

# MULTIVARIABLE CALCULUS

**This part of MA101 course will enable you to**

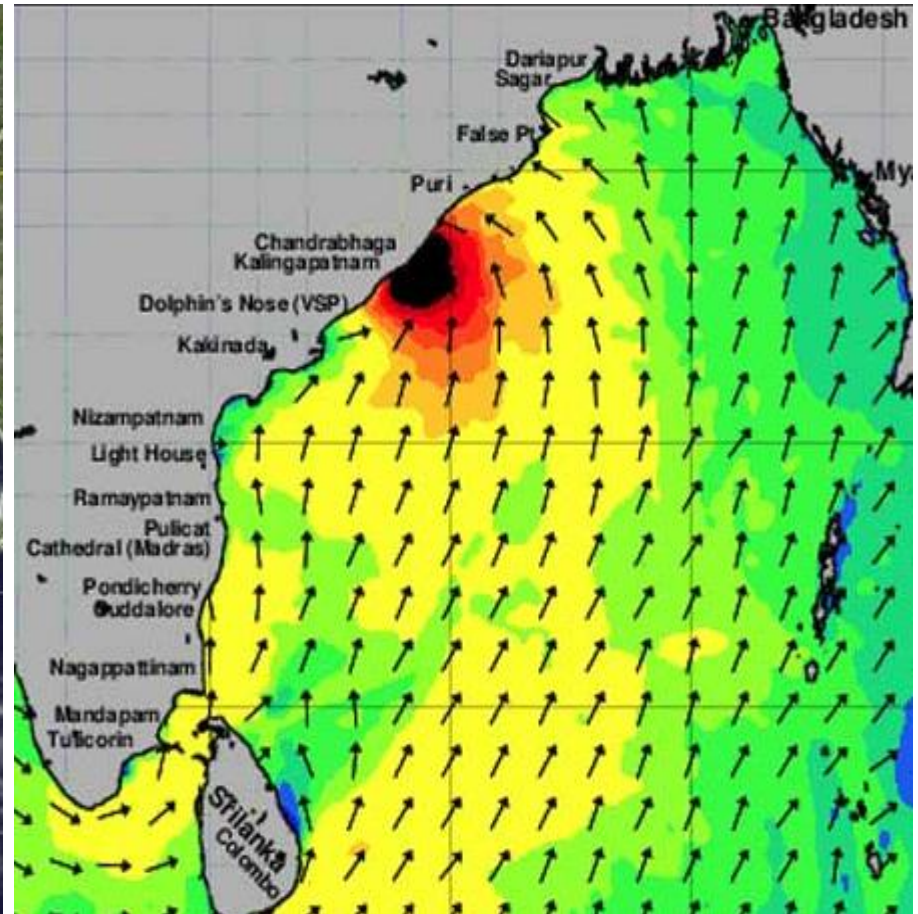
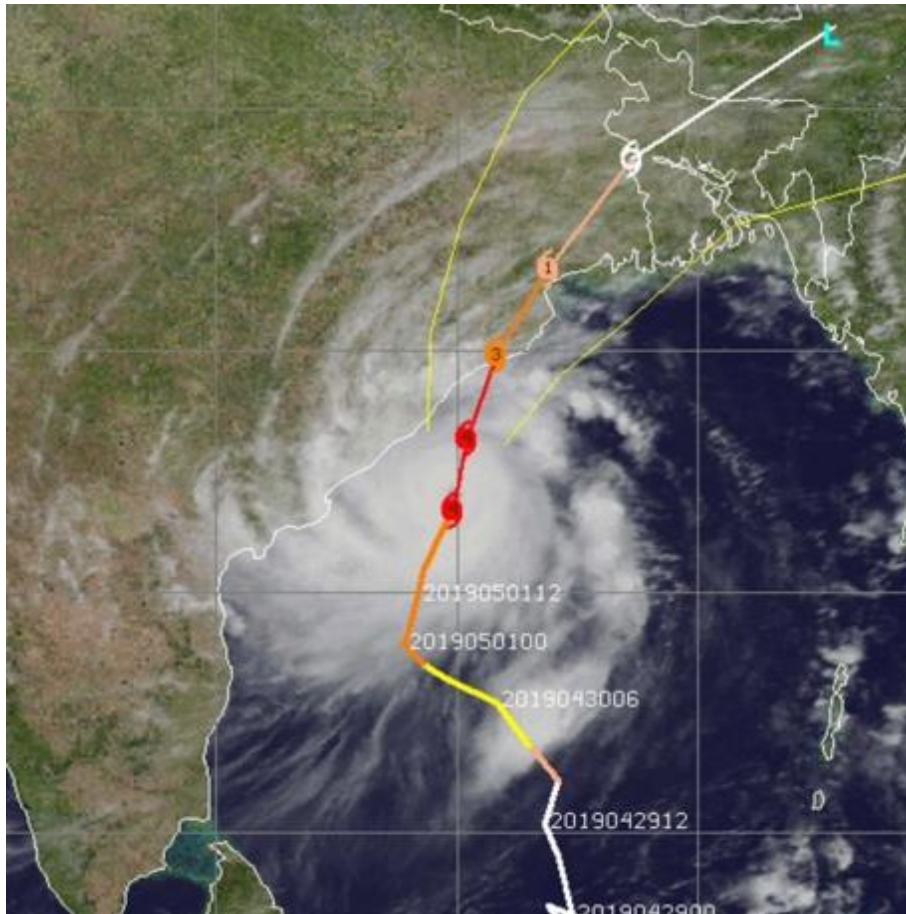
- ✚ Relate Mathematics to the real world.
- ✚ Use Mathematics as a tool in Engineering.

