Diversity Over Dominance Approach for Many-Objective Optimization On Reference-Points-based Framework

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Abstract. Evolutionary multi-objective optimization (EMO) algorithms are designed to achieve a balance between convergence and diversity. However, these algorithms confront major challenges when all of their individuals become non-dominated while solving many-objective optimization problems. Although the appreciable efforts have been made by using the reference-points-based framework coupled with the Pareto-dominance ranking in the literature, selection of a diverse set of individuals, sometimes preferring isolated and dominated individuals over non-dominated individuals, needs to be addressed. In this paper, we propose the diversity over dominance (DoD) approach in which the diversity is preserved first by making clusters of individuals that are made by associating individuals to their nearest line using the reference-points-based framework. The Pareto-dominance ranking is then used to rank the individuals separately for each cluster. The environment selection is then developed that selects individuals from each cluster. The DoD approach is tested on DTLZ and WFG problem instances and the results demonstrate its competitive performance over the existing EMO algorithms.

Keywords: Diversity · Dominance · Many objective optimization · Evolutionary Algorithm · Evolutionary Multiobjective Optimization.

1 Introduction

Many real-world problems often have multiple objectives that are conflicting in nature such as in crashworthiness of vehicle [11], bulldozer blade-design [1] to name a few. For such problems, a set of solutions is optimal, which are referred as Pareto-optimal (PO) solutions. Evolutionary multi-objective optimization (EMO) algorithms are the ideal choice for solving these problems because a set of PO solutions can be generated in one run.

From last few years, EMO algorithms for solving many-objective optimization problems (generally more than three-objective problems) are getting attention worldwide. The most successful ones like NSGA-III [3], $\theta-$DEA [15], MOEA/DD [10] to name a few, are developed using the reference-points-based framework in which the convergence is achieved by performing the Pareto-dominance ranking and the diversity is preserved by selecting individuals based on the reference points generated on a unit hyperplane [2]. Since for many-objective optimization problems, almost all individuals of a population become non-dominated, the Pareto-dominance ranking fails to provide enough selection pressure for convergence [12]. At this stage, the environment selection based on the reference-points framework plays a major role in selecting individuals. Therefore, NSGA-III introduced the niching procedure for selecting individuals representing the lines that are drawn using the reference points. To give preference to isolated individuals, MOEA/DD [10] introduced a uniform paradigm of dominance and decomposition approaches in which only one offspring individual at a time gets a chance for its survival. On the similar line, SPEA/R [8] proposed a composite fitness function so that individuals from each subregion (defined using the reference points) can be selected. In [7], an external archive was maintained for those individuals which may get eliminated using NSGA-III's environment selection. An individual with minimum distance between the ideal point and the line drawn from the reference point was selected to update the archive. In all these studies, the environmental selection gave first emphasis on dominance-based selection followed by diversity for each subregion constructed from the directions through the reference points and the origin.

On the contrary to the above EMO algorithms, Jiang and Yang [9] suggested performing diversity-first sorting approach. The environment selection for diversity was performed first and then Pareto-ranking was used to select individuals. The diversity-first sorting based evolutionary algorithm (DBEA) outperformed NSGA-III on many-objective optimization instances of WFG [6] problems. On the similar line, $\theta-$DEA [15] performs $\theta-$dominance sorting on the clusters of individuals which are made using the reference-points framework for diversity. $\theta-$DEA showed better results than NSGA-III over DTLZ [4] and WFG problem instances. Motivated by these approaches, we propose a dominance over decomposition approach, refer as DoD, in which individuals in a population are first clustered based on the directions from the reference points and the origin. Thereafter, the non-dominated sorting is performed to each clustered independently. The main contribution of DoD approach is the environment selection that selects a diverse set of individuals by preferring isolated individuals from each cluster and sometimes selecting dominated individuals over the crowded non-dominated individuals. In the remaining paper, the challenges with dominance-based EMO algorithms are discussed in section 2. The DoD approach is described followed by its implementation in section 3. The results are discussed and compared with the existing EMO algorithms in section 4. The paper is concluded in section 5 with the future work.

2 Challenges with Dominance-based Environment Selection

The challenges with the environment selection of dominance-based EMO algorithms are shown using three cases in Fig. 1. In all cases, the objective space

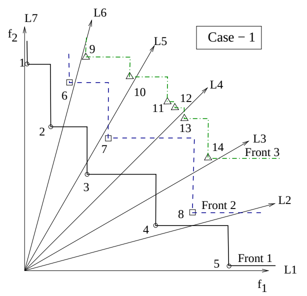

Fig. 1. Case-1: when the number of non-dominated individuals is less than N. Case-2: when the number of non-dominated individuals is equal to N. Case-3: when the number of non-dominated individuals is more than N.

has $N = 7$ reference lines $(L1, \ldots, L7)$, which are drawn using the structured reference points [2]. Among $2N$ individuals, N individuals need to be selected. In Case-1, the number of non-dominated individuals (currently five) is less than N. The dominance-based approach, which prefers dominance followed by diversity, selects all non-dominated individuals from the front-1 and the rest of two individuals will be selected from the front-2. In this case, no individual representing lines $L3$ and $L4$ is selected. In case-2, the number of non-dominated individuals is equal to N. The dominance-based approach selects all individuals from the front-1. Again, there is no individual representing lines $L3$ and $L4$. In Case-3, the number of non-dominated individuals are more than N. In this case, the dominance-based approach will select individuals based on the diversity preserving mechanism since the ranking through non-dominated sorting cannot differentiate individuals. For example, the dominance approach can select individuals nearest to their respective lines. For example, individuals marked as 1, 4, 6, 9, and 11 are selected. The remaining two individuals are selected only from the front-1. In this case also, there is no individual representing lines $L3$ and $L4$.

