IWASAWA λ-INVARIANTS OF p-ADIC MEASURES ON $\mathbb{Z}_p^n$ AND THEIR Γ-TRANSFORMS

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Abstract. In [2], we proved a relation between the λ-invariants of a p-adic measure on $\mathbb{Z}_p^n$ and its Γ-transform under a strong condition. In this paper, we determine the relation without imposing any condition. We also determine p-adic properties of certain Mahler coefficients by exploiting some combinatorial identities.
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1. **Introduction**

The theory of Γ-transform is very useful in studying the Iwasawa invariants of imaginary abelian number fields. In [8], Sinnott gave an elegant new proof of the theorem of Ferrero and Washington that the Iwasawa μ-invariant is zero for the cyclotomic $\mathbb{Z}_p$-extension of any abelian number field. Sinnott further showed how to compute the μ-invariant of the Γ-transform of a rational function. In [4], Katz showed that the p-adic L-functions of a totally real number field K do indeed arise from roughly rational function measures on $\mathbb{Z}_p^d$, where $d = [K : \mathbb{Q}]$.

It was Kida who first obtained a relation between the λ-invariant of a measure on $\mathbb{Z}_p$ and its Γ-transform [5]. In our paper [2], exploiting certain combinatorial identities we determined a relation between the λ-invariants of a p-adic measure on $\mathbb{Z}_p^n$ and its Γ-transform for any $n \geq 1$ under some restrictive hypothesis. In this paper, we generalize the results of [2].

Let p be a fixed odd prime. We have $\mathbb{Z}_p^\times = V \times U$, where V is the group of $(p-1)$st roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. Then the projections from $\mathbb{Z}_p^\times$ onto V and U are denoted by ω and $<>$, respectively. By u, we will denote a fixed topological generator of U. Then there is an isomorphism $\phi : \mathbb{Z}_p \to U$ given by $\phi(y) = u^y$. Let $n \geq 1$. By fixing a topological generator u_i $(1 \leq i \leq n)$ for each copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^n$, we obtain an isomorphism $\phi^n : \mathbb{Z}_p^n \to U^n$ given by $(y_1, \ldots, y_n) \mapsto (u_1^{y_1}, \ldots, u_n^{y_n})$.

Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ with a local parameter π. It is well-known that there is an isomorphism between the ring Λ_n of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^n$ under convolution and the power series ring $\mathcal{O}[[T_1 - 1, \ldots, T_n - 1]]$. Explicitly, for $x \in \mathbb{Z}_p$, if we put

$$T^x = \sum_{m=0}^{\infty} \binom{x}{m} (T-1)^m \in \mathcal{O}[[T-1]],$$

then the unique power series $\widehat{\alpha}(T_1,\ldots,T_n)$ associated with α is given by

$$\begin{aligned}\widehat{\alpha}(T_1,\ldots,T_n) &= \int_{\mathbb{Z}_p^n} T_1^{x_1}\cdots T_n^{x_n}\,d\alpha(x_1,\ldots,x_n)\\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\left(\int_{\mathbb{Z}_p^n}\binom{x_1}{m_1}\cdots\binom{x_n}{m_n}d\alpha(x_1,\ldots,x_n)\right)\\ &\quad\times (T_1-1)^{m_1}\cdots(T_n-1)^{m_n}. \end{aligned} \tag{1.1}$$

If $a=(a_1,\ldots,a_n)\in(\mathbb{Z}_p^\times)^n$, we denote by $\alpha\circ a$ the measure on $\mathbb{Z}_p^n$ given by $\alpha\circ a(A_1\times\cdots\times A_n)=\alpha(a_1A_1\times\cdots\times a_nA_n)$, where A_i are compact open subsets of $\mathbb{Z}_p$. Also, for compact open subsets A_i of $\mathbb{Z}_p$, we let $\alpha|_A$ denote the measure obtained by restricting α to A and extending by 0, where $A=A_1\times\cdots\times A_n$.

