MA 101 (Mathematics I)

Model Solutions of Quiz - 2

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1. Let $x_n = n + \frac{1}{n}$ and $y_n = -n$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty}\frac{1}{n} = 0$ but $\lim\limits_{n\to\infty}(x_n^3 + y_n^3) = \lim\limits_{n\to\infty}(3n + \frac{3}{n} + \frac{1}{n^3})$ does not exist (in $\mathbb{R}$) and hence $\lim\limits_{n\to\infty}(x_n^3 + y_n^3) \neq 0$. Therefore the given statement is FALSE.

   (For a similar example, where $\lim\limits_{n\to\infty}(x_n^3 + y_n^3)$ exists (in $\mathbb{R}$) but is not equal to 0, we can take $x_n = (n+1)^{\frac{1}{3}}$ and $y_n = -n^{\frac{1}{3}}$ for all $n \in \mathbb{N}$.)

   (b) *First method*: If possible, let there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\sin n) = \frac{n\pi}{2}$ for all $n \in \mathbb{N}$. Since $(\sin n)$ is a bounded sequence in $\mathbb{R}$, by Bolzano-Weierstrass theorem, there exist $\alpha \in \mathbb{R}$ and a subsequence (n_k) of (n) such that $\sin n_k \to \alpha$. Since f is continuous at α, we get $f(\sin n_k) \to f(\alpha)$ and so $\frac{n_k\pi}{2} \to f(\alpha)$. This is a contradiction, since $(\frac{n_k\pi}{2})$, being an unbounded sequence, cannot converge in $\mathbb{R}$. Therefore the given statement is FALSE.

   Second method: If possible, let there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\sin n) = \frac{n\pi}{2}$ for all $n \in \mathbb{N}$. Then f is continuous on $[-1, 1]$ and hence f is bounded on $[-1, 1]$. Therefore there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [-1, 1]$. Since $|\sin n| \le 1$ for all $n \in \mathbb{N}$, we get $|f(\sin n)| \le M$ for all $n \in \mathbb{N}$. This gives $n \le \frac{2M}{\pi}$ for all $n \in \mathbb{N}$, which is a contradiction. Therefore the given statement is FALSE.

2. We have $x_{n+1} - x_n = \frac{x_n^2+3}{2x_n} - x_n = \frac{3-x_n^2}{2x_n}$ for all $n \in \mathbb{N}$ and $x_{n+1} - \sqrt{3} = \frac{x_n^2+3}{2x_n} - \sqrt{3} = \frac{(x_n-\sqrt{3})^2}{2x_n}$ for all $n \in \mathbb{N}$. Now $x_1 = 2 > 0$ and if we assume that $x_k > 0$ for some $k \in \mathbb{N}$, then $x_{k+1} = \frac{x_k^2+3}{2x_k} > 0$. Hence by the principle of mathematical induction, $x_n > 0$ for all $n \in \mathbb{N}$. Therefore from above, we obtain $x_n \ge \sqrt{3}$ for all $n \in \mathbb{N}$ (observing that $x_1 \ge \sqrt{3}$) and so $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$. Thus the sequence (x_n) is decreasing and bounded below. Consequently (x_n) is convergent.

3. Let $x_n = \frac{n^2+2n}{(2n^4+1)^p}$ and $y_n = \frac{1}{n^{4p-2}}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty}\frac{x_n}{y_n} = \lim\limits_{n\to\infty}(1 + \frac{2}{n})\left(\frac{1}{2+\frac{1}{n^4}}\right)^p = \frac{1}{2^p} \neq 0$. Since the series $\sum\limits_{n=1}^{\infty} y_n$ is convergent iff $4p - 2 > 1$, *i.e.* iff $p > \frac{3}{4}$, by the limit comparison test, the series $\sum\limits_{n=1}^{\infty} x_n$ is convergent iff $p > \frac{3}{4}$.

4. Let $f(x) = \sin^2\frac{1}{x} + \cos^4\frac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$. If $x_n = \frac{1}{n\pi}$ and $y_n = \frac{4}{(8n+1)\pi}$ for all $n \in \mathbb{N}$, then $x_n \to 0$ and $y_n \to 0$. Since $f(x_n) = 1$ and $f(y_n) = \frac{3}{4}$ for all $n \in \mathbb{N}$, we get $f(x_n) \to 1$ and $f(y_n) \to \frac{3}{4}$. Therefore by the sequential criterion of limit, $\lim\limits_{x\to 0} f(x)$ does not exist (in $\mathbb{R}$).