The challenges described above leads to the motivation of the present work in which the DoD approach is proposed to select a diverse set of individuals, especially when a large number of individuals is non-dominated. Moreover, an additional emphasis is given to select isolated and sometimes dominated individuals over crowded non-dominated individuals for better convergence and diversity.

3 DoD Approach and its Implementation

The DoD approach is described using Fig. 2 in which the DoD approach selects diverse individuals for the same three cases as presented in Fig. 1. For Case-1, the clusters are made for every line as shown in Fig. 2. Then, the Pareto-dominance ranking is applied to rank the individuals for every cluster separately. The non-dominated individual within each cluster is then chosen. For example, all non-dominated individuals from the front-1 are selected along with individ-

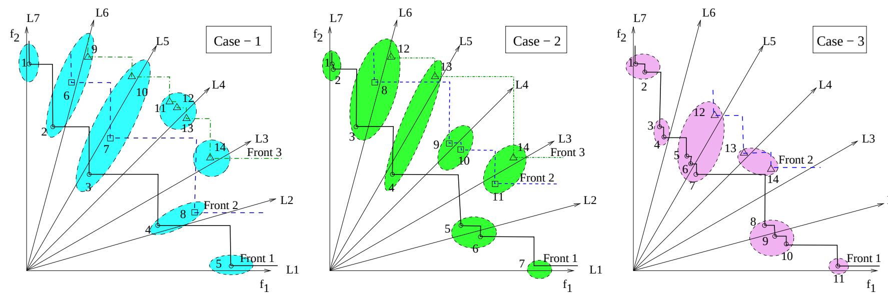

Fig. 2. DoD approach of selecting diverse individuals for three cases presented in Fig. 1.

uals (isolated, and dominated as per the dominance-based approach) marked as 13 and 14 in the figure. For Case-2, the DoD approach selects individuals marked as 1, 3, 4, 6, 7 from the front-1. It is noted that individual marked as 1 is preferred over 2 in the same cluster. Basically, when a cluster has more than one non-dominated individual, the individual nearest to the line gets selected. The other individuals marked as 10 and 11 are selected from different clusters. For Case-3, the DoD approach selects individuals marked as 1, 4, 6, 9 and 11 from the front-1 and 13 from the front-2 from their respective clusters. In this case, there is no associated individual with line $L3$. The DoD approach first associates the nearest individual to line $L3$, that is, individual marked as 14 and then selects the same individual. From the above discussion, it can be observed that the DoD approach can select a diverse set of individuals representing every line. This environment selection can keep enough selection pressure for better convergence and diversity of EMO algorithms. The major limitation of this approach is preferring dominated individuals over non-dominated individuals, which may cause convergence issue.

The DoD approach is implemented using the reference-points-based framework, which is shown in algorithm 1[1]. At any generation t, the individuals are randomly selected from the parent population (P_t) on which simulated binary crossover and polynomial mutation operators are applied to create a new population, which is referred as offspring population (Q_t). Both P_t and Q_t are merged into R_t. The next generation parent population (P_{t+1}) is then chosen using the DoD environment selection. Algorithm 2 shows steps of the DoD environment selection in which R_t is normalized and then the individuals are associated with the reference lines. In this paper, an external vector $\mathbf{e}$ is maintained for storing the extreme objective function values for normalizing the population. As it can be seen in step 2 of algorithm 1 that $\mathbf{e}$ is initialized by maximum objective function values. In normalization, $\mathbf{e}$ is updated or kept same based on the intercept and extreme points found. Algorithm 3 shows steps required for normalizing R_t, which include computing ideal point, extreme points and intercepts on each objective axis. Since some degenerate cases like unavailability of distinct extreme

[1] Source code at http://www.iitg.ac.in/dsharma/pub.html

Algorithm 1 Framework for DoD approach

Input: Parameters, $t = 1$, T : Number of generations, M : Number of objectives, N : population size, H : Number of reference points
Output: P_{t+1}

Initialize random population (P_t)
Compute external vector: $\mathbf{e} = (e_1, e_2, \ldots, e_M)^T$ such that $e_j = \max_{\mathbf{x} \in P_t} f_j(\mathbf{x})$
while $t \leq T$ **do**
  $P_t^{'} =$ Random selection (P_t)
  $Q_t =$ Recombination + Mutation $(P_t^{'})$
  $R_t = P_t \cup Q_t$
  $P_{t+1} =$ DoD Environment selection (R_t)
  $t = t + 1$
end while

points from R_t or negative intercept can occur, the Nadir point is found from R_t. The external vector $\mathbf{e}$ is updated when any component of the Nadir point is better than the corresponding component of $\mathbf{e}$ as shown in step 8 of algorithm 3. Otherwise, the external vector $\mathbf{e}$ is updated completely by the intercepts found in step 11. Finally, each objective of all individuals of R_t is normalized using $\mathbf{e}$ at step 13.

