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We write: $\lim_{x \to x_0} f(x) = \ell$.

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Result: Let $D \subseteq \mathbb{R}$ and let $x_0 \in D$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \subseteq D$. Then $f : D \to \mathbb{R}$ is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

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Sequential criterion of continuity: $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ iff for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

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Similar criterion for limit.

Example: $\lim_{n \to \infty} \frac{\sin(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} - \sqrt{n}} = 1$

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1. $f(x) = \begin{cases} 3x + 2 & \text{if } x < 1, \\ 4x^2 & \text{if } x \geq 1. \end{cases}$

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Result: Let $f, g : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then

(a) $f + g$, fg and $|f|$ are continuous at x_0,

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Intermediate value theorem: Let I be an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous. If $a, b \in I$ with $a < b$ and if $f(a) < k < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

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(a) The equation $x^2 = x \sin x + \cos x$ has at least two real roots.

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(b) If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $c \in [0, 1]$ such that $f(c) = c$.

(c) Let $f : [0, 2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Then there exist $x_1, x_2 \in [0, 2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.

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Example: There does not exist any continuous function from $[0, 1]$ onto $(0, \infty)$.

Result: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.