Discrete Mathematics (CS202/MA221) — IIT

Problem set #6 — **Guwahati, India**

November 19, 2011

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1. Prove that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.
2. Prove that the rings $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.
3. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.
4. Describe all the ideals of $\mathbb{Z}$ and describe all the homomorphic images of $\mathbb{Z}$.
5. Consider the ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$ of multivariate polynomials over a field $\mathbb{K}$ (i.e. coefficients of the polynomial come from $\mathbb{K}$). For any $T \subseteq \mathbb{K}^n$, let $I(T) = \{f : f(t) = 0 \text{ for all } t \in T\}$. Is $I(T)$ an ideal?
6. Prove or disprove. The union of ideals is an ideal.
7. Prove or disprove. The intersection of ideals is an ideal.
8. What are the ideals of a field?
9. Prove Binomial Theorem in a commutative ring with identity. Interpret the various terms appropriately.
10. Give an example for an infinite increasing chain of ideals.
11. Prove or disprove. The union of an increasing chain of ideals is an ideal.
12. Consider a convex polygon on n vertices. Choose k points in its interior. Triangulate the polygon using the n vertices and the n points. What is the number of triangles in any triangulation?
13. Let n be a natural number. Split n into natural numbers a and b. Let p_1 be ab. Choose one of a and b to split again. Let p_2 be their product. Repeat this process till every part is 1, i.e at stage i choose a number k greater than 1 from the previously generated numbers, split it arbitrarily into a_i and b_i. What is the maximum and minimum value of $\sum_i p_i$? Let P_i be defined as the product of all the parts at stage i. What is maximum and minimum values of P_i?
14. Prove that $\displaystyle\sum_{j=0}^{k} \binom{n}{k} = \sum_{j=0}^{k} \binom{n-1-j}{k-j} 2^j$.
15. Calculate the average the number of cycles for permutations of length n. For example, the permutation $(1, 5)(2, 4, 3)$ is a permutation of length 5 with 2 cycles.
16. Find $S(n, 3)$.
17. Show that $B(n) < n!$
18. Prove or disprove. For a fixed integer k, $S(n, n-k)$ is a polynomial.
19. Prove or disprove. For a fixed integer k, $S(n, k)$ is a polynomial.
20. Solve the recurrence $a_0 = 0, a_{k+1} = a_k + 2^k$

21. Solve the recurrence $a_0 = 2, a_1 = 0, a_2 = -2, a_{k+3} = 6a_{k+2} - 11a_{k+1} + 6a_n$

22. Compute the coefficient of x^n in the following generating functions

    - $\frac{1}{1-z^3}$
    - $(1+z)^n + (1-z)^n$
    - $\frac{(1-z)^2}{(1-z)^4}$
    - $\frac{1}{(1-z)(1-z^2)(1-z^3)}$

23. Count the number of Hamiltonian cycles in K_n and $K_{n,n}$

24. Can a tree have a perfect matching?

25. **König - Egervary theorem** A "vertex cover" in a graph G is a subset of vertices which contains at least one vertex of every edge. Show that in a bipartite graph the maximum matching and minimum vertex cover are of same size.