Intro.

$K/\mathbb{Q}_p$ fin. $\mathcal{O} \supset \mathfrak{p}$. $\mathcal{O}/\mathfrak{p} =: k \cong \mathbb{F}_q$.
$v : K^\times \twoheadrightarrow \mathbb{Z}$.

$G_K := \mathrm{Gal}(\bar{K}/K)$. $v : G_K \twoheadrightarrow G_k \xrightarrow{\cong} \hat{\mathbb{Z}}$, $\quad \mathrm{Frob}_K \mapsto (x \mapsto x^q)^{-1} \mapsto 1$

$I_K := \mathrm{Ker}\, v$
$W_K := v^{-1}(\mathbb{Z})$
(loc. profin. w/ I_K open)
$|\cdot| : W_K \ni \sigma \mapsto q^{-v(\sigma)} \in \mathbb{Q}^\times$.

- LCFT $\Leftarrow$ Lubin-Tate + Hasse-Arf

$\mathrm{Art}_K : W_K \twoheadrightarrow K^\times$, $\quad \Downarrow \quad (W_K^{ab} \simeq K^\times)$

$$\begin{array}{ccc} W_K & \twoheadrightarrow & K^\times \\ \uparrow & \circlearrowleft & \uparrow N_{K'/K} \\ W_{K'} & \twoheadrightarrow & K'^\times \end{array} \qquad K'/K \text{ fin}$$

$\exists$ local, "geometric" proof.

- (Classical) LLC. reps / $\mathbb{C}$ or $\bar{\mathbb{Q}}_\ell$.

$\{$irred smooth reps of $GL_n(K)\}/\cong \;\longleftrightarrow\; \{n$-dim Frob-ss WD-rep of $W_K\}/\cong$

(r, N), V: fdvs.
$r : W_K \to \mathrm{Aut}(V)$ sm.
$N \in \mathrm{End}(V)$
$r(\sigma) N r(\sigma)^{-1} = |\sigma| N \quad (\forall \sigma \in W_K)$

Frob-s.s. $\Leftrightarrow$ r : ss.

Ⓐ 1.

- ℓ-adic monodromy. $\ell \neq p$.

[choose $\phi \in W_K$, $v(\phi) = 1$, $t : I_K \twoheadrightarrow \mathbb{Z}_\ell$. $\quad WD(\phi, t) \cong WD(\phi', t')$]

$(\text{cts } W_K\text{-reps}/\bar{\mathbb{Q}}_\ell) \xrightarrow[WD]{\cong} (WD\text{-reps of } W_K / \bar{\mathbb{Q}}_\ell)$

eq. of $\otimes$-abel. cat. Ind. Res.
$\left[\begin{array}{l} WD(\rho) = (r, N) \\ \rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma) N) \end{array}\right]$

$\{V^{GL_n(\mathcal{O})} \neq 0\} \longleftrightarrow \{$unramified $\Leftrightarrow r|_{I_K} = 1, N = 0\}$.

$\{V^{Iw} \neq 0\} \longleftrightarrow \{$unipotent $\underset{def}{\Leftrightarrow} r|_{I_K} = 1\}$.

§1 admissible rep $\left(V, \text{ gen. by } V^{Iw}\right) \overset{\cong}{\longleftrightarrow} \left(\text{fin. length } \mathcal{H}\text{-mod}\right)$

$V \longmapsto V^{Iw}$.

Ⓐ (X : var/K, sst red. $\Rightarrow WD(H^*(X))$: unipotent).

$$\boxed{X \otimes * \rightsquigarrow X_*}$$

Ⓐ2. X : proper sm. var$/K$. $H^i(X) := H^i_{et}(X\otimes \bar K, \bar{\mathbb{Q}}_\ell) \circlearrowleft W_K$