**Emphasis will be more on**

- The physical interpretation of the mathematical expressions.
- **HAVING FUN WITH CALCULUS.**

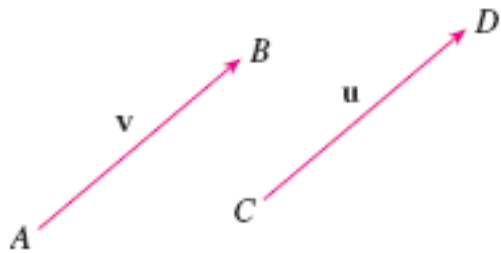
# Review of Vectors

WHAT IS A VECTOR?



Wind velocity is a vector because it has both magnitude and direction.

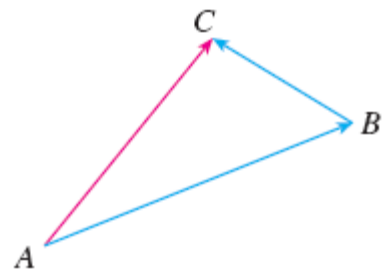
January 2022



**FIGURE 1**  
Equivalent vectors

Displacement Vector, Initial point and terminal point.

**Combining Vectors**



**FIGURE 2**

**Definition of Vector Addition** If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .

# Illustration of vector addition

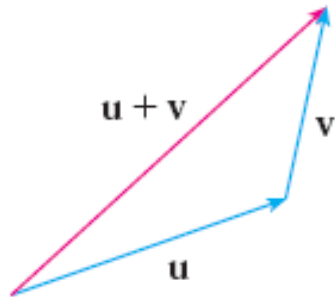


FIGURE 3 The Triangle Law

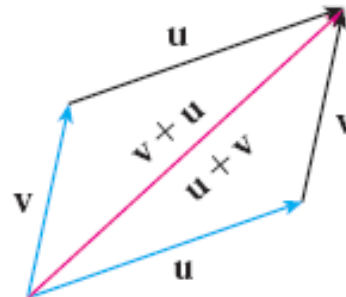


FIGURE 4 The Parallelogram Law

**Definition of Scalar Multiplication** If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .

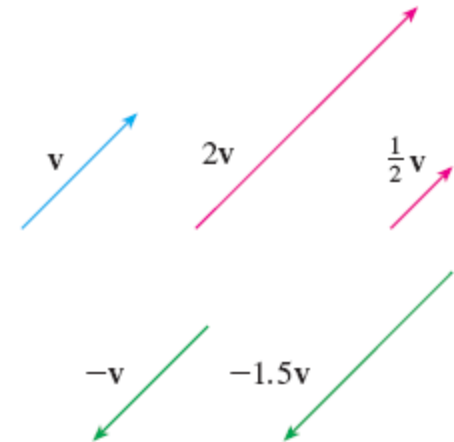
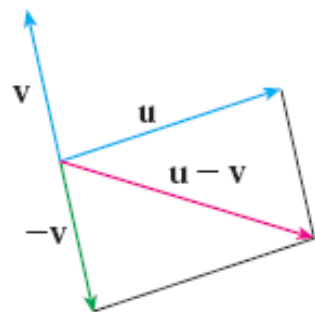
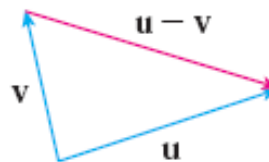