The Γ-transform of α is defined as a function of the p-adic variables $s_1,\ldots,s_n$ given by

$$\begin{aligned}\Gamma_\alpha(s_1,\ldots,s_n) &= \int\limits_{(\mathbb{Z}_p^\times)^n} <x_1>^{s_1}\cdots<x_n>^{s_n}\,d\alpha(x_1,\ldots,x_n)\\ &= \sum_{\eta_1\in V}\cdots\sum_{\eta_n\in V}\int\limits_{U^n}<\eta_1x_1>^{s_1}\cdots<\eta_nx_n>^{s_n}\,d\alpha(\eta_1x_1,\ldots,\eta_nx_n)\\ &= \int\limits_{U^n} x_1^{s_1}\cdots x_n^{s_n}\,d\beta(x_1,\ldots,x_n),\end{aligned}$$

where $d\alpha(\eta_1x_1,\ldots,\eta_nx_n)$ denotes $d(\alpha\circ(\eta_1,\ldots,\eta_n))(x_1,\ldots,x_n)$ and

$$\beta=\sum_{\eta_1\in V}\cdots\sum_{\eta_n\in V}(\alpha\circ(\eta_1,\ldots,\eta_n))|_{U^n}, \tag{1.2}$$

a measure on U^n. By the isomorphism $\phi^n:\mathbb{Z}_p^n\to U^n$, one can transport the measure β on U^n to a measure $\tilde{\beta}$ on $\mathbb{Z}_p^n$. It is clear from (1.1) that the power series $\widehat{\tilde{\beta}}$ associated with $\tilde{\beta}$ interpolates the Γ-transform of α as

$$\Gamma_\alpha(s_1,\ldots,s_n)=\int\limits_{\mathbb{Z}_p^n}(u_1^{s_1})^{y_1}\cdots(u_n^{s_n})^{y_n}\,d\tilde{\beta}(y_1,\ldots,y_n)=\widehat{\tilde{\beta}}(u_1^{s_1},\ldots,u_n^{s_n}). \tag{1.3}$$

The measure β on U^n can be extended by 0 to $\mathbb{Z}_p^n$, and the associated power series of the extended measure will be denoted by $\widehat{\beta}$.

2. Iwasawa invariants

The Iwasawa μ- and λ- invariants of a power series

$$F(T)=\sum_{n=0}^{\infty}a_n(T-1)^n\in\mathcal{O}[[T-1]]$$

are defined by

$$\mu(F(T)) = \min\{ord_\pi(a_n) : n \geq 0\}$$
$$\lambda(F(T)) = \min\{n : ord_\pi(a_n) = \mu(F(T))\}.$$

Analogously, in [2] we defined the Iwasawa μ- and λ- invariants of a power series

$$F(T_1,\dots,T_n) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} a_{m_1,\dots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}$$

in $\mathcal{O}[[T_1-1,\dots,T_n-1]]$ as follows:

$$\mu(F(T_1,\dots,T_n)) = \min\{ord_\pi(a_{m_1,\dots,m_n}) : m_i \geq 0 \quad \forall i\}$$
$$\lambda(F(T_1,\dots,T_n)) = \min\{m_1+\cdots+m_n : ord_\pi(a_{m_1,\dots,m_n}) = \mu(F(T_1,\dots,T_n))\}.$$

Definition 1. *Let $\alpha \in \Lambda_n$. The Iwasawa μ- and λ- invariants of α are defined as $\mu(\widehat{\alpha}(T_1,\dots,T_n))$ and $\lambda(\widehat{\alpha}(T_1,\dots,T_n))$ respectively. Similarly, the Iwasawa invariants of Γ_α are defined as the corresponding invariants of the power series $\widehat{\widehat{\beta}}(T_1,\dots,T_n)$.*

Let $\alpha \in \Lambda_n$. In case of $n = 1$, Sinnott in his paper [8] proved that $\mu(\Gamma_\alpha) = \mu(\alpha^* + \alpha^* \circ (-1))$, if $\widehat{\alpha}(T)$ is a rational function of T. Here $\alpha^* = \alpha|_{\mathbb{Z}_p^\times}$. It is known that $\mu(\Gamma_\alpha) = \mu(\beta)$ (see for example [3, 8]). It is easy to prove that $\mu(\Gamma_\alpha) = \mu(\beta)$ for any $n \geq 1$ (see [2, Lemma 2.2]). It would be interesting to extend it to study λ-invariants. The case $n = 1$ has been studied in [3, 5, 6, 7]. The aim of this paper is to prove the following main result.