- $\mathfrak{X}/\mathcal{O}$: proper flat. $X = \mathfrak{X}\otimes_{\mathcal{O}} K$. $Y = \mathfrak{X}\otimes k$ $\qquad H^i_{et}(Y\otimes\bar k, R\Psi\bar{\mathbb{Q}}_\ell)$
- $Y\otimes\bar k \xhookrightarrow{\bar i} \mathfrak{X}\otimes\mathcal{O}^{ur} \xhookleftarrow{\bar j} X\otimes\bar K$
- $R\Psi\bar{\mathbb{Q}}_\ell := \bar i^* R\bar j_* \bar{\mathbb{Q}}_\ell \in D^b_c(Y\otimes\bar k, \bar{\mathbb{Q}}_\ell)$

ex. $\mathfrak{X}/\mathcal{O}$ sm $\Rightarrow$ $\bar{\mathbb{Q}}_\ell \simeq R\Psi\bar{\mathbb{Q}}_\ell$. $\quad H^i(X) \simeq H^i(Y) \circlearrowleft W_K \twoheadrightarrow \mathbb{Z}$: unram

. $\mathfrak{X}/\mathcal{O}$ semistable (strictly) $\Rightarrow$ $WD(H^i(X))$: unipotent.

$\Updownarrow$

$\mathfrak{X}$: Zariski locally étale over $\mathcal{O}[X_1 \cdots X_n]/(\varpi - X_1\cdots X_m)$ $\quad 1\le m\le n$.

$m=1$: Smooth$/\mathcal{O}$

$\Rightarrow Y = \bigcup_{i\in\Delta} Y_i$, $|\Delta|<\infty$

- Y_i/k : smooth, dim $n-1$
- Y_i, Y_j share no com. comp

For $I\subset\Delta$. $\quad Y_I := \bigcap_{i\in I} Y_i$: smooth$/k$ dim $n-|I|$

$Y^{(m)} := \coprod_{|I|=m} Y_I$ $\quad (1\le m\le n)$. $\quad a_m : Y^{(m)} \xrightarrow[\text{fin}]{} Y$

- $\Lambda := \bar{\mathbb{Q}}_\ell$. $R\Psi\Lambda \in D^b_c(Y_{\bar k}, \Lambda) \supset \mathrm{Perv}(Y_{\bar k}, \Lambda)$ abelian subcat.

$R\Psi\Lambda[n-1] \in \mathrm{Perv}(Y_{\bar k}, \Lambda)$

Weight filt (= monodromy filt) ... increasing $F^w R\Psi\Lambda$ $\quad w\in\mathbb{Z}$.

$Gr^w := F^w R\Psi\Lambda / F^{w-1} R\Psi\Lambda$.

For $1\le m\le n$. $\Lambda_m := a_{m*}\Lambda_{Y^{(m)}}$ $\quad \left(\Rightarrow H^*(Y_{\bar k}, \Lambda_m) = H^*(Y^{(m)})\right)$.

$$Gr^i \simeq \bigoplus_{\substack{s-t=i\\ s,t\ge 0}} \Lambda_{s+t+1}[-(s+t)](-s)$$

$$E_1^{i,j} := H^{i+j}(Y_{\bar k}, Gr^{-i}) \Rightarrow H^{i+j}(Y_{\bar k}, R\Psi\Lambda) = H^{i+j}(X)$$

$$\| $$

$$H^{i+j}\Big(Y_{\bar k}, \bigoplus_{s-\underbrace{(i+s)}_{=t}=-i} \Lambda_{i+2s+1}[-(i+2s)](-s)\Big)$$

$$\| $$

$$\bigoplus_{s\ge 0,-i} H^{j-2s}(Y^{(i+2s+1)})(-s).$$

i	
2	$\Lambda_3[-2](-2)$
1	$\Lambda_2[-1](-1)$
0	$\Lambda_3[-2](-1)$ $\quad \Lambda_1$
-1	$\Lambda_2[-1]$
-2	$\Lambda_3[-2]$

Λ_Y

Groups, Algebras ⟳ X ... alg. corres. $\Gamma \in Z_{n-1}(X \times X)$.

$$[\Gamma]^* : H^i(X) \xrightarrow{pr_2^*} H^i(X \times X)$$
$$\cap \qquad\qquad \Big\downarrow [\Gamma_{\bar{k}}]\cup$$
$$\mathrm{End}(H^i(X)). \qquad H^{i+2(n-1)}(X \times X)(n-1) \xrightarrow{pr_{1*}} H^i(X)$$

($Z_i :=$ i-cycles. $A_i := Z_i/\text{rat.eq.}$)

Use $\bar{\Gamma} \in Z_n(\mathfrak{X} \times_{\mathcal{O}} \mathfrak{X})$: closure of Γ.

