MA 101 (Mathematics I)

Series : Summary of Lectures

An infinite series in $\mathbb{R}$ is an expression $\sum_{n=1}^{\infty} x_n$, where (x_n) is a sequence in $\mathbb{R}$.
More formally, it is an ordered pair $((x_n),(s_n))$, where (x_n) is a sequence in $\mathbb{R}$ and $s_n = x_1 + \cdots + x_n$ for all $n \in \mathbb{N}$.

x_n : nth term of the series
s_n : nth partial sum of the series

Convergence of series: $\sum_{n=1}^{\infty} x_n$ is convergent if (s_n) is convergent. In this case, the sum of the series is $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$.
A series which is not convergent is called divergent.

Examples:

1. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ (where $a \neq 0$) converges iff $|r| < 1$.
2. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent with sum 1.
3. The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.

Algebraic operations on series: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent with sums x and y respectively. Then
(a) $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent with sum $x + y$
(b) $\sum_{n=1}^{\infty} \alpha x_n$ is convergent with sum αx, where $\alpha \in \mathbb{R}$

Monotonic criterion: A series $\sum_{n=1}^{\infty} x_n$ of non-negative terms is convergent iff the sequence (s_n) is bounded above.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Cauchy criterion: A series $\sum_{n=1}^{\infty} x_n$ is convergent iff for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_{m+1} + \cdots + x_n| < \varepsilon$ for all $m > n \geq n_0$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Result: If $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n \to 0$.

Hence if $x_n \not\to 0$, then $\sum_{n=1}^{\infty} x_n$ cannot be convergent.

Examples: The following series are not convergent.
(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

Comparison test: Let (x_n) and (y_n) be sequences in $\mathbb{R}$ such that for some $n_0 \in \mathbb{N}$,

$0 \le x_n \le y_n$ for all $n \ge n_0$.
Then

(a) $\sum\limits_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ is convergent,

(b) $\sum\limits_{n=1}^{\infty} x_n$ is divergent $\Rightarrow \sum\limits_{n=1}^{\infty} y_n$ is divergent.

Limit comparison test: Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \to \ell \in \mathbb{R}$.

(a) If $\ell \neq 0$, then $\sum\limits_{n=1}^{\infty} x_n$ is convergent iff $\sum\limits_{n=1}^{\infty} y_n$ is convergent.

(b) If $\ell = 0$, then $\sum\limits_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ is convergent.

Examples: (a) $\sum\limits_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ (b) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n+n}$ (c) $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ (d) $\sum\limits_{n=1}^{\infty} \frac{n}{4n^3-2}$

Cauchy's condensation test: Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum\limits_{n=1}^{\infty} x_n$ is convergent iff $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

Examples:

(a) p-series: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff $p > 1$.

(b) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent iff $p > 1$.

Definitions: $\sum\limits_{n=1}^{\infty} x_n$ is called absolutely convergent if $\sum\limits_{n=1}^{\infty} |x_n|$ is convergent.
$\sum\limits_{n=1}^{\infty} x_n$ is called conditionally convergent if $\sum\limits_{n=1}^{\infty} x_n$ is convergent but $\sum\limits_{n=1}^{\infty} |x_n|$ is divergent.

Result: Every absolutely convergent series is convergent.

Ratio test: Let (x_n) be a sequence of nonzero real numbers such that $|\frac{x_{n+1}}{x_n}| \to \ell$.

(a) If $\ell < 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.

(b) If $\ell > 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is divergent.

Examples: (a) $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ (b) $\sum\limits_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Root test: Let (x_n) be a sequence in $\mathbb{R}$ such that $|x_n|^{\frac{1}{n}} \to \ell$.

(a) If $\ell < 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.

(b) If $\ell > 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is divergent.

Examples: (a) $\sum\limits_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ (b) $\sum\limits_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$

Leibniz's test: Let (x_n) be a decreasing sequence of positive real numbers such that

$x_n \to 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent.

Examples: (a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$, $p \in \mathbb{R}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$

Result: Grouping of terms of a convergent series does not change the convergence and the sum.
However, a divergent series can become convergent after grouping of terms.

Result: Rearrangement of terms does not change the convergence and the sum of an absolutely convergent series.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = s$
$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$

Riemann's rearrangement theorem: Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series.

(a) If $s \in \mathbb{R}$, then there exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series has the sum s.

(b) There exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series diverges.