FIGURE 7  
Scalar multiples of  $\mathbf{v}$

# Illustration of vector subtraction (difference of vectors)



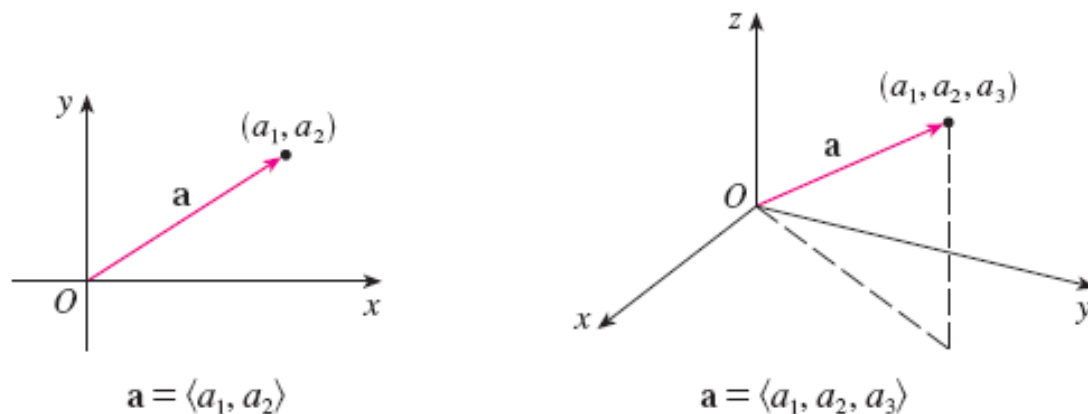
(a)



(b)

## Components

**(1) DEFINITION** A two-dimensional vector is an ordered pair  $\mathbf{a} = \langle a_1, a_2 \rangle$  of real numbers. A three-dimensional vector is an ordered triple  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  of real numbers. The numbers  $a_1$ ,  $a_2$ , and  $a_3$  are called the **components** of  $\mathbf{a}$ .



We denote by  $V_2$  the set of all two-dimensional vectors and by  $V_3$  the set of all three-dimensional vectors. More generally, we will later need to consider the set  $V_n$  of all  $n$ -dimensional vectors. An  $n$ -dimensional vector is an ordered  $n$ -tuple:

$$\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle$$

where  $a_1, a_2, \dots, a_n$  are real numbers that are called the components of  $\mathbf{a}$ .

# Magnitude Or Length

The **magnitude** or **length** of the vector  $\mathbf{v}$  is the length of any of its representations and is denoted by the symbol  $|\mathbf{v}|$  or  $\|\mathbf{v}\|$ . By using the distance formula to compute the length of a segment  $OP$ , we obtain the following formulas.

The length of the two-dimensional vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