Thm 1) (T. Saito). $Y_{I,I'} := Y_I \times Y_{I'}$ ~~$\exists \Gamma_{I,I'} \in A_{n-m}(Y_{I,I'})$~~

$\exists \Gamma_{I,I'} \in A_{n-m}(Y_{I,I'})$ for $|I| = |I'| = m$.

$$\Gamma^{(m)} := \sum_{|I|=|I'|=m} \Gamma_{I,I'} \in A_{n-m}(Y^{(m)} \times Y^{(m)})$$

$$\bigoplus_s [\Gamma^{(i+2s+1)}]^* \circlearrowright E_1^{i,j} \Rightarrow H^{i+j}(X) \circlearrowleft [\Gamma]^* \quad \text{cptible}$$

2) $I = \{i_1, \dots, i_m\}$. $I' = \{i'_1, \dots, i'_{m-1}\}$.

$$\sum_{h=1}^{m} (-1)^h Y_{I,I'} \cdot \Gamma_{I \setminus \{i_h\}, I'} = \sum_{i' \in \Delta \setminus I'} (-1)^{h(i')} \Gamma_{I, I' \cup \{i'\}} \quad (*)$$

in $A_{n-m}(Y_{I,I'} \cap |\bar{\Gamma}|)$. $h(i')$: position of i' in $J \cup \{i'\}$

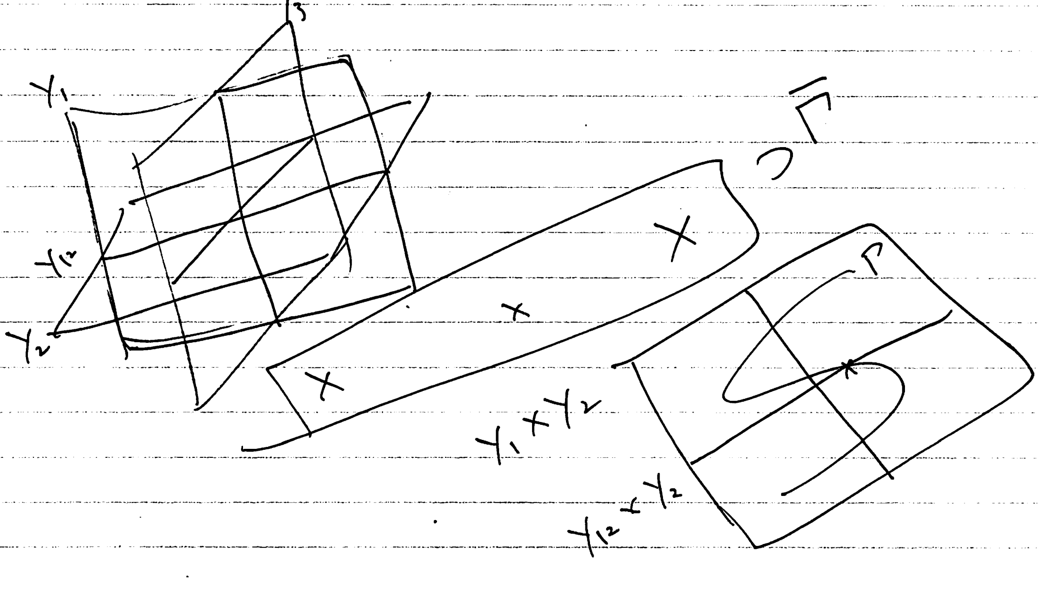
[$i'_{h(i')-1} < i' < i'_{h(i')}$ ($i'_m = \infty$)]

3) If $pr_1, pr_2 : |\bar{\Gamma}| \to \mathfrak{X}$ finite.

$\Gamma_{I,I'} \in Z_{n-m}(Y_{I,I'} \cap |\bar{\Gamma}|)$ for $\forall I, I'$. $|I| = |I'| = m$

determined by

- $\Gamma_{i,j}$ = closure of $Y_{i,j}^{\circ} \cdot \bar{\Gamma}|_{(\mathfrak{X} \times_{\mathcal{O}} \mathfrak{X})^{sm}}$ in $Y_{i,j}$ and $(*)$



B2

$\Lambda \cong \mathcal{O}^n$. $V = \Lambda \otimes_{\mathcal{O}} K$. $G = \mathrm{Aut}(V) \cong GL_n(K)$

$U = \mathrm{Stab}(\Lambda) \cong GL_n(\mathcal{O})$.

