$\boxed{1}$

$$\frac{dx}{dt} = -x + y + y^2$$

$$\frac{dy}{dt} = x - xy$$

(a) critical pts: $x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$

$x = 0 \Rightarrow y^2 + y = 0 \Rightarrow (0,0)\,,\ (0,-1)$

$y = 1 \Rightarrow x = 2$ ie $(2, 1)$ [2]

(b) (i) At $(0,0)$: the linearized system has matrix $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$,

eigenvalues: $\lambda(\lambda+1) - 1 = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{5}}{2}$,

of opposite signs, $\therefore$ unstable, saddle pt. [2]

(ii) At $(0,-1)$: $X = x$ $Y = y+1$ ie $Y - 1 = y$

The translated system is
$$\frac{dX}{dt} = -X + Y - 1 + (Y-1)^2 = -X - Y + Y^2$$
$$\frac{dY}{dt} = X - X(Y-1) = 2X - XY$$

with matrix $\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$, eigenvalues: $\lambda(\lambda+1) + 2 = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{-7}}{2}$

both complex, negative real part [2]

$\therefore$ stable, spiral pt.

1. (b) (iii) at $(2, 1)$, $X = x - 2$ $Y = y - 1$

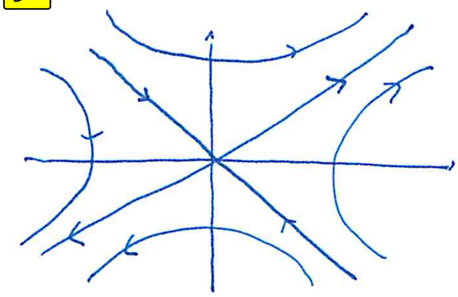
translated system is $\frac{dX}{dt} = -(X+2) + Y + 1 + (Y+1)^2 = -X + 3Y + Y^2$

$\frac{dY}{dt} = +(X+2) - (X+2)(Y+1) = -2Y - XY$

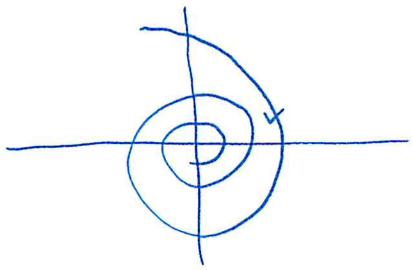
with matrix $\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$, eigenvalues $\lambda = -1, -2$

stable, node. [2]

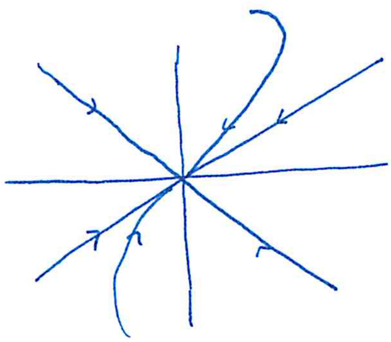
1 (c) (i)



(ii)



(iii)



[3]

$\boxed{2}$

(a) $\quad \vec{X} = (yz,\ x^2y - xz,\ x^2z - xy)$

$$\operatorname{curl} \vec{X} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & x^2y - xz & x^2z - xy \end{vmatrix} = \left(-x + x,\ y + y + 2xz,\ 2xy - z - z\right)$$

$$= (0,\ 2y + 2xz,\ 2xy - 2z)$$

$$\therefore \vec{X} \cdot \operatorname{curl} \vec{X} = (x^2y - xz)(2y + 2xz) + (x^2z - xy)(2xy - 2z)$$

$$= 2x^2y^2 - 2xyz + 2x^3yz - 2x^2z^2 + 2x^3yz - 2x^2y^2 - 2x^2z^2 + 2xyz$$

$= 0$

$\therefore$ integrable. [2]

(b) $\quad yz\,dx + (x^2y - xz)\,dy + (x^2z - xy)\,dz = 0 \qquad (*)$

treat x as constant first, get

$$x^2(y\,dy + z\,dz) - x(z\,dy + y\,dz) = 0$$

$$\Rightarrow \quad x(y^2 + z^2) - 2yz = f(x) \longrightarrow (1)$$

Now put $y = 0$ (in $(*)$) and get $x^2z\,dz = 0$

$$\Rightarrow \quad z = b \longrightarrow (1')$$

Put $y = 0$ in (1) and get $f(x) = xz^2 = b^2x$

$\therefore$ sol^n: $\quad x(y^2 + z^2) - 2yz = b^2x$ [4]