Theorem 2.1. *Let α be an $\mathcal{O}$-valued measure on $\mathbb{Z}_p^n$. Define a measure β on U^n by (1.2) and let $\widehat{\beta}(T_1,\dots,T_n)$ be the power series associated with the measure β on U^n extended to $\mathbb{Z}_p^n$ by zero. Then $\lambda(\beta) = p\lambda(\Gamma_\alpha)$.*

Remark 2.2. *The above theorem was proved under some restrictive hypothesis in* [2, Lemma 2.2].

3. Mahler coefficients and proof of the Theorem 2.1

A crucial ingredient in our proof of theorem 2.1 is a relation (see lemma 3.1) between coefficients of the power series $\widehat{\beta}$ and $\widehat{\widehat{\beta}}$ via Mahler coefficients. A classical theorem of Mahler states that any continuous function $f : \mathbb{Z}_p \to \mathcal{O}$ can be written uniquely in the form

$$f(x) = \sum_{j=0}^{\infty} a_j(f)\binom{x}{j}, \tag{3.1}$$

where $a_j(f) \in \mathcal{O}, a_j(f) \to 0$ as $j \to \infty$. In fact

$$a_j(f) = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}f(i). \tag{3.2}$$

Furthermore, if $f:\mathbb{Z}_p^n\to\mathcal{O}$ is continuous, we may write (by repeated application of (3.1))

$$f(x_1,\ldots,x_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}a_{m_1,\ldots,m_n}(f)\binom{x_1}{m_1}\cdots\binom{x_n}{m_n}, \tag{3.3}$$

where

$$\begin{aligned}a_{m_1,\ldots,m_n}(f)=\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}&(-1)^{m_1-j_1}\cdots(-1)^{m_n-j_n}\binom{m_1}{j_1}\cdots\binom{m_n}{j_n}\\ &\times f(j_1,\ldots,j_n)\in\mathcal{O}.\end{aligned}$$

The constants $a_{m_1,\ldots,m_n}(f)$ are called the Mahler coefficients of the function f.

Let us consider the continuous functions $f_m:\mathbb{Z}_p\to\mathbb{Z}_p$ and $f_{m_1,\ldots,m_n}:\mathbb{Z}_p^n\to\mathbb{Z}_p$ defined by

$$f_m(x)=\binom{u^x}{m}\quad\text{and}\quad f_{m_1,\ldots,m_n}(x_1,\ldots,x_n)=f_{m_1}(x_1)\cdots f_{m_n}(x_n).$$

Now, if $a_m(f_k)$ are the Mahler coefficients of $f_k(x)=\binom{u^x}{k}=\sum_{m=0}^{\infty}a_m(f_k)\binom{x}{m}$, then

$$a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n})=a_{j_1}(f_{m_1})\cdots a_{j_n}(f_{m_n}). \tag{3.4}$$

Suppose

$$\widehat{\beta}(T_1,\ldots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}b_{m_1,\ldots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}$$

and

$$\widehat{\widehat{\beta}}(T_1,\ldots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}g_{m_1,\ldots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}.$$

Then we have the following important lemma (see lemma 2.3 in [2]) which relates the coefficients of $\widehat{\beta}(T_1,\ldots,T_n),\widehat{\widehat{\beta}}(T_1,\ldots,T_n)$, and certain Mahler coefficients.

Lemma 3.1. *Modulo* $p^{n+k_1+\cdots+k_n}\mathcal{O}$*, we have*

$$m_1!\cdots m_n!b_{m_1,\ldots,m_n}\equiv m_1!\cdots m_n!\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n}g_{j_1,\ldots,j_n}a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n}),$$

where $a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n})$ *are the Mahler coefficients of* $f_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$.