- lattice flag in V = chain of lattices

$\mathcal{L}: \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_n = \varpi^{-1}\Lambda_0$.

fix $\mathcal{L}$ w/ $\Lambda_0 = \Lambda$.

$I := \mathrm{Stab}(\mathcal{L}) \cong Iw_n := \{ g \in GL_n(\mathcal{O}) \mid g \bmod \varpi \text{ upper triang} \}$

- $\begin{cases} \varpi : \text{unif.} \\ \text{Basis } (e_1, \cdots e_n) \text{ of } \Lambda. \quad \Lambda_i = \varpi^{-1}e_1 \oplus \cdots \oplus \varpi^{-1}e_i \oplus \mathcal{O}e_{i+1} \oplus \cdots \oplus \mathcal{O}e_n. \end{cases}$

$G \ni t_i \ (1 \le i \le n-1)$. $e_i \leftrightarrow e_{i+1}$. $W := \langle t_i \rangle \sim \mathfrak{S}_n$.

$G \ni z_i \ (1 \le i \le n)$ $e_1 \mapsto \varpi^{-1}e_1, \ldots, e_i \mapsto \varpi^{-1}e_i$ $A(\varpi) := \langle z_i \rangle \sim \mathbb{Z}^n$

$G \supset \widetilde{W} = W \cdot A(\varpi) \cong \mathfrak{S}_n \ltimes \mathbb{Z}^n$ ext'd affine Weyl gp.

$G = \coprod_{w \in \widetilde{W}} IwI$ Iwahori-Bruhat decomp.

- $\mathcal{H} := \overline{\mathbb{Q}}_\ell [I \backslash G / I] \cong \overline{\mathbb{Q}}_\ell [Iw_n \backslash GL_n(K) / Iw_n]$.

$T_i := It_iI$, $Z_i = Iz_iI$ double coset alg.

$\mathcal{H}_W := \overline{\mathbb{Q}}_\ell[T_i]$: fin. Hecke alg. $\begin{cases} T_iT_j = T_jT_i \ (|i-j| \neq 1) \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \\ T_i^2 = (q-1)T_i + q \end{cases}$

$X_i := q^{-\frac{(n-2i+1)}{2}} Z_i Z_{i-1}^{-1}$

$\mathcal{H}_A = \overline{\mathbb{Q}}_\ell [X_1^{\pm}, \ldots, X_n^{\pm}]$: Laurent pol. alg. (commutative)

$\mathcal{H} = \mathcal{H}_W \otimes \mathcal{H}_A$ as $\overline{\mathbb{Q}}_\ell$-v.s. $\mathcal{H}_W, \mathcal{H}_A$ sub alg.

$T_iX_iT_i = qX_{i+1}$. $(1 \le i \le n-1)$

$q = 1 \Rightarrow \overline{\mathbb{Q}}_\ell[W]$ — Reps of W — $GL_n(k)$ in $\mathrm{Ind}_B^{GL_n} \mathbb{1}$.

- Center of $\mathcal{H} = \mathcal{H}_A^W$. irred $\mathcal{H}$-mod's are f.d./$\overline{\mathbb{Q}}_\ell$
- $\mathrm{Rep}\,\mathcal{H} :=$ (fin. length $\mathcal{H}$-mod)
- char of $\mathcal{H}_A^W \longleftrightarrow s \in (\overline{\mathbb{Q}}_\ell^\times)^n / \mathfrak{S}_n$: central char

$\mathrm{Rep}\,\mathcal{H} = \bigoplus_s \mathrm{Rep}_s \mathcal{H}$. $\mathrm{Rep}_s \mathcal{H} := \left(\begin{matrix} \text{all JH const.} \\ \text{have central char } s \end{matrix} \right)$

- parab. ind. $\underline{n} := (n_1, \ldots, n_r)$. $n = \sum n_j$. $n_j > 0$.

$\mathcal{H}_{\underline{n}} := \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_r} \hookrightarrow \mathcal{H}_n = \bigoplus_{w \in W/W_{\underline{n}}} S_w \cdot \mathcal{H}_{\underline{n}}$

$W_{\underline{n}} \sim \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$.