$\boxed{3}$

$f: \quad p^2+q^2-1=0$

$g: \quad (p^2+q^2)x - pz = 0$

Consider $\quad \dfrac{\partial(f,g)}{\partial(x,p)} + p\,\dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\,\dfrac{\partial(f,g)}{\partial(z,q)}$ [2]

$$= \begin{vmatrix} 0 & p^2+q^2 \\ 2p & \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 2p & 2px-z \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ & \end{vmatrix} + q\begin{vmatrix} 0 & -p \\ 2q & \end{vmatrix}$$

$$= -2p(p^2+q^2) + p \cdot 2p^2 + q \cdot 2qp$$

$$= -2p + 2p \qquad (\text{as } p^2+q^2=1)$$

$$= 0$$

$\therefore$ compatible. [2]

4. (a) $\quad 2z = ax^2+by^2-ab$

$\Rightarrow \quad 2p = 2ax \quad \Rightarrow \quad a = \dfrac{p}{x}$

and $\quad 2q = 2by \quad \Rightarrow \quad b = \dfrac{q}{y}$

$\therefore$ PDE is $\quad 2z = px+qy-\dfrac{pq}{xy}$ [2]

(b) Let $\quad f: \; ax^2+by^2-ab-2z=0 \quad$ be the given family

$f_a: \quad x^2 - b = 0$

$f_b: \quad y^2 - a = 0$ [2]

$\therefore \quad a = y^2 \quad b = x^2 \quad$ in the given family yields

$$x^2y^2 + x^2y^2 - x^2y^2 = 2z$$

$$\Rightarrow \quad z = \tfrac{1}{2}x^2y^2$$ [1]

Check: $\quad p = xy^2 \qquad q = x^2y$

and $\quad px+qy-\dfrac{pq}{xy} = x^2y^2+x^2y^2-\dfrac{x^3y^3}{xy} = x^2y^2 = 2z.$ [1]

$\boxed{5}$ Let $(p, q, -1)$ denote the direction of the normal at a pt (a, b, c) on the given surface.

The eq^n of the normal is $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{-1} \; (= \lambda \text{ say})$ [2]

$$\Rightarrow \quad x = a + p\lambda$$
$$y = b + q\lambda$$
$$z = c - \lambda$$

For pts A, B, the corresponding λ's are given by the roots λ_1, λ_2

of $\quad 3(a+p\lambda)^2 + 2(b+q\lambda)^2 + (c-\lambda)^2 = 6$

$$\Rightarrow \quad \lambda^2(3p^2 + 2q^2 + 1) + \lambda(6ap + 4bq - 2c) + * = 0$$

so that $\quad \lambda_1 + \lambda_2 = \dfrac{-(6ap + 4bq - 2c)}{3p^2 + 2q^2 + 1} \longrightarrow$ (A) [2]

But $z = 0$ bisects the line AB, $\therefore$ the z-co-ords of A and B must add up to 0. $\left(\frac{z_1 + z_2}{2} = 0 \right)$

$$\Rightarrow \quad c - \lambda_1 + c - \lambda_2 = 0$$
$$\Rightarrow \quad \lambda_1 + \lambda_2 = 2c \longrightarrow \text{(B)}$$ [2]

(A) and (B) $\Rightarrow \quad 2c - 6ap - 4bq = 2c(3p^2 + 2q^2 + 1)$

$$\Rightarrow \quad c(3p^2 + 2q^2) + 3ap + 2bq = 0$$

i.e., the $\quad z(3p^2 + 2q^2) + 3px + 2qy = 0$ [2]

6. (a) $f: \quad pxy + qy^2 - 2yz - p^2 = 0$

Consider $\dfrac{dy}{f_q} = \dfrac{dp}{-(f_x + p f_z)}$ [1]