Once R_t is normalized, the individuals are then associated with their nearest reference lines. The association procedure is shown in algorithm 4 for which the normalized $\bar{R}_t$ and H are required. The structured reference points are created on a unit hyperplane using Das and Dennis approach [2]. These reference points are then used to compute reference lines (step 2) which pass from the origin and the reference point. These reference lines are stored in Z^r for associating individuals of R_t in the normalized objective space. As can be seen from step 4, a set $C_j \in C$ is initialized empty, which will store the individuals associated with a reference line j. Also, the niche count ρ_j for all reference lines is set zero that signifies a number of individuals associated with a reference line j. Inside the loop at step 5, each individual $\mathbf{r}$ is associated with its nearest reference line $(\pi(\mathbf{r}))$ based in its distance as shown in step 9. Thereafter, an individual $\mathbf{r}$ is stored in the cluster of $\pi(\mathbf{r})$ reference line. Also, the niche count of reference line $\pi(\mathbf{r})$ is incremented by one.

After normalization and association, the non-dominated sorting is performed for the individuals stored in a cluster C_j for a reference line j, which has at least one associated individual (refer step 3 of algorithm 2). If a number of non-dominated individuals in C_j is more than one, then an individual $\mathbf{x}$ is selected, which is nearest to a reference line j. Otherwise, the only non-dominated individual is selected. The selected individual is then copied to P_{t+1} and it is removed from R_t. These steps are followed for all the lines, which has a niche count $\rho_j > 0$.

In addition to the above steps, if any reference line has no individual associated (meaning $\rho_j = 0$ and $C_j = \phi$, refer step 12 of algorithm 2), the individual $\mathbf{x}$ nearest to a reference line j is chosen from the remaining individuals of R_t.

Algorithm 2 DoD Environment Selection (R_t)

Input: R_t, H, $\mathbf{e}$
Output: P_{t+1}

$\bar{R}_t :=$ Normalize $(R_t, \mathbf{e})$
$(C, \rho, Z^r) :=$ Associate $(\bar{R}_t, H)$ $\quad \%C = \{C_1, \dots, C_H\}$, $\rho = (\rho_1, \dots, \rho_H)^T$, Z^r : set of reference lines.
for each $j \in Z^r$ and $\rho_j > 0$ **do**
  Non-dominated sorting of individuals $\in C_j$
  if Number of the best ranked individuals > 1 **then**
    Select the individual $\mathbf{x} \in C_j$ which is nearest to the reference line j
  else
    Select the best ranked individual $\mathbf{x} \in C_j$
  end if
  Include $P_{t+1} = P_{t+1} \cup \mathbf{x}$ and update $R_t = R_t \setminus \mathbf{x}$
end for
for each $j \in Z^r$ and $\rho_j == 0$ **do**
  Associate the closest individual $\mathbf{x}$ from the remaining R_t to the reference line j and update $I_j = I_j \cup \mathbf{x}$, and $\rho_j = 1$
  Include $P_{t+1} = P_{t+1} \cup \mathbf{x}$ and update $R_t = R_t \setminus \mathbf{x}$
end for
while $|P_{t+1}| < N$ **do**
  Select a random reference line $j \in Z^r : \rho_j > 1$
  Select the best individual $\mathbf{x}$ associated to the reference line j such that $\mathbf{x} \notin P_{t+1}$
  Include $P_{t+1} = P_{t+1} \cup \mathbf{x}$ and update $R_t = R_t \setminus \mathbf{x}$
end while

Algorithm 3 Normalize (R_t)

Input: $R_t, \mathbf{e}$
Output: $\bar{R}_t$: Normalized population

Determine ideal point, $\mathbf{z^I} = (z_1^I, z_2^I, \dots, z_M^I)^T$ such that $z_j^I = \min_{\mathbf{r} \in R_t} f_j(\mathbf{r})$
Translate objectives, $\mathbf{f}^{'}(\mathbf{r}) = (f_1^{'}(\mathbf{r}), f_2^{'}(\mathbf{r}), \dots, f_M^{'}(\mathbf{r}))^T$ such that $f_j^{'}(\mathbf{r}) = f_j(\mathbf{r}) - z_j^I, \ \forall \mathbf{r} \in R_t$
Compute extreme solutions, $Z = (\mathbf{z_1^e}, \mathbf{z_2^e}, \dots, \mathbf{z_M^e})$ such that $\mathbf{z}_j^e = \mathbf{f}^{'}(\mathbf{r}), \mathbf{r} : \min_{\mathbf{r} \in R_t} \left(\max_{i=1}^{M} f_i^{'}(\mathbf{r})/w_i \right)$
Compute intercept a_j for $j = 1, \dots, M$.
if Degenerate case or negative intercept found **then**
  Compute Nadir point, $\mathbf{z^N} = (z_1^N, z_2^N, \dots, z_M^N)^T$ such that $z_j^N = \max_{\mathbf{r} \in R_t^*} f_j(\mathbf{r})$ and $R_t^* \in R_t$ is the set of the non-dominated individuals.
  if $z_j^N < e_j$, where $j \in \{i, \dots, M\}$ **then**
    $e_j = z_j^N$
  end if
else
  Update $e_j = a_j, \ \forall j \in \{i, \dots, M\}$
end if
Normalize objective $\bar{f}_j(\mathbf{r}) = f_j^{'}(\mathbf{r})/e_j, \forall \mathbf{r} \in R_t, \ \forall j \in \{i, \dots, M\}$ and return $\bar{R}_t$