$Z(\mathcal{H}_{\underline{n}}) = \mathcal{H}_A^{W_{\underline{n}}}$

$\mathrm{Rep}\,\mathcal{H}_{\underline{n}} \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathrm{Rep}\,\mathcal{H}_n$. $\mathrm{Ind} := \mathcal{H}_n \otimes_{\mathcal{H}_{\underline{n}}} -$ $\left(M \times N := \mathrm{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} M \boxtimes N \right)$

if $\prod_{i \neq j} (s_i - qs_j) \neq 0$ for $s = (s_1, \ldots, s_n)$. $\Rightarrow \mathrm{Res}_s \mathcal{H}_{(1,\ldots,1)} \underset{\mathrm{Ind}}{\cong} \mathrm{Rep}_s \mathcal{H}_n$.

$$- \begin{pmatrix} \text{adm. } G\text{-rep } V. \\ \text{gen. by } V^I \end{pmatrix} \xrightarrow{\cong} \operatorname{Rep} \mathcal{H}.$$

$$V \longmapsto V^I.$$

$$\left.\begin{array}{l} St_n \longmapsto st_n : T_i \mapsto -1.\ Y_i \mapsto 1. \\ \mathbb{1}_n \longmapsto \mathbb{1}_n : T_i \mapsto q.\ Y_i \mapsto q^{i(n-i)} \end{array}\right\} \text{1-dim}/\overline{\mathbb{Q}_\ell}$$

$\operatorname{Rep}_{(1,q,q^2,\dots,q^{n-1})} \mathcal{H} = \operatorname{Rep}_q \mathcal{H}$

In $K_0(\operatorname{Rep}_q \mathcal{H})$. $\displaystyle\sum_{m=0}^{n} (-1)^m st_m \times \mathbb{1}_{n-m} = 0.$

$$0 \to \underbrace{st_{m-1} \boxplus \mathbb{1}_{n-m+1}}_{\substack{\text{irred} \\ \binom{n-1}{m-1} \text{ dim}}} \to \underbrace{st_m \times \mathbb{1}_{n-m}}_{\binom{n}{m}\text{-dim}} \to \underbrace{st_m \boxplus \mathbb{1}_{n-m}}_{\substack{\text{irred} \\ \binom{n-1}{m} \text{ dim.}}} \to 0$$

LT ~~tower~~ sp.

Σ_0: formal BT $\mathcal{O}$-mod of ht n. / $\kappa/\mathbb{k}$ ($\exists!$ up to $\cong$ / $\overline{\mathbb{F}_q}$)

(RZ-space)

$\mathfrak{X}_0$: deform. sp of Σ_0 ~~= Spec $\hat{\mathcal{O}}$~~

$\cdot\cdot \cong \coprod_{\mathbb{Z}} \operatorname{Spec} \mathcal{O}_K[[T_1 \cdots T_{n-1}]]$. $\quad \mathcal{O}_K/\mathcal{O}$ unram.

$\mathcal{O}_K$: CDVR. res field κ

$\mathfrak{X}$: moduli of $(\Sigma, \mathcal{L})$.

$\mathcal{L}$: $0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_n = \Sigma[p]$.

$\mathcal{C}_i$: fin flat of deg q^i. $\mathcal{O}$-mod.

$\cong \coprod_{\mathbb{Z}} \operatorname{Spec} \mathcal{O}_K[[X_1 \cdots X_n]] / (\varpi - X_1 \cdots X_n)$.

$\left\{ \begin{array}{l} Y_i = \{X_i = 0\} \\ X_i = \operatorname{Lie} f_i \text{ for} \end{array} \right.$

$$\Sigma \xrightarrow[f_1]{} \Sigma/\mathcal{C}_1 \to \cdots \to \Sigma/\mathcal{C}_{i-1} \xrightarrow[f_i]{} \Sigma/\mathcal{C}_i$$

isog. deg q.

$X = LT^I$.

$\mathfrak{X} \otimes \hat{K}^{ur} = X$ $\circlearrowleft$ LT-tower $\Big\rangle$ $\to$ Cohom. $H^*(LT)$ realizes LLC.

$\mathcal{O}_K$. G

$\mathcal{H} \curvearrowright X$ by alg. corresp.