Note that when $\mathrm{ord}_p(m_1!\cdots m_n!)\le k_1+\cdots+k_n$, then

$$b_{m_1,\ldots,m_n}\equiv\sum_{j_1=0}^{k_1}\cdots\sum_{j_n=0}^{k_n}g_{j_1,\ldots,j_n}a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n})\pmod{p^n\mathcal{O}}. \tag{3.5}$$

In order to prove the theorem 2.1, we need to investigate p-adic properties of the Mahler coefficients $a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n})$. We shall now study these coefficients using certain combinatorial identities.

Let us fix a topological generator $u=1+t_1p+t_2p^2+\cdots$ of $1+p\mathbb{Z}_p$. Hence t_1 is a unit. In fact t_1 is an integer lying between 1 and $p-1$. We now state a binomial expansion in the following lemma. One can find a simple proof using the fact that $(1+T)^{p^i} \equiv (1+T^{p^i}) \pmod p$ for $i \geq 1$.

Lemma 3.2. *For $n \geq 1$, we have*

$$(1+T)^{u^n} \equiv (1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{a_{n,2}+nt_2} \cdots (1+T^{p^j})^{a_{n,j}+nt_j} \cdots \pmod p, \quad (3.6)$$

where $a_{n,j} \geq 0$ for all $j \geq 2$.

In the following lemma, we prove another binomial expansion.

Lemma 3.3. *For $k \geq 1$, let $m = l_k p^k + l_{k-1}p^{k-1} + \cdots + l_1 p + l_0$, where $0 \leq l_i < p$ for all $i = 0, 1, \ldots, k$. Then we have*

$$\begin{aligned}(1+T)^{u^m} \equiv\ & (1+T)(1+T^p)^{l_0t_1}(1+T^{p^2})^{l_1t_1+l_0t_2+a_{m,2}}(1+T^{p^3})^{l_2t_1+l_1t_2+l_0t_3+a_{m,3}} \cdots \\ & (1+T^{p^k})^{l_{k-1}t_1+l_{k-2}t_2+\cdots+l_1t_{k-1}+l_0t_k+a_{m,k}}(1+T^{p^{k+1}})^{l_kt_1+l_{k-1}t_2+\cdots+l_1t_k+l_0t_{k+1}+a_{m,k+1}} \\ & \cdots (1+T^{p^{k+j}})^{l_kt_j+l_{k-1}t_{j+1}+\cdots+l_1t_{k+j-1}+l_0t_{k+j}+a_{m,k+j}} \cdots \pmod p, \qquad (3.7)\end{aligned}$$

where $a_{m,j} \geq 0$ for all $j \geq 2$.
Furthermore, for $j \geq 0$,

$$a_{l_{k+j}p^{k+j}+l_{k+j-1}p^{k+j-1}+\cdots+l_1p+l_0,k+1} = a_{l_{k-1}p^{k-1}+\cdots+l_1p+l_0,k+1} \qquad (3.8)$$

Proof. One can easily deduce (3.7) from (3.6). We now give a proof of (3.8). Let $m_1 = l_{k-1}p^{k-1} + \cdots + l_1 p + l_0$. ¿From (3.7), the exponent of $(1+T^{p^{k+1}})$ in the expansion of $(1+T)^{u^{m_1}}$ and $(1+T)^{u^{l_{k+j}p^{k+j}+\cdots+l_kp^k+m_1}}$ are, respectively

$$l_{k-1}t_2 + \cdots + l_1t_k + l_0t_{k+1} + a_{m_1,k+1} \qquad (3.9)$$

$$l_kt_1 + l_{k-1}t_2 + \cdots + l_1t_k + l_0t_{k+1} + a_{l_{k+j}p^{k+j}+\cdots+l_kp^k+m_1,k+1}. \qquad (3.10)$$

Again,

$$u^{l_{k+j}p^{k+j}+\cdots+l_kp^k} = 1 + l_kt_1p^{k+1} + \cdots .$$

Hence,

$$(1+T)^{u^{l_{k+j}p^{k+j}+\cdots+l_kp^k}} \equiv (1+T)(1+T^{p^{k+1}})^{l_kt_1} \cdots \ (mod\ p).$$

