COEFFICIENTS OF A p-ADIC MEASURE AND IWASAWA λ-INVARIANT OF ITS Γ-TRANSFORM

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Abstract. In this paper we relate the coefficients of a p-adic valued measure α on $\mathbb{Z}_p^2$ to the λ-invariant of the Iwasawa series of the Γ-transform of α.
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1. Introduction

Fix an odd prime p. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ with a local parameter π. We write $\mathbb{Z}_p^\times = V \times U$ where V is the group of $(p-1)$st roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. Let u be a topological generator of U. The projections from $\mathbb{Z}_p^\times$ onto V and U are denoted by ω and $<>$ respectively. We have an isomorphism $\phi : \mathbb{Z}_p \to U$ given by $\phi(y) = u^y$.

Let $\Lambda_{(n)}$ denote the $\mathcal{O}$-valued measures on $\mathbb{Z}_p^n$. It is well-known, (see e.g. [1], [2]), that $\Lambda_{(n)}$ is a ring under convolution, and is isomorphic to the formal power series ring $\mathcal{O}[[T_1 - 1, \cdots, T_n - 1]]$. This correspondence is given by

$$\begin{aligned}\widehat{\alpha}(T_1, \cdots, T_n) &= \int_{\mathbb{Z}_p^n} T_1^{x_1} \cdots T_n^{x_n} d\alpha(x_1, \cdots, x_n) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \left(\int_{\mathbb{Z}_p^n} \binom{x_1}{m_1} \cdots \binom{x_n}{m_n} d\alpha(x_1, \cdots, x_n) \right) \\ &\quad \times (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \end{aligned} \tag{1.1}$$

To integrate any continuous function $f : \mathbb{Z}_p \to \mathcal{O}$ with respect to a measure $\alpha \in \Lambda_{(1)}$, we use a theorem of Mahler, (see [5], [8]):

Theorem 1.1. *If $f : \mathbb{Z}_p \to \mathcal{O}$ is continuous, then*

$$f(x) = \sum_{j=0}^{\infty} m_j(f) \binom{x}{j},$$

where $m_j(f) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(k) \to 0$ in $\mathcal{O}$.

Using Mahler's theorem, if $\widehat{\alpha}(T_1) = \sum_{j=0}^{\infty} a_j (T_1 - 1)^j$, then $\int_{\mathbb{Z}_p} f(x) d\alpha(x) = \sum_{j=0}^{\infty} a_j m_j(f)$.

Furthermore, if $f : \mathbb{Z}_p^n \mapsto \mathcal{O}$ is continuous, we may write (by repeated application of Mahler's theorem)

$$f(x_1, \cdots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} a_{m_1,\cdots,m_n}(f) \binom{x_1}{m_1} \cdots \binom{x_n}{m_n}, \tag{1.2}$$

where

$$\begin{aligned} a_{m_1,\cdots,m_n}(f) &= \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} (-1)^{m_1 - j_1} \cdots (-1)^{m_n - j_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} f(j_1, \cdots, j_n) \\ &\to 0 \quad \text{in} \quad \mathcal{O}. \end{aligned} \tag{1.3}$$

The constants $a_{m_1,\cdots,m_n}(f)$ are called the Mahler coefficients of the function f.

Similar to the case $n = 1$, we have the following integration formulas:

$$\int_{\mathbb{Z}_p^n} x_1^{m_1} \cdots x_n^{m_n} d\alpha(x_1, \cdots, x_n) = \left(T_1 \frac{d}{dT_1}\right)^{m_1} \cdots \left(T_n \frac{d}{dT_n}\right)^{m_n} \widehat{\alpha}(T_1, \cdots, T_n)|_{T_1 = \cdots = T_n = 1} \tag{1.4}$$

Let α be a measure on $\mathbb{Z}_p^n$. For $(a_1, \cdots, a_n) \in (\mathbb{Z}_p^\times)^n$, denote by $\alpha \circ (a_1, \cdots, a_n)$ the measure on $\mathbb{Z}_p^n$ given by

$$\alpha \circ (a_1, \cdots, a_n)(A_1 \times \cdots \times A_n) = \alpha(a_1 A_1, \cdots, a_n A_n),$$

where A_i are compact open subsets of $\mathbb{Z}_p$. Also, if $A = (A_1, \cdots, A_n) \subseteq \mathbb{Z}_p^n$, where all A_i are compact open subsets of $\mathbb{Z}_p$, we let $\alpha|_A$ denote the measure obtained by restricting α to A and extending by 0.