$\Rightarrow \quad \dfrac{dy}{y^2} = \dfrac{dp}{-(py + p\cdot(-2y))}$

$\Rightarrow \quad \dfrac{dy}{y} = \dfrac{dp}{p}$

$\Rightarrow \quad p = ay$

$$q = \frac{a^2y^2 + 2yz - axy^2}{y^2}$$

$$= a^2 - ax + 2\frac{z}{y}$$ [1]

Consider $p\,dx + q\,dy = dz$

$\Rightarrow \quad ay\,dx + \left(a^2 - ax + \dfrac{2z}{y}\right)dy = dz \longrightarrow (1)$

treat y as constant and get $ay\,dx = dz$

$\Rightarrow \quad axy - z = f(y) \longrightarrow$ (A) [2]

put $x = a$ in (1) and get $\dfrac{2z}{y}\,dy = dz$

$\Rightarrow \quad by^2 = z \longrightarrow$ (B)

Putting $x = a$ in (A), we get $f(y) = a^2y - z$

$= a^2y - by^2$ (from (B))

$\therefore$ the reqd complete integral is

$$axy - z = a^2y - by^2$$

$\Rightarrow \quad z = axy + by^2 - a^2y.$ [2]

(b) $$y(x+hy)^2 = 4(z-ky^2) \longrightarrow ©$$

$y=1 \Rightarrow z-k = \frac{1}{4}(x+h)^2$

$\Rightarrow z = k + \frac{1}{4}(x+h)^2$, $y=1$ is a curve on ©

[2]

For intersection of © and $z = axy + by^2 - a^2y$ to be tangential, we need equal roots in the eqn

$$k + \frac{1}{4}(x+h)^2 = ax + b - a^2$$

ie, $$x^2 + 2xh - 4ax + h^2 + 4(a^2+k-b) = 0$$

ie, $$(2h-4a)^2 - 4(h^2 + 4(a^2+k-b)) = 0$$

$$\Rightarrow \cancel{h^2} + \cancel{4a^2} - 4ah - \cancel{h^2} - \cancel{4a^2} - 4k + 4b = 0$$

$$\Rightarrow b = k + ah.$$

[2]

Putting $b = k+ah$ in $z = axy + by^2 - a^2y$, we find

$$0 = axy + (k+ah)y^2 - a^2y - z \quad :(f)$$

$f_a = 0 \Rightarrow xy + hy^2 - 2ay = 0$

$\Rightarrow a = \frac{x+hy}{2}$

$\therefore$ the envelope is $z = xy\,\frac{x+hy}{2} + \left(k + \frac{x+hy}{2}h\right)y^2 - \left(\frac{x+hy}{2}\right)^2 y$

$$\Rightarrow z = y\,\frac{x+hy}{2}\left(x + hy - \frac{x+hy}{2}\right) + ky^2$$

$$\Rightarrow z - ky^2 = \frac{y(x+hy)^2}{4}$$

$$\Rightarrow y(x+hy)^2 = 4(z-ky^2)$$

[2]

7. $\left(D^2 + 2DD' - 3D'^2\right) z = 2x(x+y)$

(i. $(D+3D')(D-D')z = 0$

$\Rightarrow z_{cf} = \varphi_1(x+y) + \varphi_2(3x-y)$ [2]

P.I.: Let $(D+3D')z = z_1$

So, $(D-D')z_1 = 2x(x+y)$

$\Rightarrow \frac{dx}{1} = \frac{dy}{-1} = \frac{dz_1}{2x(x+y)}$

$\Rightarrow x+y = a, \quad dz_1 = 2xa\,dx$

$\Rightarrow z_1 = ax^2$

$\Rightarrow z_1 = (x+y)x^2$ [2]

Now $(D+3D')z = (x+y)x^2$

$\Rightarrow \frac{dx}{1} = \frac{dy}{3} = \frac{dz}{(x+y)x^2}$

$\Rightarrow 3x-y = b, \quad dz = (x+y)x^2\,dx$

$dz = (4x-b)x^2\,dx$

$\Rightarrow z = x^4 - \frac{x^3}{3}b$

$= x^4 - \frac{x^3}{3}(3x-y)$

$\Rightarrow z = \frac{1}{3}x^3y$ [2]

$\therefore$ General sol^n: $\frac{1}{3}x^3y + \varphi_1(x+y) + \varphi_2(3x-y)$ [2]