This implies that modulo p, the exponent of $(1+T^{p^{k+1}})$ in the expansion of $(1+T)^{u^{l_{k+j}p^{k+j}+\cdots+l_kp^k+m_1}}$ is

$$l_kt_1 + l_{k-1}t_2 + \cdots + l_1t_k + l_0t_{k+1} + a_{m_1,k+1}. \qquad (3.11)$$

¿From (3.10) and (3.11), we complete the proof of the lemma. □

In [1], we proved the following two theorems.

Theorem 3.4. *For non-negative integers n, t, k, we have*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{ti+k}{n} = t^n. \qquad (3.12)$$

Theorem 3.5. *For non-negative integers n, t, k, j with $n \geq j \geq 1$, we have*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{ti+k}{n-j} = 0. \tag{3.13}$$

We now state a result from [9].

Result 3.6. *Suppose that $n \geq m$ satisfy*

$$n = a_0 + a_1p + a_2p^2 + \cdots + a_kp^k$$
$$m = b_0 + b_1p + b_2p^2 + \cdots + b_kp^k$$

where $a_j, b_j \in \{0, 1, \ldots, p-1\}$. Then

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_k}{b_k} \pmod{p}.$$

Lemma 3.7. *Let $l, m \geq 0$. Suppose that $l = l_0 + l_1p + \cdots + l_kp^k$ with $0 \leq l_i < p$ for $i = 1, 2, \ldots, k$. Then*

$$a_l(f_m) \equiv \begin{cases} 0 \mod p, & \text{if } m < pl; \\ t_1^{l_0+l_1+\cdots+l_k} \mod p, & \text{if } m = pl. \end{cases}$$

Proof. Since $m < pl$, we can write $m = m_0 + m_1p + \cdots + m_{k+1}p^{k+1}$, where $0 \leq m_i < p$. From (3.2), we have

$$a_{l_0+l_1p+\cdots+l_kp^k}(f_m) = \sum_{j=0}^{l_0+l_1p+\cdots+l_kp^k}(-1)^{l_0+l_1p+\cdots+l_kp^k-j}\binom{l_0+l_1p+\cdots+l_kp^k}{j}\binom{u^j}{m}. \tag{3.14}$$

But, $\binom{u^j}{m}$ is the coefficient of T^m in the expansion of $(1+T)^{u^j}$. Clearly from (3.7), the coefficient of T^m modulo p in $(1+T)^{u^j}$ is zero if $1 < m_0 < p$. Also, the coefficients of T^m modulo p in $(1+T)^{u^j}$ are equal for $m_0 = 0, 1$. So, we can assume that $m_0 = 0$. Letting $j = i_kp^k + i_{k-1}p^{k-1} + \cdots + i_1p + i_0$ and using (3.7) and then (3.8), we find that modulo p,

$$\begin{aligned}
&\binom{u^j}{m} \\
&\equiv \binom{i_0t_1}{m_1}\binom{i_1t_1 + i_0t_2 + a_{j,2}}{m_2} \times \cdots \times \binom{i_kt_1 + i_{k-1}t_2 + \cdots + i_1t_k + i_0t_{k+1} + a_{j,k+1}}{m_{k+1}} \\
&\equiv \binom{i_0t_1}{m_1}\binom{i_1t_1 + i_0t_2 + a_{i_0,2}}{m_2} \times \cdots \times \binom{i_kt_1 + \cdots + i_1t_k + i_0t_{k+1} + a_{i_{k-1}p^{k-1}+\cdots+i_0,k+1}}{m_{k+1}}
\end{aligned}$$