For $s_1, \cdots, s_n \in \mathbb{Z}_p$, let each of $k_1, \cdots, k_n$ vary through a sequence of positive integers satisfying $k_j \to s_j$ p-adically and $k_j \equiv 0 \pmod{p-1}$. Then Γ-transform of a measure $\alpha \in \Lambda_{(n)}$ is defined as a function of the p-adic variables $s_1, \cdots, s_n$ given by

$$\begin{aligned} \Gamma_\alpha(s_1, \cdots, s_n) &= \lim_{k_1,\cdots,k_n} \int_{\mathbb{Z}_p^n} x_1^{k_1} \cdots x_n^{k_n} d\alpha(x_1, \cdots, x_n) \\ &= \int_{(\mathbb{Z}_p^\times)^n} < x_1 >^{s_1} \cdots < x_n >^{s_n} d\alpha(x_1, \cdots, x_n). \end{aligned} \tag{1.5}$$

If we put $d\alpha(a_1 x_1, \cdots, a_n x_n)$ for $d(\alpha \circ (a_1, \cdots, a_n))(x_1, \cdots, x_n)$, splitting up the integral, we can also write

$$\begin{aligned} \Gamma_\alpha(s_1, \cdots, s_n) &= \sum_{\eta_1 \in V} \cdots \sum_{\eta_n \in V} \int_{U^n} < \eta_1 x_1 >^{s_1} \cdots < \eta_n x_n >^{s_n} d\alpha(\eta_1 x_1, \cdots, \eta_n x_n) \\ &= \int_{U^n} x_1^{s_1} \cdots x_n^{s_n} d\beta(x_1, \cdots, x_n), \end{aligned}$$

where

$$\beta = \sum_{\eta_1 \in V} \cdots \sum_{\eta_n \in V} (\alpha \circ (\eta_1, \cdots, \eta_n))|_{U^n},$$

a measure on U^n. We extend β to $\mathbb{Z}_p^n$ by 0 and then we get a power series

$$\widehat{\beta}(T_1,\cdots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}b_{m_1,\cdots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}. \tag{1.6}$$

Again, the measure β may be viewed as a measure on $\mathbb{Z}_p^n$ via the isomorphism ϕ:

$$\tilde{\beta}(A_1,\cdots,A_n)=\beta(\phi(A_1),\cdots,\phi(A_n)).$$

Let us write $d\beta(u^{y_1},\cdots,u^{y_n})$ for $d\tilde{\beta}(y_1,\cdots,y_n)$. Let $G(T_1,\cdots,T_n)$ be the power series associated to $\tilde{\beta}$, that is,

$$G(T_1,\cdots,T_n)=\int_{\mathbb{Z}_p^n}T_1^{y_1}\cdots T_n^{y_n}d\beta(u^{y_1},\cdots,u^{y_n}).$$

Then $\Gamma_\alpha(s_1,\cdots,s_n)=G(u^{s_1},\cdots,u^{s_n})$.

The Iwasawa μ and λ- invariants of a power series

$$F(T_1,\cdots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}a_{m_1,\cdots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}\in\mathcal{O}[[T_1-1,\cdots,T_n-1]]$$

are defined as follows:

$$\mu(F(T_1,\cdots,T_n))=\min\{ord(a_{m_1,\cdots,m_n}):m_i\geq 0\quad\forall i\}$$
$$\lambda(F(T_1,\cdots,T_n))=\min\{m_1+\cdots+m_n:ord(a_{m_1,\cdots,m_n})=\mu(F(T_1,\cdots,T_n))\}.$$

For a measure $\alpha\in\Lambda_{(n)}$, we understand $\mu(\alpha)$ and $\lambda(\alpha)$ to mean $\mu(\widehat{\alpha}(T_1,\cdots,T_n))$ and $\lambda(\widehat{\alpha}(T_1,\cdots,T_n))$.