Algorithm 4 Associate (R_t) with the reference lines

Input: $\bar{R}_t,\ H$
Output: C, ρ, Z^r
for all $\mathbf{r} \in H$ **do**
  Compute reference line $\mathbf{w}$ and $Z^r = Z^r \cup \mathbf{w}$
end for
Initialize $C_j = \emptyset\ \forall j \in H$ and $\rho = (0, 0, \ldots, 0)^T$
for all $\mathbf{r} \in R_t$ **do**
  for all $\mathbf{w} \in Z^r$ **do**
    Compute $dist(\mathbf{r}, \mathbf{w}) = ||(\mathbf{r} - \mathbf{w}^T \mathbf{r} \mathbf{w}/||\mathbf{w}||^2)||$
  end for
  $\pi(\mathbf{r}) = \mathbf{w} : \text{argmin}\, dist(\mathbf{r}, \mathbf{w})$
  $d(\mathbf{r}) = dist(\mathbf{r}, \pi(\mathbf{r}))$
  $C_{\pi(\mathbf{r})} = C_{\pi(\mathbf{r})} \cup \mathbf{r}$
  $\rho_{\pi(\mathbf{r})} = \rho_{\pi(\mathbf{r})} + 1$
end for

This individual $\mathbf{x}$ is then stored in C_j and the niche count is increased by one. The same individual $\mathbf{x}$ is then copied to P_{t+1} and it is removed from R_t. These steps are then followed for those reference lines which have their ρ_j's zero.

Since the structure reference points are generated for reference lines, sometimes H (number of reference points) is less than N (population size). In this scenario, a few of individuals are selected from those lines which has $\rho_j > 1$. It is because the best individual from each cluster is already selected earlier, which cannot be copied again to P_{t+1}. Satisfying the conditions given in step 17 of algorithm 2, a random reference line j is chosen and then select the best individual $\mathbf{x}$ associated to the reference line j such that $\mathbf{x} \notin P_{t+1}$. The selected individual is then copied to P_{t+1} and it is removed from R_t. The loop at step 16 of algorithm 2 is active till $|P_{t+1}|$ become N.

It can be observed that the condition at step 3 of algorithm 2 is imposed to select the nearest non-dominated individual from each cluster of all reference lines. It means that the diversity driven by the reference-points approach is maintained. Since every reference line is important for maintaining diversity among the individuals of a given population, the condition at step 12 of algorithm 2 is imposed to select individuals for empty reference lines, which can be isolated and sometimes, dominated individual.

The computational complexity of the DoD approach remains same as NSGA-III that is $\max(O(N^2 log^{M-2} N), O(N^2 M))$ when almost all individuals are associated with a single line. First association requires $O(N^2 M)$ operations and then, the non-dominated sorting for this cluster requires $O(N^2 log^{M-2} N)$ operations.

4 Results and Discussion

The proposed DoD approach is compared with the existing EMO algorithms, such as NSGA-III [3] and MOEA/D [16] on DTLZ problems [4] with $M \in$

$\{3, 5, 8, 10, 15\}$ objective test instances and WFG problems [6] with $M \in \{3, 5, 8, 10\}$ objective instances. For DTLZ problems, the number of decision variables is given as $n = M + k - 1$, where $k = 5$ for DTLZ1, and $k = 10$ for DTLZ2-4 problems. For WFG6-7 problems, the number of decision variables is set to $n = k + l$ in which the position-related variable is $k = 2 \times (M - 1)$, and the distance-related variable is $l = 20$. The inverse generalized distance (IGD) indicator [16] and hypervolume (HV) indicator [13] are used for the performance evaluation of EMO algorithms. All EMO algorithms are run for 20 times with different initial population. Moreover, a difference for statistical significance is tested using the Wilcoxon signed-rank test [14] at 5% significance level for the assessment of obtained results from competing EMO algorithms.

Table 1 presents the population sizes, divisions and a number of reference points for EMO algorithms. For more than 5-objective instances, the two-layered reference points are generated similar to [3]. The table also summarizes termination conditions for all problems, which is kept similar to [3].

Table 2 presents the IGD values obtained from three EMO algorithms. A smaller IGD value refers better performance. It can be seen that the DOD approach is superior to both EMO algorithms in DTLZ2, DTLZ4, and WFG7 instances. For DTLZ3, the DoD approach is found to be better in lower objective instances. NSGA-III is better than both EMO algorithms in WFG6 instances. Table 3 presents HV values in which it can be seen that the DoD approach is better than both EMO algorithms in almost all instances of DTLZ and WFG problems. Since HV values are close to one, the DoD approach showed its efficacy in selecting a diverse set of solutions.

The non-dominated solutions obtained corresponding to the median IGD value run are shown in Fig. 3 for DTLZ problems. A well-distributed front can be seen from the DoD approach, whereas MOEA/D is unable to generate similar fonts for DTLZ1 and DTLZ4 problems. Fig. 4 presents parallel coordinates for 10-objective DTLZ problems. It can be seen that a well-distributed set of solutions is generated by the DoD approach against MOEA/D.

