MA 101 (Mathematics I)

Integration : Summary of Lectures

Riemann Integral: Motivation

Partition of $[a, b]$: A finite set $\{x_0, x_1, ..., x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

Upper sum & Lower sum: Let $f : [a, b] \to \mathbb{R}$ be bounded. For a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1, 2, ..., n$.
$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ – Upper sum of f for the partition P
$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ – Lower sum of f for the partition P
We have $m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a)$, where $M = \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$.

Example: Let $f(x) = x^4 - 4x^3 + 10$ for all $x \in [1, 4]$. Then for the partition $P = \{1, 2, 3, 4\}$ of $[1, 4]$, $U(f, P) = 11$ and $L(f, P) = -40$.

Upper integral: $\bar{\int_a^b} f = \inf_P U(f, P)$

Lower integral: $\underline{\int_a^b} f = \sup_P L(f, P)$

Riemann integral: If $\bar{\int_a^b} f = \underline{\int_a^b} f$, then f is said to be Riemann integrable on $[a, b]$ and the common value is the Riemann integral of f on $[a, b]$, denoted by $\int_a^b f$.

Examples:

(a) $f(x) = k$ for all $x \in [0, 1]$.
(b) Let $f(x) = \begin{cases} 0 & \text{if } x \in (0, 1], \\ 1 & \text{if } x = 0. \end{cases}$
(c) Let $f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$
(d) $f(x) = x$ for all $x \in [0, 1]$.
(e) $f(x) = x^2$ for all $x \in [0, 1]$.

Remark: Let $f : [a, b] \to \mathbb{R}$ be bounded. Let there exist a sequence (P_n) of partitions of $[a, b]$ such that $L(f, P_n) \to \alpha$ and $U(f, P_n) \to \alpha$. Then $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = \alpha$.

Riemann's criterion for integrability: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ iff for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Some Riemann integrable functions:

(a) A continuous function on $[a, b]$
(b) A bounded function on $[a, b]$ which is continuous except at finitely many points in $[a, b]$
(c) A monotonic function on $[a, b]$

Properties of Riemann integrable functions:

Example: $\frac{1}{3\sqrt{2}} \leq \int\limits_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \leq \frac{1}{3}$

First fundamental theorem of calculus: Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and let $F(x) = \int\limits_a^x f(t)\,dt$ for all $x \in [a,b]$. Then $F : [a,b] \to \mathbb{R}$ is continuous. Also, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Second fundamental theorem of calculus: Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. If there exists a differentiable function $F : [a,b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a,b]$, then $\int\limits_a^b f(x)\,dx = F(b) - F(a)$.

Riemann sum: $S(f,P) = \sum\limits_{i=1}^{n} f(c_i)(x_i - x_{i-1})$,
where $f : [a,b] \to \mathbb{R}$ is bounded, $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a,b]$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, ..., n$.

Result: A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ iff $\lim\limits_{\|P\|\to 0} S(f,P)$ exists in $\mathbb{R}$.
Also, in this case, $\int\limits_a^b f = \lim\limits_{\|P\|\to 0} S(f,P)$.

Example: $\lim\limits_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}] = \log 2$.

Improper integrals:

(a) Type I : The interval of integration is infinite
(b) Type II : The integrand is unbounded in the (finite) interval of integration

Also, combination of Type I and Type II is possible.

Convergence of Type I improper integrals:
Let $f \in \mathcal{R}[a,x]$ for all $x > a$. If $\lim\limits_{x\to\infty} \int\limits_a^x f(t)\,dt$ exists in $\mathbb{R}$, then $\int\limits_a^\infty f(t)\,dt$ converges and $\int\limits_a^\infty f(t)\,dt = \lim\limits_{x\to\infty} \int\limits_a^x f(t)\,dt$. Otherwise, $\int\limits_a^\infty f(t)\,dt$ is divergent.

Similarly, we define convergence of $\int\limits_{-\infty}^b f(t)\,dt$ and $\int\limits_{-\infty}^\infty f(t)\,dt$.

Examples: (a) $\int\limits_1^\infty \frac{1}{t^p}\,dt$ converges iff $p > 1$. (b) $\int\limits_{-\infty}^\infty e^t\,dt$ (c) $\int\limits_0^\infty \frac{1}{1+t^2}\,dt$

Comparison test: Let $0 \leq f(t) \leq g(t)$ for all $x \geq a$. If $\int\limits_a^\infty g(t)\,dt$ converges, then $\int\limits_a^\infty f(t)\,dt$ converges.

Limit comparison test: Let $f(t) \geq 0$ let $g(t) > 0$ for all $t \geq a$ and let $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = \ell \in \mathbb{R}$.