We now use result 2.6 to simplify (3.14) and deduce that

$$\begin{aligned}
&a_{l_0+l_1p+\cdots+l_kp^k}(f_{m_1p+\cdots+m_{k+1}p^{k+1}}) \\
&\equiv \sum_{i_0=0}^{l_0}(-1)^{l_0-i_0}\binom{l_0}{i_0}\binom{i_0t_1}{m_1} \times \Bigg[\sum_{i_1=0}^{l_1}(-1)^{l_1-i_1}\binom{l_1}{i_1}\binom{i_1t_1+i_0t_2+a_{i_0,2}}{m_2} \\
&\times \Bigg[\sum_{i_2=0}^{l_2}(-1)^{l_2-i_2}\binom{l_2}{i_2}\binom{i_2t_1+i_1t_2+i_0t_3+a_{i_1p+i_0,3}}{m_3} \times\cdots\times \Bigg[\sum_{i_k=0}^{l_k}(-1)^{l_k-i_k}\binom{l_k}{i_k} \\
&\times \binom{i_kt_1+i_{k-1}t_2+\cdots+i_1t_k+i_0t_{k+1}+a_{i_{k-1}p^{k-1}+\cdots+i_1p+i_0,k+1}}{m_{k+1}}\Bigg]\cdots\Bigg]\Bigg]. \qquad (3.15)
\end{aligned}$$

If $m < pl$, then there exists j such that $m_j < l_{j-1}$ with $j \in \{0, 1, \ldots, k+1\}$ and $m_i = l_{i-1}$ for $i > j$. Using (3.12) in (3.15), we obtain

$$\begin{aligned}
&a_{l_0+l_1p+\cdots+l_kp^k}(f_{m_1p+\cdots+m_{k+1}p^{k+1}}) \\
&\equiv t_1^{l_k+\cdots+l_j}\Bigg[\sum_{i_0=0}^{l_0}(-1)^{l_0-i_0}\binom{l_0}{i_0}\binom{i_0t_1}{m_1} \times \Bigg[\sum_{i_1=0}^{l_1}(-1)^{l_1-i_1}\binom{l_1}{i_1}\binom{i_1t_1+a_{i_0,2}+i_0t_2}{m_2} \\
&\times \Bigg[\sum_{i_2=0}^{l_2}(-1)^{l_2-i_2}\binom{l_2}{i_2}\binom{i_2t_1+i_1t_2+a_{i_1p+i_0,3}+i_0t_3}{m_3} \times\cdots\times \Bigg[\sum_{i_{j-1}=0}^{l_{j-1}}(-1)^{l_{j-1}-i_{j-1}}\binom{l_{j-1}}{i_{j-1}} \\
&\times \binom{i_{j-1}t_1+\cdots+i_1t_{j-1}+a_{i_{j-2}p^{j-2}+\cdots+i_1p+i_0,j}+i_0t_j}{m_j}\Bigg]\cdots\Bigg]\Bigg]\Bigg]. \qquad (3.16)
\end{aligned}$$

But, $m_j = l_{j-1} - (l_{j-1} - m_j)$ with $l_{j-1} - m_j > 0$. Using (3.13), we have

$$a_{l_0+l_1p+\cdots+l_kp^k}(f_{m_1p+\cdots+m_{k+1}p^{k+1}}) \equiv 0 \ (mod\ p).$$

If $m = pl$, then $m_i = l_{i-1}$ for all $i = 1, \ldots, k+1$. Hence from (3.12), we have

$$a_{l_0+l_1p+\cdots+l_kp^k}(f_{l_0p+l_1p^2+\cdots+l_kp^{k+1}}) \equiv t_1^{l_0+l_1+\cdots+l_k} \ (mod\ p).$$

This completes the proof of the theorem. □

Lemma 3.8. *Let $j_i, m_i \geq 0$ for $i = 1, \ldots, n$. Then*

$$a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n}) \equiv \begin{cases} 0 \ \ mod\ p, & if\ m_i < pj_i\ for\ some\ i; \\ a\ p\text{-}adic\ unit\ \ mod\ p, & if\ m_i = pj_i\ for\ all\ i. \end{cases}$$

Proof. The proof follows from the lemma 3.7 and the fact that

$$a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n}) = \prod_{i=1}^{n} a_{j_i}(f_{m_i}).$$