5 Conclusions

The purpose of DoD approach was to select a diverse set of individuals in the environment selection using the reference-points-based framework. Since almost all individuals became non-dominated for many-objective optimization, the DoD approach showed its superiority over the environmental selection of NSGA-III by solving many test instances of DTLZ and WFG problems. The IGD values obtained using the DoD approach were found to be better than NSGA-III in many instances and better in all instances against MOEA/D. The HV values indicated that the DoD approach served its purpose of selecting diverse individuals and showed its efficacy against NSGA-III and MOEA/D in almost all test instances. In future, the DoD approach can be improved further to design selection rules that can emphasis non-dominated individuals over dominated individuals without losing its core idea.

Table 1. Input parameters for EMO algorithms.

Population				Termination				
M	divisions	H	N	DTLZ1	DTLZ2	DTLZ3	DTLZ4	WFG6-7
3	12	91	92	400	250	1000	600	1000
5	6	210	210	600	350	1000	1000	1250
8	$(3, 2)$	156	156	750	500	1000	1250	1500
10	$(3, 2)$	275	276	1000	750	1500	2000	2000
15	$(2, 1)$	135	136	1500	1000	2000	3000	3000

Table 2. Best, median and worst IGD values obtained by DoD approach and other algorithms on DTLZ and WFG instances with different number of objectives. Best performances are highlighted in bold face with gray background. NSGA-III results are obtained from [3], and MOEA/D results are obtained using [5].

	M	NSGA-III	MOEA/D	DoD		M	NSGA-III	MOEA/D	DoD
DTLZ1	3	4.880E-04	2.607E-02	**3.333E-04**	DTLZ3	3	9.751E-04	2.491E-02	**6.966E-04**
		1.308E-03	4.713E-02$^+$	**1.106E-03**			4.007E-03	4.477E-01$^+$	**2.144E-03**
		4.880E-03	3.954E-01	5.770E-03			6.665E-03	1.691E+01	**5.898E-03**
	5	**5.116E-04**	1.582E-02	5.567E-04		5	3.086E-03	2.311E-02	**1.698E-03**
		9.799E-04	3.071E-02$^+$	1.354E-03			5.960E-03	2.303E-01$^+$	**5.181E-03**
		1.979E-03	6.377E-02	1.167E-02			**1.196E-02**	4.304E-01	7.762E-02
	8	**2.044E-03**	1.798E-02	2.176E-03		8	**1.244E-02**	7.445E-02	1.828E-02
		3.979E-03	2.721E-02$^+$	**3.546E-03**			**2.375E-02**	6.251E-01$^+$	3.478E-02
		8.721E-03	5.509E-02	9.393E-03			**9.649E-02**	1.151E+00	2.033E+00
	10	**2.215E-03**	2.168E-02	2.219E-03		10	**8.849E-03**	4.514E-02	9.529E-03
		3.462E-03	3.007E-02$^+$	**3.034E-03**			**1.188E-02**	2.744E-01$^+$	1.579E-02
		6.869E-03	4.202E-02	**6.482E-03**			**2.083E-02**	1.161E+00	2.598E-02
	15	**2.649E-03**	4.782E-02	3.740E-03		15	1.401E-02	1.864E-01	**1.004E-02**
		5.063E-03	5.338E-02$^-$	2.778E-01			2.145E-02	1.281E+00$^+$	**1.672E-02**
		1.123E-02	6.177E-02	3.878E-01			**4.195E-02**	1.300E+00	6.676E-01
DTLZ2	3	1.262E-03	1.056E-02	**1.162E-03**	DTLZ4	3	**2.915E-04**	7.446E-03	3.535E-04
		1.357E-03	1.469E-02$^+$	1.509E-03			5.970E-04	5.307E-01$^+$	**4.469E-04**
		2.114E-03	2.243E-02	5.328E-03			4.286E-01	9.503E-01	**6.577E-04**
	5	4.254E-03	1.321E-02	**3.797E-03**		5	9.849E-04	1.475E-02	**3.741E-04**
		4.982E-03	1.675E-02$^+$	**4.630E-03**			1.255E-03	3.095E-02$^+$	**4.632E-04**
		5.862E-03	2.295E-02	**5.562E-03**			1.721E-03	6.050E-01	**5.623E-04**
	8	1.371E-02	3.001E-02	**1.141E-02**		8	5.079E-03	3.161E-02	**3.123E-03**
		1.571E-02	3.453E-02$^+$	**1.410E-02**			7.054E-03	2.936E-01$^+$	**3.546E-03**
		1.811E-02	4.046E-02	1.848E-02			6.051E-01	6.410E-01	**4.695E-03**
	10	1.350E-02	2.509E-02	**1.116E-02**		10	5.694E-03	4.741E-02	**3.448E-03**
		1.528E-02	3.974E-02$^+$	**1.265E-02**			6.337E-03	1.856E-01$^+$	**4.252E-03**
		1.697E-02	4.348E-02	**1.532E-02**			1.076E-01	3.959E-01	**5.031E-03**
	15	1.360E-02	2.248E-02	**1.063E-02**		15	7.110E-03	5.447E-02	**5.404E-03**
		1.726E-02	6.526E-02$^+$	**1.304E-02**			3.431E-01	2.656E-01$^+$	**7.290E-03**
		2.114E-02	1.917E-01	**1.686E-02**			1.073E+00	6.714E-01	**9.265E-03**

