MA 101 (Mathematics I)

Sequence : Summary of Lectures

A sequence of real numbers or a sequence in $\mathbb{R}$ is a mapping $f : \mathbb{N} \to \mathbb{R}$.
Notation: We write x_n for $f(n)$, $n \in \mathbb{N}$ and the notation for a sequence is (x_n).

Examples:

1. Constant sequence: $(a, a, a, ...)$, where $a \in \mathbb{R}$
2. Sequence defined by listing: $(1, 4, 8, 11, 52, ...)$
3. Sequence defined by rule: (x_n), where $x_n = 3n^2$ for all $n \in \mathbb{N}$
4. Sequence defined recursively: (x_n), where $x_1 = 4$ and $x_{n+1} = 2x_n - 5$ for all $n \in \mathbb{N}$

Convergence: What does it mean?

Think of the examples:
$(2, 2, 2, ...)$
$(\frac{1}{n})$
$((-1)^n \frac{1}{n})$
$(1, 2, 1, 2, ...)$
$((-1)^n (1 - \frac{1}{n}))$
$(n^2 - 1)$

Definition: The sequence (x_n) is convergent if there exists $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $|x_n - \ell| < \varepsilon$ for all $n \geq n_0$.
We say: ℓ is a limit of (x_n): $\lim\limits_{n\to\infty} x_n = \ell$ or $x_n \to \ell$.
A sequence which is not convergent is called divergent.

Result: The limit of a convergent sequence is unique.

Examples: (a) $(\frac{n+1}{2n+3})$ (b) $(1, 2, 1, 2, ...)$ (c) $(n^3 + 1)$

Definition: The sequence (x_n) is bounded if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.
Otherwise (x_n) is called unbounded (not bounded).

Examples: (a) $(\frac{3n+2}{2n+5})$ (b) $(1, 2, 1, 3, 1, 4, ...)$

Result: Every convergent sequence is bounded.
So, Not bounded implies Not convergent.

Limit rules for convergent sequences: Let $x_n \to x$ and $y_n \to y$. Then
(a) $x_n + y_n \to x + y$.
(b) $\alpha x_n \to \alpha x$ for all $\alpha \in \mathbb{R}$.
(c) $|x_n| \to |x|$.
(d) $x_n y_n \to xy$.
(e) $\frac{x_n}{y_n} \to \frac{x}{y}$ if $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$.

Examples: (a) $(\frac{2n^2-3n}{3n^2+5n+3})$ (b) $(\sqrt{n+1} - \sqrt{n})$

Standard examples: (a) (α^n), where $|\alpha| < 1$ (b) $(\alpha^{\frac{1}{n}})$, where $\alpha > 0$ (ii) $(n^{\frac{1}{n}})$

Sandwich theorem: Let (x_n), (y_n), (z_n) be sequences such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.
If both (x_n) and (z_n) converge to the same limit ℓ, then (y_n) also converges to ℓ.

Examples: (a) $((2^n+3^n)^{\frac{1}{n}})$ (b) $\left(\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

Result: Let $x_n \neq 0$ for all $n \in \mathbb{N}$ and let $L = \lim\limits_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|$ exist.

(a) If $L < 1$, then $x_n \to 0$.
(b) If $L > 1$, then (x_n) is divergent.

Examples: (a) $(\frac{\alpha^n}{n!})$, where $\alpha \in \mathbb{R}$ (b) $(\frac{2^n}{n^4})$

Definition: (x_n) is increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.
(x_n) is decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.
(x_n) is monotonic if it is either increasing or decreasing.

Examples: (a) $(1-\frac{1}{n})$ (b) $(n+\frac{1}{n})$ (c) $(\cos\frac{n\pi}{3})$

Definition: Let $S(\neq \emptyset) \subset \mathbb{R}$ and $u \in \mathbb{R}$.
u is an upper bound of S in $\mathbb{R}$ if $x \leq u$ for all $x \in S$.
u is the supremum (least upper bound) of S in $\mathbb{R}$ if

(a) u is an upper bound of S in $\mathbb{R}$, and
(b) u is the least among all the upper bounds of S in $\mathbb{R}$, *i.e.* if u' is any upper bound of S in $\mathbb{R}$, then $u \leq u'$.

Lower bound and infimum (greatest lower bound) are defined similarly.

Result: An increasing sequence (x_n) which is bounded above converges to $\sup\{x_n : n \in \mathbb{N}\}$.
A decreasing sequence (x_n) which is bounded below converges to $\inf\{x_n : n \in \mathbb{N}\}$.
So a monotonic sequence converges iff it is bounded.

Example: Let $x_1 = 1$ and $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$. Then (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

Cauchy sequence: A sequence (x_n) is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_0$.

Result: A sequence in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.

Example: Let $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ for all $n \in \mathbb{N}$. Then (x_n) is convergent.

Example: Let (x_n) satisfy either of the following conditions:

(a) $|x_{n+1} - x_n| \leq \alpha^n$ for all $n \in \mathbb{N}$
(b) $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$,

where $0 < \alpha < 1$.
Then (x_n) is a Cauchy sequence.

Example: Let $x_1 = 1$ and let $x_{n+1} = \frac{1}{x_n+2}$ for all $n \in \mathbb{N}$. Then (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \sqrt{2} - 1$.

Subsequence: Let (x_n) be a sequence in $\mathbb{R}$. If (n_k) is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then (x_{n_k}) is called a subsequence of (x_n).

Examples: Think of some divergent sequences and their convergent subsequences.

Result: If a sequence (x_n) converges to ℓ, then every subsequence of (x_n) must converge to ℓ.
So, if (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \not\to \ell$, then $x_n \not\to \ell$.
Also, if (x_n) has two subsequences converging to two different limits, then (x_n) cannot be convergent.

Example: Let $x_n = (-1)^n(1 - \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $x_n \not\to 1$.
In fact, (x_n) is not convergent.

Remark: Let (x_n) be a sequence such that $x_{2n} \to \ell$ and $x_{2n-1} \to \ell$. Then $x_n \to \ell$.

Example: The sequence $(1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, ...)$ converges to 1.

Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Examples: If $x \in \mathbb{R}$, then there exists a sequence (r_n) of rationals converging to x.
Similarly, if $x \in \mathbb{R}$, then there exists a sequence (t_n) of irrationals converging to x.