In case of $n=1$, Childress in her paper [3] showed how the coefficients of the power series associated to a p-adic valued measure α on $\mathbb{Z}_p$ are related to the coefficients of the measure β. She proved congruences modulo p amongst these coefficients. Finally, using these congruences and the results of [2], [4] and [7], she related the coefficients of α to the λ-invariant of the Iwasawa series of the Γ-transform of α. One can easily generalize the congruences modulo p amongst the coefficients to any p-adic valued measure α on $\mathbb{Z}_p^n$. In case $n>1$, to relate the coefficients of α to the λ-invariant of the Iwasawa series of the Γ-transform of α, one needs the results of [1]. In this paper, we relate the coefficients of $\alpha\in\Lambda_{(2)}$ to the λ-invariant of the Iwasawa series of the Γ-transform of α using the results of [1]. However, one can produce similar results for any $\alpha\in\Lambda_{(n)}$, but the number of coefficients of $\widehat{\alpha}(T_1-1,\cdots,T_n-1)$ which are involved will increase with n.

2. The series associated to β

Let $\alpha\in\Lambda_{(2)}$ and $\eta,\nu\in V$. Let us fix a primitive p^{th} root of unity ζ. The characteristic function of $\eta U\times\nu U$ is $\left[\frac{1}{p}\sum_{j_1=1}^{p}\zeta^{j_1(x_1-\eta)}\right]\times\left[\frac{1}{p}\sum_{j_2=1}^{p}\zeta^{j_2(x_2-\nu)}\right]$. Using this and Theorem 5 proved by Childress in her paper [3], we have the following result.

Theorem 2.1. *Let $\alpha\in\Lambda_{(2)}$ and let $\widehat{\alpha}(T_1,T_2)=\sum\sum a_{i_1,i_2}(T_1-1)^{i_1}(T_2-1)^{i_2}$ be the associated power series. Let η and ν be fixed $(p-1)^{th}$ root of unity in $\mathbb{Z}_p$. Given non-negative integers k_1,k_2, let m_1,m_2 be the integers such that $m_1p\leq k_1<(m_1+1)p$ and $m_2p\leq k_2<(m_2+1)p$. Put $k_1=m_1p+k_1^0$ and $k_2=m_2p+k_2^0$. Let $\eta_0<p$ and $\nu_0<p$ be the positive integers*

such that $\eta \equiv \eta_0$ *(mod p) and* $\nu \equiv \nu_0$ *(mod p). Then the coefficient of* $(T_1 - 1)^{k_1}(T_2 - 1)^{k_2}$ *in* $\widehat{\alpha|_{\eta U \times \nu U}}(T_1, T_2)$ *is* $e^{\eta,\nu}_{k_1,k_2}$*, where, modulo p, we have*

$$e^{\eta,\nu}_{k_1,k_2} \equiv \binom{\eta_0}{k_1^0}\binom{\nu_0}{k_2^0} \sum_{j_1=0}^{p-\eta_0-1} \sum_{j_2=0}^{p-\nu_0-1} \binom{j_1 + \eta_0}{j_1}\binom{j_2 + \nu_0}{j_2} (-1)^{j_1+j_2} a_{pm_1+\eta_0+j_1, pm_2+\nu_0+j_2}. \tag{2.1}$$

Now, we note that $\widehat{\alpha \circ (\eta, \nu)}(T_1, T_2) = \widehat{\alpha}(T_1^{\overline{\eta}}, T_2^{\overline{\nu}})$, where $\overline{\eta} = \eta^{-1}$ and $\overline{\nu} = \nu^{-1}$. Also, $(\alpha \circ (\eta, \nu))|_{U^2} = (\alpha|_{\eta U \times \nu U}) \circ (\eta, \nu)$. Therefore,

$$\widehat{\beta}(T_1, T_2) = \sum_{\eta \in V} \sum_{\nu \in V} \widehat{\alpha|_{\eta U \times \nu U}}(T_1^{\overline{\eta}}, T_2^{\overline{\nu}}). \tag{2.2}$$

In case $\alpha \in \Lambda_{(1)}$, Childress in her paper [3] proved certain congruences modulo p amongst the coefficients of $\widehat{\alpha}(T)$ and $\widehat{\beta}(T)$. Using her approach, we shall prove Theorem (2.3) below. Let us now state a useful Lemma from [3] and one can give a proof of the Lemma by induction on k.