	M	NSGA-III	MOEA/D	DoD		M	NSGA-III	MOEA/D	DoD
WFG6	3	**4.828E-03**	7.550E-02	1.962E-02	WFG7	3	2.789E-03	9.344E-02	**2.309E-03**
		1.224E-02	8.163E-02	2.847E-02			3.692E-03	1.049E-01	**2.891E-03**
		5.486E-02	1.242E-01	**3.633E-02**			4.787E-03	1.182E-01	**3.696E-03**
	5	**5.065E-03**	3.159E-01	2.604E-02		5	8.249E-03	3.613E-01	**6.549E-03**
		1.965E-02	4.418E-01	3.381E-02			9.111E-03	3.950E-01	**8.103E-03**
		4.475E-02	5.407E-01	**4.237E-02**			**1.050E-02**	4.315E-01	2.224E-02
	8	**1.009E-02**	9.031E-01	3.465E-02		8	2.452E-02	8.977E-01	**1.665E-02**
		2.922E-02	9.362E-01	4.114E-02			2.911E-02	9.303E-01	**2.089E-02**
		7.098E-02	9.716E-01	**5.024E-02**			6.198E-02	9.595E-01	**2.399E-02**
	10	**1.060E-02**	9.487E-01	2.781E-02		10	3.228E-02	9.368E-01	**2.091E-02**
		2.491E-02	1.008E+00	3.562E-02			4.292E-02	9.533E-01	**2.309E-02**
		6.129E-02	1.031E+00	**4.480E-02**			9.071E-02	1.006E+00	**2.533E-02**
	15	**1.368E-02**	1.120E+00	2.486E-02		15	**3.457E-02**	1.212E+00	8.945E-02
		2.877E-02	1.240E+00	3.522E-02			**5.450E-02**	1.216E+00	5.559E-01
		6.970E-02	1.250E+00	2.028E-01			**8.826E-02**	1.222E+00	6.990E-01

+,− and = indicate that DoD approach performs significantly better, significantly bad, and equivalent to the corresponding EMO algorithm.

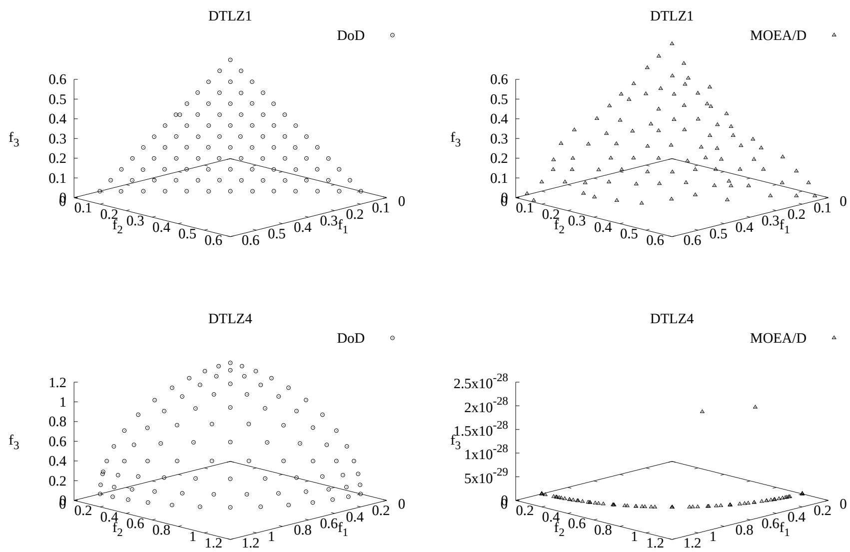

Fig. 3. Non-dominated solutions obtained using the DoD approach and MOEA/D for DTLZ1 and DTLZ4 problems.

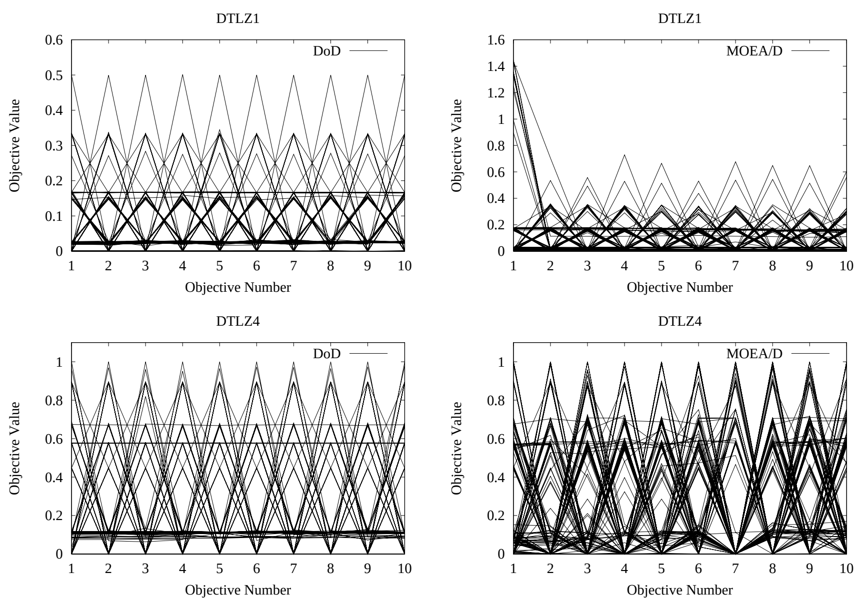

Fig. 4. Parallel coordinates of non-dominated front obtained from the DoD approach and MOEA/D for DTLZ1 and DTLZ4 problems.