# Vector Addition And Scalar Multiplication

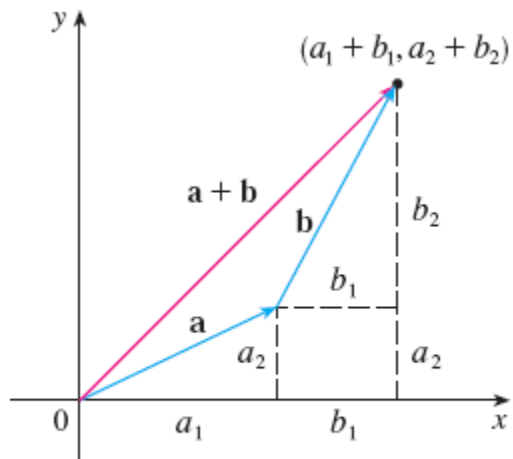


FIGURE 14

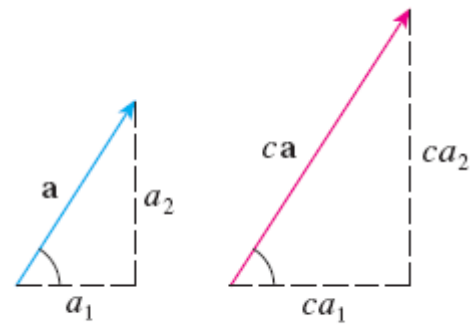
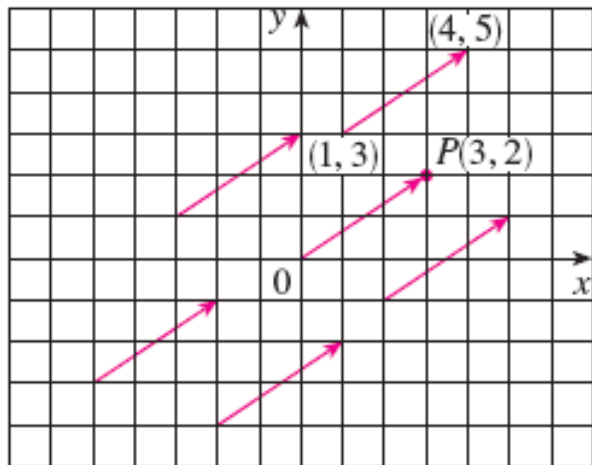


FIGURE 15

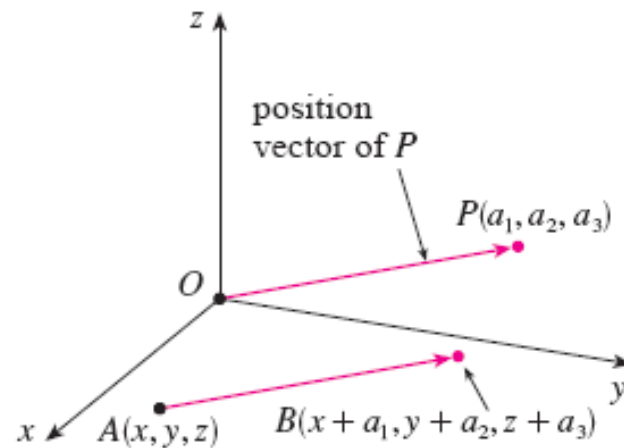
Addition and scalar multiplication in  $V_n$  is defined in terms of components just as in the case of  $n = 2$  and 3.



# Position Vectors:



**FIGURE 12**  
Representations of the vector  $\mathbf{a} = \langle 3, 2 \rangle$



**FIGURE 13**  
Representations of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

**1** Given the points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\vec{AB}$  is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Properties of Vectors** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars, then

1.  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

2.  $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3.  $\mathbf{a} + \mathbf{0} = \mathbf{a}$

4.  $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5.  $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

6.  $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7.  $(cd)\mathbf{a} = c(d\mathbf{a})$

8.  $1\mathbf{a} = \mathbf{a}$

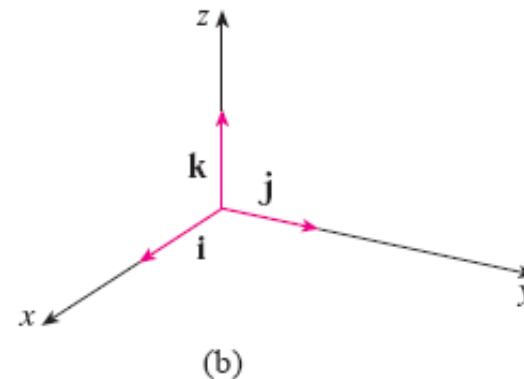
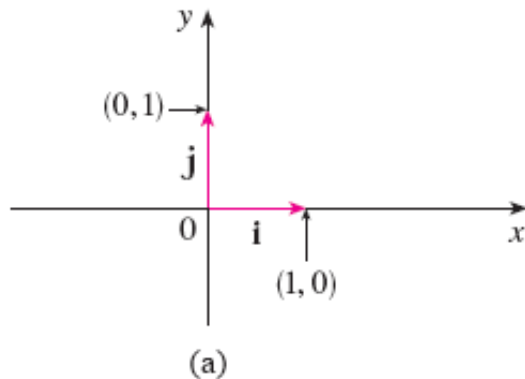
## Standard Basis Vectors

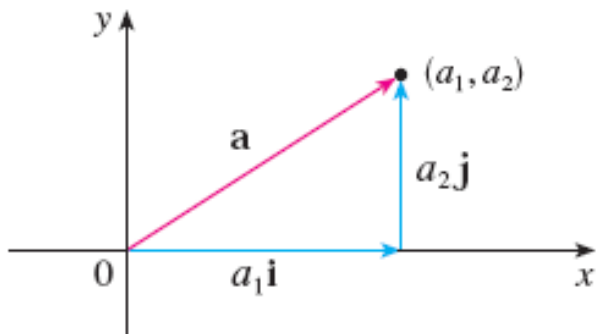
Three vectors in  $V_3$  play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