Lemma 2.2. *For any positive integer k,*

$$\left[\sum_{j=1}^{\infty} \binom{\eta}{j} Y^j\right]^k = \sum_{j=k}^{\infty} \rho_\eta(j, k) Y^j,$$

where $\rho_\eta(j, k)$ *is defined by:* $\rho_\eta(j, 1) = \binom{\eta}{j}, \rho_\eta(j, k) = \sum_{i=1}^{j-1} \binom{\eta}{i} \rho_\eta(j - i, k - 1)$.

For notational convenience, we set $\rho_\eta(j, 0) = \rho_\eta(0, k) = 0$ when $jk \neq 0$ and $\rho_\eta(0, 0) = 1$.

Theorem 2.3. *For* $j_1 \geq 0, j_2 \geq 0$,

$$b_{j_1 p, j_2 p} \equiv \sum_{\eta \in V} \sum_{\nu \in V} \sum_{i_1=\eta_0}^{p-1} \sum_{i_2=\nu_0}^{p-1} (-1)^{i_1+i_2-\eta_0-\nu_0} \binom{i_1}{\eta_0}\binom{i_2}{\nu_0} \sum_{r_1=0}^{j_1} \sum_{r_2=0}^{j_2} a_{pr_1+i_1, pr_2+i_2} \times$$
$$\sum_{t_1=0}^{r_1} \sum_{t_2=0}^{r_2} (-1)^{r_1+t_1+r_2+t_2} \binom{r_1}{t_1}\binom{r_2}{t_2} \binom{\overline{\eta}(t_1 + \frac{\eta_0-\eta}{p})}{j_1} \binom{\overline{\nu}(t_2 + \frac{\nu_0-\nu}{p})}{j_2} \pmod{p}. \tag{2.3}$$

Proof: Note that

$$\begin{aligned}
\widehat{\beta}(T_1, T_2) &= \sum_{\eta \in V} \sum_{\nu \in V} \widehat{\alpha|_{\eta U \times \nu U}}(T_1^{\overline{\eta}}, T_2^{\overline{\nu}}) \\
&= \sum_{\eta \in V} \sum_{\nu \in V} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} e^{\eta,\nu}_{k_1,k_2} (T_1^{\overline{\eta}} - 1)^{k_1} (T_2^{\overline{\nu}} - 1)^{k_2} \\
&= \sum_{\eta \in V} \sum_{\nu \in V} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} e^{\eta,\nu}_{k_1,k_2} \left[\sum_{j_1}^{\infty} \binom{\overline{\eta}}{j_1} (T_1 - 1)^{j_1}\right]^{k_1} \left[\sum_{j_2}^{\infty} \binom{\overline{\nu}}{j_2} (T_2 - 1)^{j_2}\right]^{k_2} \\
&= \sum_{\eta \in V} \sum_{\nu \in V} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} e^{\eta,\nu}_{k_1,k_2} \sum_{j_1=k_1}^{\infty} \sum_{j_2=k_2}^{\infty} \rho_{\overline{\eta}}(j_1, k_1) \rho_{\overline{\nu}}(j_2, k_2) (T_1 - 1)^{j_1} (T_2 - 1)^{j_2}
\end{aligned} \tag{2.4}$$

To obtain (2.4), we applied Lemma (2.2). From (2.4), we finally obtain

$$\widehat{\beta}(T_1,T_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}(T_1-1)^{j_1}(T_2-1)^{j_2}\left[\sum_{k_1=0}^{j_1}\sum_{k_2=0}^{j_2}\sum_{\eta\in V}\sum_{\nu\in V}e^{\eta,\nu}_{k_1,k_2}\rho_{\overline{\eta}}(j_1,k_1)\rho_{\overline{\nu}}(j_2,k_2)\right]. \tag{2.5}$$