Can compute $[T_i]^*$ $(1 \le i \le n-1)$. $[Z_i]^*$ $(1 \le i \le n)$. $\curvearrowright X$.

moduli interpretation.

$T_i \subset X \times_K X : \{(\Sigma. \mathcal{L}. \mathcal{L}') \mid \mathcal{C}_i \subset \mathcal{C}_i' \subset \mathcal{C}_{i+1}.\ \mathcal{C}_j = \mathcal{C}_j'\ (j \neq i)\}$

$$[T_i]_{I.I} = \begin{cases} (-1)[\Gamma_{id}] & \text{if } i, i+1 \in I. \\ (q-1)[\Gamma_{id}]. & \text{if } i \notin I.\ i+1 \in I. \\ [\{\mathcal{C}_{i-1} \subset \mathcal{C}_i' \subset \mathcal{C}_{i+1}\} - \Gamma_{id}.] & i, i+1 \notin I. \end{cases}$$

$[T_i]_{I.\, I \cup \{i\} \setminus \{i+1\}} = [\{\mathcal{C}_i' = \mathcal{C}_{i-1}^{(q)}\}]$. $\quad [T_i]_{I;\, I \setminus \{i\} \cup \{i+1\}} = [\{\mathcal{C}_i = \mathcal{C}_{i-1}'^{(q)}\}]$

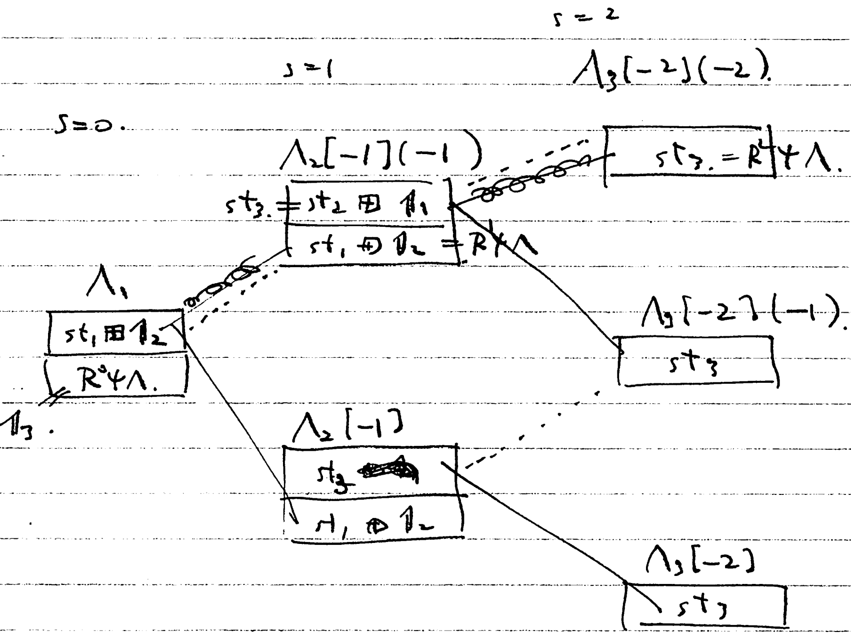
$m = s+1. \quad n-m$

$\mathcal{H} \curvearrowright Gr^{\bullet i} = \bigoplus \Lambda_{s+t+1}[(-s+t)](-s)$

$\curvearrowright \Lambda_m$ via $st_m \times \mathbb{1}_{n-m}$ $\quad H^*(Y_{\bar{k}}, \Lambda_n) = H^*(Y^{(m)})$

$= \coprod_{|I|=m} H^*(Y_I)$.

$\begin{cases} T_1, \dots, T_{m-1} \curvearrowright -1. \\ T_{m+1}, \dots, T_{n-1} \curvearrowright q. \end{cases}$ on $Y_{\{1,\dots,m\}}$.



$st_1 \times \mathbb{1}_2 \qquad st_2 \times \mathbb{1}_1 \qquad st_3.$

$\underline{n = 3.}$

Stalk at cl. pt of $\mathrm{Spec}\, \mathcal{O}[[X_1, X_2, X_3]] / (\varpi - X_1 X_2 X_3)$

$\circlearrowright \mathcal{H}_3.$