Table 3. Best, median and worst HV values obtained by DoD approach and other algorithms on DTLZ and WFG instances with different number of objectives. Best performances are highlighted in bold face with gray background. NSGA-III results are obtained from [10], and MOEA/D results are obtained using [5].

	M	NSGA-III	MOEA/D	DoD
DTLZ1	3	9.73519E-01	9.66870E-01	**9.73627E-01**
		9.73217E-01	9.57697E-01$^{=}$	**9.73509E-01**
		9.71931E-01	6.14190E-01	**9.73198E-01**
	5	9.98971E-01	9.98629E-01	**9.98981E-01**
		9.98963E-01	9.98330E-01$^{=}$	**9.98971E-01**
		9.98673E-01	9.97441E-01	**9.98942E-01**
	8	**9.99975E-01**	9.99645E-01	9.99974E-01
		9.93549E-01	9.99370E-01$^{=}$	**9.99970E-01**
		9.66432E-01	9.98375E-01	**9.99962E-01**
	10	9.99991E-01	9.99934E-01	**9.99998E-01**
		9.99985E-01	9.99875E-01$^{=}$	**9.99997E-01**
		9.99969E-01	9.99672E-01	**9.99994E-01**
DTLZ2	3	9.26626E-01	9.25292E-01	**9.26666E-01**
		9.26536E-01	9.24412E-01$^{=}$	**9.26632E-01**
		9.26395E-01	9.22765E-01	**9.26497E-01**
	5	9.90459E-01	9.90426E-01	**9.90493E-01**
		9.90400E-01	9.90271E-01$^{=}$	**9.90460E-01**
		9.90328E-01	9.90013E-01	**9.90431E-01**
	8	9.99320E-01	9.99323E-01	**9.99335E-01**
		9.78936E-01	9.99315E-01$^{=}$	**9.99327E-01**
		9.19680E-01	9.99298E-01	**9.99319E-01**
	10	9.99918E-01	**9.99919E-01**	9.99919E-01
		9.99916E-01	9.99876E-01$^{=}$	**9.99918E-01**
		9.99915E-01	9.99868E-01	**9.99916E-01**

	M	NSGA-III	MOEA/D	DoD
DTLZ3	3	9.26480E-01	3.41019E-03	**9.26669E-01**
		9.25805E-01	6.94952E-03$^{=}$	**9.26328E-01**
		9.24234E-01	1.89791E-01	**9.25428E-01**
	5	9.90453E-01	9.90009E-01	**9.90565E-01**
		9.90344E-01	9.76349E-01$^{=}$	**9.90446E-01**
		9.89510E-01	9.43850E-01	**9.90256E-01**
	8	9.99300E-01	9.99122E-01	**9.99308E-01**
		9.24059E-01	7.76470E-01$^{=}$	**9.99253E-01**
		9.04182E-01	5.03871E-01	6.43785E-02
	10	**9.99921E-01**	9.99865E-01	9.99920E-01
		9.99918E-01	9.99144E-01$^{=}$	9.99916E-01
		9.99910E-01	5.10243E-01	9.99908E-01
DTLZ4	3	9.26659E-01	9.26587E-01	**9.26774E-01**
		9.26705E-01	8.00983E-01$^{=}$	**9.26728E-01**
		7.99572E-01	5.00000E-01	**9.26716E-01**
	5	**9.91102E-01**	9.90611E-01	9.90586E-01
		9.90413E-01	9.90564E-01$^{=}$	**9.90575E-01**
		9.90156E-01	9.12068E-01	**9.90570E-01**
	8	9.99363E-01	**9.99383E-01**	9.99364E-01
		9.99361E-01	9.99131E-01$^{=}$	**9.99364E-01**
		9.94784E-01	9.86416E-01	**9.99363E-01**
	10	9.99915E-01	**9.99926E-01**	9.99924E-01
		9.99910E-01	9.99917E-01$^{=}$	**9.99923E-01**
		9.99827E-01	9.99430E-01	**9.99923E-01**

	M	MOEA/D	DoD
WFG6	3	8.90380E-01	**9.11580E-01**
		8.83410E-01$^{=}$	**9.04850E-01**
		8.10510E-01	**8.98480E-01**
	5	9.13140E-01	**9.68380E-01**
		8.16580E-01$^{=}$	**9.61910E-01**
		7.14410E-01	**9.54090E-01**
	8	7.08280E-01	**9.72600E-01**
		6.55160E-01$^{=}$	**9.63970E-01**
		6.08160E-01	**9.52200E-01**
	10	7.69000E-01	**9.75300E-01**
		6.71860E-01$^{=}$	**9.66450E-01**
		6.18670E-01	**9.58170E-01**