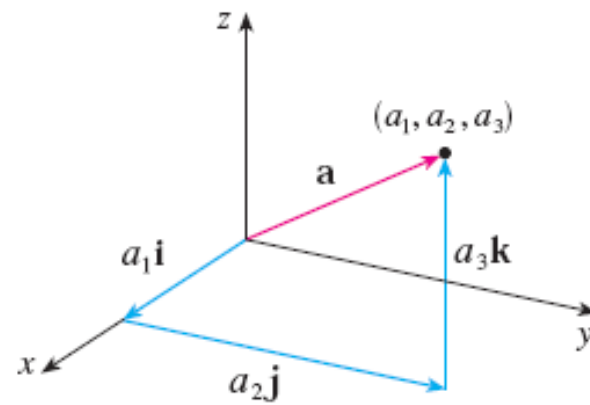
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$





(a)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b)  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ , then we can write

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle \end{aligned}$$

**2**

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

# Unit Vector

A **unit vector** is a vector whose length is 1. For instance,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are all unit vectors. In general, if  $\mathbf{a} \neq \mathbf{0}$ , then the unit vector that has the same direction as  $\mathbf{a}$  is

4

$$\mathbf{u} = \frac{1}{|\mathbf{a}|} \mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

## Applications

**EXAMPLE 7** A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces)  $T_1$  and  $T_2$  in both wires and the magnitudes of the tensions.

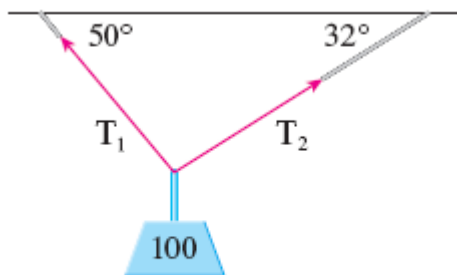


FIGURE 19

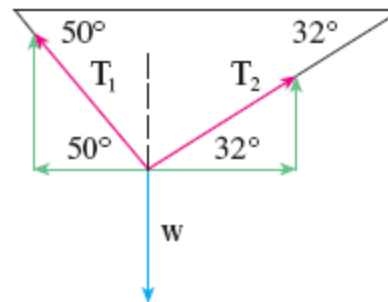


FIGURE 20

## The Dot Product

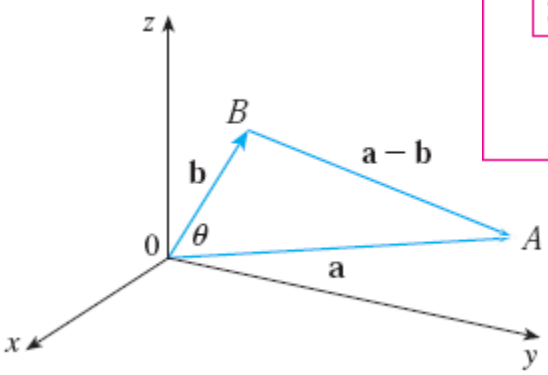
**1 Definition** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the number  $\mathbf{a} \cdot \mathbf{b}$  given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

**2 Properties of the Dot Product** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_3$  and  $c$  is a scalar, then

1.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4.  $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5.  $\mathbf{0} \cdot \mathbf{a} = 0$

# Geometric Interpretation of the dot product:



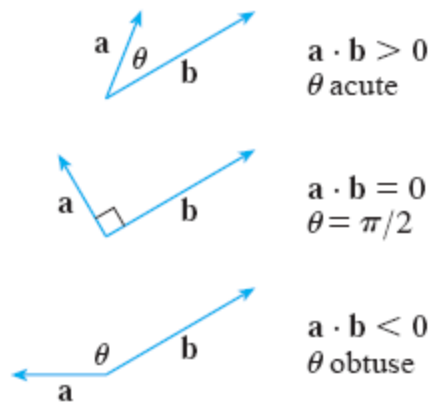
**3 Theorem** If  $\theta$  is the angle between the vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