□

Proof of Theorem 2.1: Recall that

$$\widehat{\beta}(T_1, \ldots, T_n) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} b_{m_1,\ldots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}$$

and

$$\widehat{\widetilde{\beta}}(T_1,\ldots,T_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}g_{m_1,\ldots,m_n}(T_1-1)^{m_1}\cdots(T_n-1)^{m_n}.$$

We know that $\mu(\Gamma_\alpha)=\mu(\beta)$, that is, $\mu(\widehat{\widetilde{\beta}}(T_1,\ldots,T_n))=\mu(\beta)$. For any power series $F(T_1,\ldots,T_n)\in\mathcal{O}[[T_1-1,\ldots,T_n-1]]$, if $\pi|F(T_1,\ldots,T_n)$ then $\lambda(\pi^{-1}F(T_1,\ldots,T_n))=\lambda(F(T_1,\ldots,T_n))$. So, we may assume that $\mu(\widehat{\widetilde{\beta}}(T_1,\ldots,T_n))=0$.

Suppose that $\lambda(\Gamma_\alpha)=k$, that is, $\lambda(\widehat{\widetilde{\beta}}(T_1,\ldots,T_n))=k$. If $k=0$, then $g_{0,\ldots,0}$ and $b_{0,\ldots,0}$ are units in $\mathcal{O}$ and hence $\lambda(\Gamma_\alpha)=0=p\lambda(\beta)$. If $k\geq 1$, then there exists a partition $k_1+\cdots+k_n$ of k such that $g_{k_1,\ldots,k_n}$ is a unit in $\mathcal{O}$ and for every $m_i\geq 0$ satisfying $m_1+\cdots+m_n<k$, $g_{m_1,\ldots,m_n}\equiv 0 \pmod{\pi}$. Let $r<pk$. Let $r=r_1+\cdots+r_n$ and $k=i_1+\cdots+i_n$ be any partitions of r and k, respectively. If $l_i=\mathrm{ord}_p(r_i!)$, then from (3.5) we get

$$b_{r_1,\ldots,r_n}\equiv\sum_{j_1=0}^{l_1}\cdots\sum_{j_n=0}^{l_n}g_{j_1,\ldots,j_n}a_{j_1,\ldots,j_n}(f_{r_1,\ldots,r_n}) \pmod{\pi}. \tag{3.17}$$

If $j_1+\cdots+j_n\geq k$, then $pj_1+\cdots+pj_n\geq pk>r$. Hence $r_i<pj_i$ for some i and lemma 3.8 implies that

$$a_{j_1,\ldots,j_n}(f_{r_1,\ldots,r_n})\equiv 0 \pmod{\pi}. \tag{3.18}$$

Again if $j_1+\cdots+j_n<k$, then $g_{j_1,\ldots,j_n}\equiv 0 \pmod{\pi}$. Thus if $r<pk$, then (3.18) and (3.17) imply that

$$b_{r_1,\ldots,r_n}\equiv 0 \pmod{\pi} \tag{3.19}$$

for every partition $r=r_1+\cdots+r_n$.

Now let $r=pk$. Consider the partition $k_1+\cdots+k_n$ of k. Then $pk_1+\cdots+pk_n$ is a partition of pk such that $\mathrm{ord}_p((pk_i)!)=\mathrm{ord}_p(r_i!)=l_i\geq k_i$. From (3.5) and lemma 3.8, we find that

$$\begin{aligned}b_{pk_1,\ldots,pk_n}&\equiv\sum_{j_1=0}^{l_1}\cdots\sum_{j_n=0}^{l_n}g_{j_1,\ldots,j_n}a_{j_1,\ldots,j_n}(f_{pk_1,\ldots,pk_n})\\&\equiv g_{k_1,\ldots,k_n}a_{k_1,\ldots,k_n}(f_{pk_1,\ldots,pk_n}) \pmod{\pi},\end{aligned} \tag{3.20}$$

which is a unit in $\mathcal{O}$. This proves that $\lambda(\beta)=pk_1+\cdots+pk_n=pk=p\lambda(\widehat{\widetilde{\beta}}(T_1,\ldots,T_n))$. This completes the proof of the main theorem. □

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