	M	MOEA/D	DoD
WFG7	3	9.08050E-01	**9.25330E-01**
		8.96660E-01$^{=}$	**9.24910E-01**
		8.67760E-01	**9.24030E-01**
	5	9.33350E-01	**9.87230E-01**
		9.03530E-01$^{=}$	**9.86740E-01**
		8.60340E-01	**9.85380E-01**
	8	7.38580E-01	**9.95360E-01**
		6.73050E-01$^{=}$	**9.93610E-01**
		6.22420E-01	**9.91490E-01**
	10	8.03100E-01	**9.96890E-01**
		7.81430E-01$^{=}$	**9.96310E-01**
		6.85380E-01	**9.95490E-01**

+,− and = indicate that DoD approach performs significantly better, significantly bad, and equivalent to the corresponding EMO algorithm.

References

1. Barakat, N., Sharma, D.: Evolutionary multi-objective optimization for bulldozer and its blade in soil cutting. International Journal of Management Science and Engineering Management **0**(0), 1–11 (2018). https://doi.org/10.1080/17509653.2018.1500953
2. Das, I., Dennis, J.E.: Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems. SIAM Journal on Optimization **8**(3), 631–657 (1998). https://doi.org/10.1137/S1052623496307510
3. Deb, K., Jain, H.: An evolutionary many-objective optimization algorithm using reference-point-based nondominated sorting approach, part i: Solving problems

with box constraints. IEEE Transactions on Evolutionary Computation **18**(4), 577–601 (Aug 2014). https://doi.org/10.1109/TEVC.2013.2281535
4. Deb, K., Thiele, L., Laumanns, M., Zitzler, E.: Scalable test problems for evolutionary multiobjective optimization. In: Abraham, A., Jain, L., Goldberg, R. (eds.) Evolutionary multiobjective optimization: theoritical advances and applications, pp. 105–145. Springer-Verlag, London (2005)
5. Durillo, J.J., Nebro, A.J.: jmetal: A java framework for multi-objective optimization. Advances in Engineering Software **42**, 760–771 (2011). https://doi.org/DOI: 10.1016/j.advengsoft.2011.05.014, http://www.sciencedirect.com/science/article/pii/S0965997811001219
6. Huband, S., Hingston, P., Barone, L., While, L.: A review of multiobjective test problems and a scalable test problem toolkit. IEEE Transactions on Evolutionary Computation **10**(5), 477–506 (Oct 2006). https://doi.org/10.1109/TEVC.2005.861417
7. Ibrahim, A., Rahnamayan, S., Martin, M.V., Deb, K.: Elitensga-iii: An improved evolutionary many-objective optimization algorithm. In: 2016 IEEE Congress on Evolutionary Computation (CEC). pp. 973–982 (July 2016). https://doi.org/10.1109/CEC.2016.7743895
8. Jiang, S., Yang, S.: A strength pareto evolutionary algorithm based on reference direction for multiobjective and many-objective optimization. IEEE Transactions on Evolutionary Computation **21**(3), 329–346 (June 2017). https://doi.org/10.1109/TEVC.2016.2592479
9. Jiang, S., Yang, S.: Convergence versus diversity in multiobjective optimization. In: Handl, J., Hart, E., Lewis, P.R., López-Ibáñez, M., Ochoa, G., Paechter, B. (eds.) Parallel Problem Solving from Nature – PPSN XIV. pp. 984–993. Springer International Publishing, Cham (2016)
10. Li, K., Deb, K., Zhang, Q., Kwong, S.: An evolutionary many-objective optimization algorithm based on dominance and decomposition. IEEE Transactions on Evolutionary Computation **19**(5), 694–716 (Oct 2015). https://doi.org/10.1109/TEVC.2014.2373386
11. Liao, X., Li, Q., Zhang, W., Yang, X.: Multiobjective optimization for crash safety design of vehicle using stepwise regression model. Structural Multidisciplinary Optimization **35**(6), 261–569 (2008)
12. Purshouse, R.C., Fleming, P.J.: On the evolutionary optimization of many conflicting objectives. IEEE Transactions on Evolutionary Computation **11**(6), 770–784 (Dec 2007). https://doi.org/10.1109/TEVC.2007.910138
13. While, L., Bradstreet, L., Barone, L.: A fast way of calculating exact hypervolumes. IEEE Transactions on Evolutionary Computation **16**(1), 86–95 (Feb 2012). https://doi.org/10.1109/TEVC.2010.2077298
14. Wilcoxon, F.: Individual comparisons by ranking methods. Biometrics Bulletin **1**(6), 80–83 (December 1945)
15. Yuan, Y., Xu, H., Wang, B., Yao, X.: A new dominance relation-based evolutionary algorithm for many-objective optimization. IEEE Transactions on Evolutionary Computation **20**(1), 16–37 (Feb 2016). https://doi.org/10.1109/TEVC.2015.2420112
16. Zhang, Q., Li, H.: Moea/d: A multiobjective evolutionary algorithm based on decomposition. IEEE Transactions on Evolutionary Computation **11**(6), 712–731 (Dec 2007). https://doi.org/10.1109/TEVC.2007.892759