going beyond SM or wants to explain some perceived discrepancy of experimental data with SM, one creates a new scalar sector.
- **Theorem: One can construct a scalar sector to solve any problem in HEP (even cosmology).**
- Consequently, 100's or 1000's of models have been constructed in the last 25 years, many with 100's or 1000's of scalar bosons.
- But only a single one, the original one required in electroweak theory has been seen experimentally!

Fermion Masses

- In the SM, all the 12 fermion masses are arbitrary parameters fixed only by experiments.
- Perhaps one has to extend SM to include a theory of families for understanding the pattern of fermion masses.
- Enormous amount of theoretical work has been done to attack this problem, but there is no memorable result.
- ν masses alone?
- Charged lepton masses vs the rest

Koide Relation

- In 1982, Yoshjo Koide (Univ of Shizuoka, Japan) found an empirical relation:

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2$$

which is satisfied to an accuracy of 2 parts in 10^5.

$$\begin{aligned} m_e &= 0.510998902 \pm 0.000000021 \text{ MeV} \\ m_\mu &= 105.658357 \pm 0.000005 \text{ MeV} \\ m_\tau &= 1776.99^{+0.29}_{-0.26} \text{ MeV} \\ \Rightarrow \frac{\text{L.H.S}}{\text{R.H.S}} &= 1.0000 \pm 0.00002 \end{aligned}$$

Koide Relation

- The Koide relation is truly a miracle relation! There does not exist any other relation of comparable accuracy in all of HEP (except, of course, the precision calculations in QED and to a certain extent, in EWD).
- **The Koide relation is crying out for a derivation !**
- In fact, when Koide proposed his relation τ-mass had not been measured correctly. The relation can be solved for m_τ to predict it. The existing world-average of m_τ did not quite agree with the predicted value. Koide thought of fudging the formula a little, to make it agree.
- Soon, after the Beijing accelerator was built, an intensive study of τ revealed that the true m_τ was different from then-existing world-average. The new corrected value of $(m_\tau)_{\text{exp}}$ agreed perfectly with $(m_\tau)_{\text{Koide}}$.
- This was a triumph of Koide formula.

Koide Relation: Geometrical interpretation

- Consider the two vectors:

$$\vec{m} \equiv (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$$
$$\vec{s} \equiv (1, 1, 1)$$

- Koide's relation says that the angle between the two vectors is 45° i.e.

$$\frac{\vec{m}\,.\,\vec{s}}{|\vec{m}||\vec{s}|} = \frac{1}{\sqrt{2}}$$
$$\frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{3}(m_e + m_\mu + m_\tau)^{1/2}} = \frac{1}{\sqrt{2}}$$

Is this the Balmer Formula?

- Is the Koide relation the much needed Balmer formula which can guide us towards finding the hard ground amidst the swampland of theories of fermion masses and discovering the correct theory of families?
- See-saw for all fermions?
- Scale invariant?