(a) If $\ell \neq 0$, then $\int\limits_a^\infty f(t)\,dt$ converges iff $\int\limits_a^\infty g(t)\,dt$ converges.

(b) If $\ell = 0$, then $\int\limits_a^\infty f(t)\,dt$ converges if $\int\limits_a^\infty g(t)\,dt$ converges.

Examples: (a) $\int\limits_1^\infty \frac{\sin^2 t}{t^2}\, dt$ (b) $\int\limits_1^\infty \frac{dt}{t\sqrt{1+t^2}}$

Absolute convergence: If $\int\limits_a^\infty |f(t)|\, dt$ converges, then $\int\limits_a^\infty f(t)\, dt$ converges.

Example: $\int\limits_0^\infty \frac{\cos t}{1+t^2}\, dt$ converges.

Integral test for series: Let $f : [1, \infty) \to \mathbb{R}$ be a positive decreasing function. Then $\sum\limits_{n=1}^{\infty} f(n)$ converges iff $\int\limits_1^\infty f(t)\, dt$ converges.

Dirichlet's test: Let $f : [a, \infty) \to \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$ such that

(a) f is decreasing and $\lim\limits_{t\to\infty} f(t) = 0$, and

(b) g is continuous and there exists $M > 0$ such that $\left|\int\limits_a^x g(t)\, dt\right| \leq M$ for all $x \geq a$.

Then $\int\limits_a^\infty f(t)g(t)\, dt$ converges.

Example: $\int\limits_1^\infty \frac{\sin t}{t}\, dt$ converges.

Convergence of Type II and mixed type improper integrals:

Example: $\int\limits_0^1 \frac{1}{t^p}\, dt$ converges iff $p < 1$.

Lengths of smooth curves:

(a) Let $y = f(x)$, where $f : [a, b] \to \mathbb{R}$ is such that f' is continuous.
 Then $L = \int\limits_a^b \sqrt{1 + (f'(x))^2}\, dx$

(b) Let $x = \varphi(t)$, $y = \psi(t)$, where $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [a, b] \to \mathbb{R}$ are such that φ' and ψ' are continuous.
 Then $L = \int\limits_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\, dt$

(c) Let $r = f(\theta)$, where $f : [\alpha, \beta] \to \mathbb{R}$ is such that f' is continuous.
 Then $L = \int\limits_\alpha^\beta \sqrt{r^2 + (f'(\theta))^2}\, d\theta$

Examples:

(a) The length of the curve $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$ is 12.

(b) The perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(c) The length of the curve $x = e^t \sin t$, $y = e^t \cos t$, $0 \leq t \leq \frac{\pi}{2}$, is $\sqrt{2}(e^{\frac{\pi}{2}} - 1)$.

(d) The length of the cardioid $r = 1 - \cos\theta$ is 8.

Area between two curves: If $f, g : [a, b] \to \mathbb{R}$ are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, then we define the area between $y = f(x)$ and $y = g(x)$ from a to b to be $\int\limits_a^b (f(x) - g(x))\, dx$.

Example: The area above the x-axis which is included between the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 2ax$, where $a > 0$, is $(\frac{3\pi-8}{12})a^2$.

Area in polar coordinates: Let $f;[\alpha,\beta]\to\mathbb{R}$ be continuous. We define the area bounded by $r=f(\theta)$ and the lines $\theta=\alpha$ and $\theta=\beta$ to be $\frac{1}{2}\int\limits_{\alpha}^{\beta}(f(\theta))^2\,d\theta$.

Example: The area of the region that is inside the cardioid $r=a(1+\cos\theta)$ and also inside the circle $r=\frac{3}{2}a$.

Volume by slicing: $V=\int\limits_a^b A(x)\,dx$.

Example: A solid lies between planes perpendicular to the x-axis at $x=0$ and $x=4$. The cross sections perpendicular to the axis on the interval $0\leq x\leq 4$ are squares whose diagonals run from the parabola $y=-\sqrt{x}$ to the parabola $y=\sqrt{x}$. Then the volume of the solid is 16.

Volume of solid of revolution: $V=\int\limits_a^b \pi(f(x))^2\,dx$.

Example: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Volume by washer method: $V=\int\limits_a^b \pi((f(x))^2-(g(x))^2)\,dx$

Example: A round hole of radius $\sqrt{3}$ is bored through the centre of a solid sphere of radius 2. Then the volume of the portion bored out is $\frac{28}{3}\pi$.

Area of surface of revolution: $S=\int\limits_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$.

Example: The volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2=4ax$ about the x-axis, and bounded by the section $x=x_1$.