Modulo p, we have

$$\begin{aligned}
b_{0,0} &\equiv \sum_{\eta\in V}\sum_{\nu\in V}e^{\eta,\nu}_{0,0}\\
&\equiv \sum_{\eta_0=1}^{p-1}\sum_{\nu_0=1}^{p-1}\sum_{j_1=0}^{p-1-\eta_0}\sum_{j_2=0}^{p-1-\nu_0}\binom{j_1+\eta_0}{j_1}\binom{j_2+\nu_0}{j_2}(-1)^{j_1+j_2}a_{\eta_0+j_1,\nu_0+j_2}\\
&\equiv \sum_{j_1=0}^{p-2}\sum_{j_2=0}^{p-2}(-1)^{j_1+j_2}\sum_{\eta_0=1}^{p-1-j_1}\sum_{\nu_0=1}^{p-1-j_2}\binom{j_1+\eta_0}{j_1}\binom{j_2+\nu_0}{j_2}a_{\eta_0+j_1,\nu_0+j_2}\\
&\equiv \sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}a_{k_1,k_2}\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2-1}\binom{k_1}{i_1}\binom{k_2}{i_2}(-1)^{i_1+i_2}\\
&\equiv \sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}(-1)^{k_1+k_2}a_{k_1,k_2}.
\end{aligned} \tag{2.6}$$

Similarly, if j_1 or $j_2\geq 1$, then following the approach of Childress [3], modulo p we have

$$\begin{aligned}
b_{pj_1,pj_2} \equiv{}& \sum_{\eta\in V}\sum_{\nu\in V}\sum_{i_1=\eta_0}^{p-1}\sum_{i_2=\nu_0}^{p-1}\binom{i_1}{\eta_0}\binom{i_2}{\nu_0}(-1)^{i_1+i_2-\eta_0-\nu_0}\sum_{r_1=0}^{j_1}\sum_{r_2=0}^{j_2}a_{pr_1+i_1,pr_2+i_2}\times\\
&\sum_{k_1=0}^{\eta_0}\sum_{k_2=0}^{\nu_0}\binom{\eta_0}{k_1}\binom{\nu_0}{k_2}\rho_{\overline{\eta}}(pj_1,pr_1+k_1)\rho_{\overline{\nu}}(pj_2,pr_2+k_2).
\end{aligned} \tag{2.7}$$

¿From the definition of ρ,
$\sum_{k_1=0}^{\eta_0}\binom{\eta_0}{k_1}\rho_{\overline{\eta}}(pj_1,pr_1+k_1)$ is the coefficient of $Y_1^{j_1p}$ in $\sum_{k_1=0}^{\eta_0}\binom{\eta_0}{k_1}\left[\sum_{t_1}^{\infty}\binom{\overline{\eta}}{t_1}Y_1^{t_1}\right]^{r_1p+k_1}$ and hence it is the coefficient of $Y_1^{j_1p}$ in $(1+Y_1)^{\overline{\eta}\eta_0}\left((1+Y_1)^{\overline{\eta}}-1\right)^{r_1p}$. Let $x_1=\frac{\overline{\eta}(\eta_0-\eta)}{p}$ and clearly $x_1\in\mathbb{Z}_p$. Now, we have

$$\begin{aligned}
&\sum_{k_1=0}^{\eta_0}\binom{\eta_0}{k_1}\rho_{\overline{\eta}}(pj_1,pr_1+k_1)\\
&\qquad= \text{coefficient of } Y_1^{pj_1} \text{ in } (1+Y_1)^{p_1x_1+1}\left((1+Y_1)^{\overline{\eta}}-1\right)^{r_1p}\\
&\qquad\equiv \text{coefficient of } Y_1^{pj_1} \text{ in } (1+Y_1)(1+Y_1^p)^{x_1}\left((1+Y_1)^{\overline{\eta}}-1\right)^{r_1}\\
&\qquad\equiv \text{coefficient of } Y_1^{j_1} \text{ in } (1+Y_1)^{x_1}\left((1+Y_1)^{\overline{\eta}}-1\right)^{r_1}\\
&\qquad\equiv \sum_{t_1=0}^{r_1}(-1)^{r_1+t_1}\binom{r_1}{t_1}\binom{x_1+\overline{\eta}t_1}{j_1} \pmod p.
\end{aligned} \tag{2.8}$$

¿From (2.6)-(2.8), we complete the proof of the theorem.