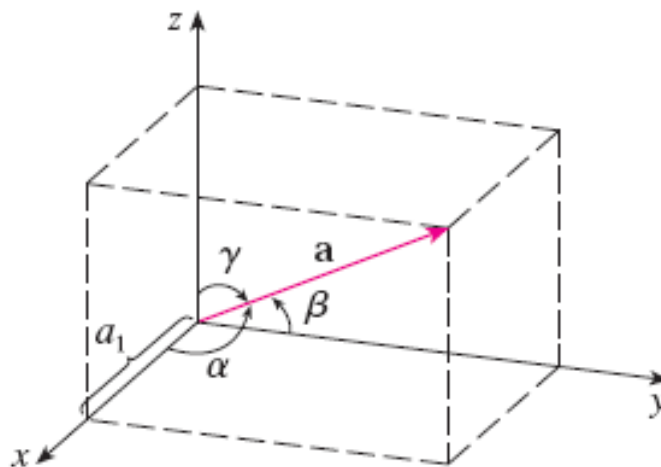
**6 Corollary** If  $\theta$  is the angle between the nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

**7** Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$ .



## Direction Angles and Direction Cosines

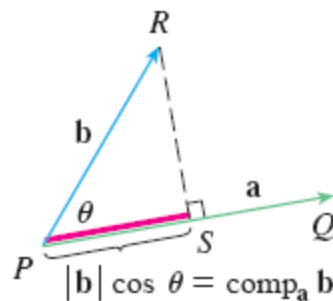


The **direction angles** of a nonzero vector  $\mathbf{a}$  are the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  (in the interval  $[0, \pi]$ ) that  $\mathbf{a}$  makes with the positive  $x$ -,  $y$ -, and  $z$ -axes. (See Figure 3.)

# Projection: Scalar Projection and Vector Projection



Figure 4 shows representations  $P\vec{Q}$  and  $P\vec{R}$  of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  with the same initial point  $P$ . If  $S$  is the foot of the perpendicular from  $R$  to the line containing  $\overrightarrow{PQ}$ , then the vector with representation  $\overrightarrow{PS}$  is called the **vector projection** of  $\mathbf{b}$  onto  $\mathbf{a}$  and is denoted by  $\text{proj}_a \mathbf{b}$ . (You can think of it as a shadow of  $\mathbf{b}$ ).



The **scalar projection** of  $\mathbf{b}$  onto  $\mathbf{a}$  (also called the **component of  $\mathbf{b}$  along  $\mathbf{a}$** ) is defined to be the signed magnitude of the vector projection, which is the number  $|\mathbf{b}| \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . (See Figure 5.) This is denoted by  $\text{comp}_a \mathbf{b}$ .



**11 Theorem** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors and  $c$  is a scalar, then

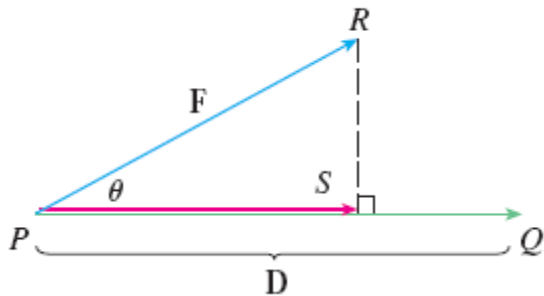
1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2.  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4.  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5.  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$



Scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

## Applications



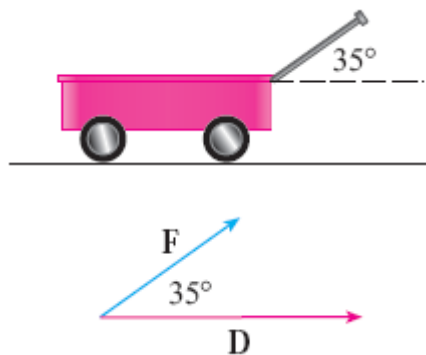
Is work done  $W = Fd$ ?