8. $R = 1 \qquad S = 0 \qquad T = -4x^2$

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow \quad \lambda^2 - 4x^2 = 0$$

$$\Rightarrow \quad \lambda = \pm 2x \qquad [2]$$

$\frac{dy}{dx} \pm 2x = 0$ yields substitution

$$\xi = y + x^2 \qquad \eta = y - x^2 \qquad [1]$$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} 2x + \frac{\partial z}{\partial \eta}(-2x)$ [1]

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2}(2x)^2 + \frac{\partial^2 z}{\partial \eta^2}(-2x)^2 + \frac{\partial^2 z}{\partial \xi \partial \eta}(2x)(2x) + \frac{\partial^2 z}{\partial \xi \partial \eta}(-2x)(2x)$$

$$+ 2\frac{\partial z}{\partial \xi} - 2\frac{\partial z}{\partial \eta} \qquad [1]$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \qquad [1]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} \qquad [1]$$

$\therefore$ The given eqn becomes:

$$\cancel{4x^2 \frac{\partial^2 z}{\partial \xi^2}} + \cancel{4x^2 \frac{\partial^2 z}{\partial \eta^2}} - 8x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} + 2\cancel{\frac{\partial z}{\partial \xi}} - 2\cancel{\frac{\partial z}{\partial \eta}}$$

$$- \cancel{4x^2 \frac{\partial^2 z}{\partial \xi^2}} - \cancel{4x^2 \frac{\partial^2 z}{\partial \eta^2}} - 8x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{x}\cdot\left(\cancel{\frac{\partial z}{\partial \xi}}(2x) - \cancel{\frac{\partial z}{\partial \eta}}(-2x)\right)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial \eta} = \varphi(\eta)$$

$$\Rightarrow \quad z = \psi(\eta) + \psi_1(\xi)$$

$$= \psi(y + x^2) + \psi_1(y - x^2) \qquad [2]$$

9.

Put $u = X(x)\, Y(y)$

the given eqn becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = (\lambda \text{ say})$$ [2]

ie $$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad \frac{d^2Y}{dy^2} - \lambda Y = 0$$

$\lambda < 0$

say $\lambda = m^2$

$X = A_m \cosh mx + B_m \sinh mx$

$u(\cancel{0, \pi/2})$

$u(0, y) = 0 \quad \forall\, y \le \pi/2 \Rightarrow A_m = 0$

$u(\pi/2, y) = 0 \quad \forall\, 0 \le y \le \pi/2 \Rightarrow B_m = 0$

$\therefore \lambda \not< 0$ [1]

$\lambda = 0$

$X = Ax + B$

$u(0, y) = 0 \quad \forall\, y \le \pi/2$

$\Rightarrow A = 0$

$u(\pi/2, y) = 0 \quad \forall\, y \le \pi/2$

$\Rightarrow B = 0$

$\therefore \lambda \neq 0$ [1]

$\lambda > 0$

So let $\lambda = m^2 > 0$

Now $X = A_m \cos mx + B_m \sin mx$

$u(0, y) = 0 \Rightarrow A_m = 0$

For nontrivial soln, $B_m \neq 0 \quad \therefore u(\pi/2, y) = 0$

$\Rightarrow \sin(m\pi/2) = 0$

$\Rightarrow m = 2, 4, 6, \ldots$ [1]

$\frac{d^2Y}{dy^2} - m^2 Y = 0$

$\Rightarrow Y = A_m \cosh my + B_m \sinh my$

$u(x, 0) = 0 \Rightarrow A_m = 0$

$\therefore$ soln satisfying hgs BC is $\sum_{m \in 2\mathbb{N}} B_m \sin mx \sinh my$

From $u(x, \pi/2) = \sin 4x$ we get

$$\sum B_m \sin mx \sinh m\pi/2 = \sin 4x$$ [1]

Using $\int_0^{\pi/2} \sin mx \sin nx = \frac{\pi}{4}\delta_{mn}$, we get

$B_m = 0 \quad m \neq 4$

$B_4 = \frac{1}{\sinh 2\pi}$ $\therefore$ soln $\operatorname{cosech} 2\pi \cdot \sin 4x \sinh 4y$ [2]