3. An application to λ-invariants

In this section we give criteria for the value of the λ-invariant of the power series $\widehat{\beta}(T_1, T_2)$ in terms of the coefficients of $\widehat{\alpha}(T_1, T_2)$. If $\alpha \in \Lambda_{(1)}$, then the λ-invariant of $\widehat{\beta}(T)$ is p times the λ-invariant of the Iwasawa series of G as in [2], [4], [6], [7]. In [1], we proved the following theorem.

Theorem 3.1. *Suppose $\lambda(G(T_1, \cdots, T_n)) \leq 2p$, then $\lambda(\beta) = p\lambda(G(T_1, \cdots, T_n))$.*

Suppose that

$$G(T_1, \cdots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} g_{m_1,\cdots,m_n}(T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \tag{3.1}$$

If $\lambda(G(T_1, \cdots, T_n)) = k$, then for a partition $k_1 + \cdots + k_n$ of k, $g_{k_1,\cdots,k_n}$ is a unit in $\mathcal{O}$. In the proof of the Theorem (3.1), we showed that $b_{pk_1,\cdots,pk_n}$ is also a unit in $\mathcal{O}$. Using this and Theorem (2.3), we will give criteria for the λ-invariant of the power series $\widehat{\beta}(T_1, T_2)$ in terms of the coefficients of $\widehat{\alpha}(T_1, T_2)$.

Example 3.2. *Let $p = 3$ and $\alpha \in \Lambda_{(2)}$ be such that $\mu(G(T_1, T_2)) = 0$. Then we have:*

(1) $\lambda(G) = 0$ *if and only if* $a_{1,1} + a_{2,2} \not\equiv a_{1,2} + a_{2,1}$ *(mod π).*
(2) *If* $\lambda(G) > 0$, *then* $\lambda(G) = 1$ *if and only if either* $a_{2,1} + a_{4,1} \not\equiv a_{2,2} + a_{4,2}$ *(mod π) or* $a_{1,2} + a_{1,4} \not\equiv a_{2,2} + a_{2,4}$ *(mod π).*
(3) *If* $\lambda(G) > 1$, *then* $\lambda(G) = 2$ *if and only if any one of the following is true:*
  (a) $a_{2,2} + a_{4,4} \not\equiv a_{2,4} + a_{4,2}$ *(mod π)*
  (b) $a_{2,2} + a_{5,1} + a_{7,2} + a_{8,1} \not\equiv a_{2,1} + a_{5,2} + a_{7,1} + a_{8,2}$ *(mod π)*
  (c) $a_{2,2} + a_{1,5} + a_{2,7} + a_{1,8} \not\equiv a_{1,2} + a_{2,5} + a_{1,7} + a_{2,8}$ *(mod π).*
(4) *If* $\lambda(G) > 2$, *then* $\lambda(G) = 3$ *if and only if any one of the following is true:*
  (a) $a_{5,1} + a_{10,1} \not\equiv a_{5,2} + a_{10,2}$ *(mod π)*
  (b) $a_{1,5} + a_{1,10} \not\equiv a_{2,5} + a_{2,10}$ *(mod π)*
  (c) $a_{2,2} + a_{4,5} + a_{4,8} \not\equiv a_{4,2} + a_{2,7} + a_{4,7} + a_{2,8}$ *(mod π)*
  (d) $a_{2,2} + a_{5,4} + a_{8,4} \not\equiv a_{2,4} + a_{7,2} + a_{7,4} + a_{8,2}$ *(mod π).*

In this way, in case $p = 3$, we can find criteria for the λ invariant of $G(T_1, T_2)$ in terms of the coefficients of $\widehat{\alpha}(T_1, T_2)$ if $\lambda(G) \leq 6$.

We may produce similar results for any prime p. The number of coefficients of $\widehat{\alpha}(T_1, T_2)$ which are involved will increase with p.

Example 3.3. *Let $p = 3$ and consider the measure $\alpha \in \Lambda_{(2)}$ given by the power series $\sum_{k=1}^{\infty} T_1^{4k} T_2$. Using* (1.4) *and* (1.5), *we find that $G(T_1, T_2) = T_1 + \sum_{k=1}^{\infty} T_2^{2k}$. Clearly $\lambda(G) = 1$ and this can also be verified using (1) and (2) of Example* (3.2).

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