One use of projections occurs in physics in calculating work. In Section 5.4 we defined the work done by a constant force  $F$  in moving an object through a distance  $d$  as  $W = Fd$ , but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector  $\mathbf{F} = \overrightarrow{PR}$  pointing in some other direction, as in Figure 6. If the force moves the object from  $P$  to  $Q$ , then the **displacement vector** is  $\mathbf{D} = \overrightarrow{PQ}$ . The **work** done by this force is defined to be the product of the component of the force along  $\mathbf{D}$  and the distance moved:

$$W = (|\mathbf{F}| \cos \theta) |\mathbf{D}|$$

But then, from Theorem 3, we have

$$\boxed{12} \quad W = |\mathbf{F}| |\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$



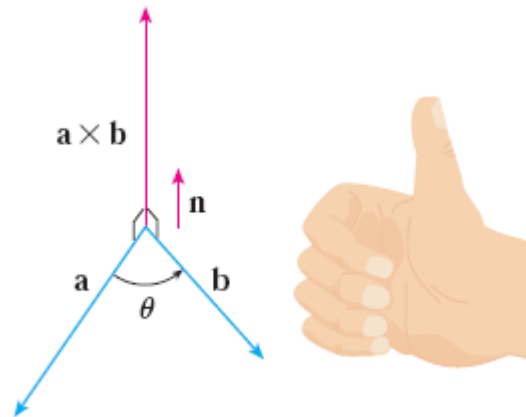
**EXAMPLE 7** A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of  $35^\circ$  above the horizontal. Find the work done by the force.

# The Vector Cross Product

**4 Definition** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

**8 Theorem** The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ .

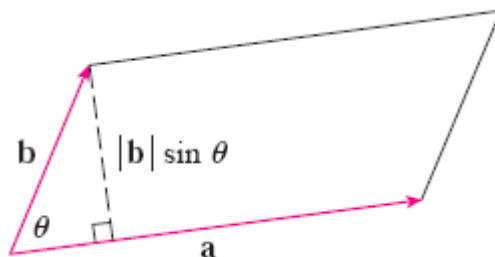


**9 Theorem** If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  (so  $0 \leq \theta \leq \pi$ ), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

**10 Corollary** Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if

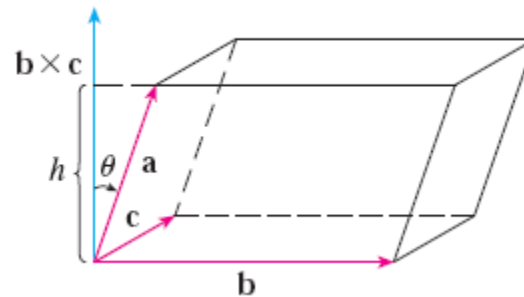
$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$



The length of the cross product  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ .

# Scalar triple product

The product  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  that occurs in Property 5 is called the **scalar triple product** of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

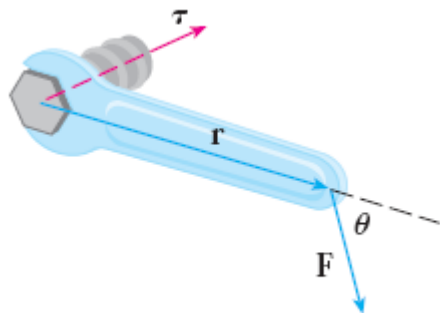


**14** The volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

# Applications

The idea of a cross product occurs often in physics. In particular, we consider a force  $\mathbf{F}$  acting on a rigid body at a point given by a position vector  $\mathbf{r}$ . (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque**  $\boldsymbol{\tau}$  (relative to the origin) is defined to be the cross product of the position and force vectors

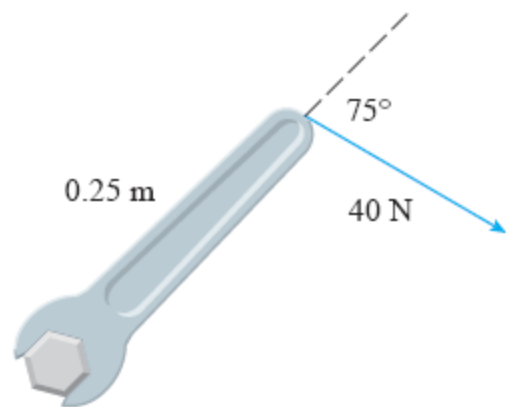


and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 6, the magnitude of the torque vector is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta$$

where  $\theta$  is the angle between the position and force vectors. Observe that the only component of  $\mathbf{F}$  that can cause a rotation is the one perpendicular to  $\mathbf{r}$ , that is,  $|\mathbf{F}| \sin \theta$ . The magnitude of the torque is equal to the area of the parallelogram determined by  $\mathbf{r}$  and  $\mathbf{F}$ .



**EXAMPLE 6** A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

FIGURE 5