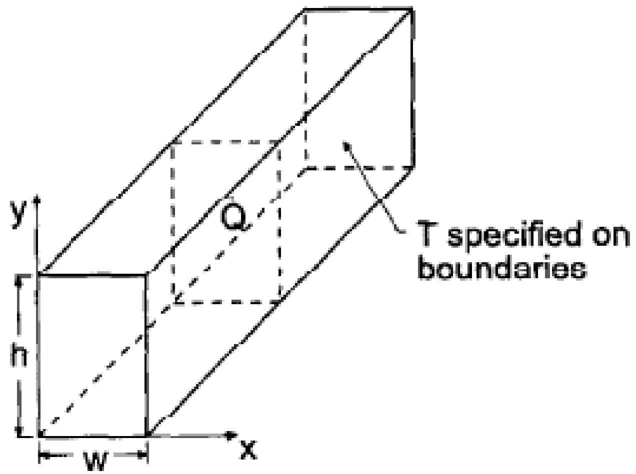
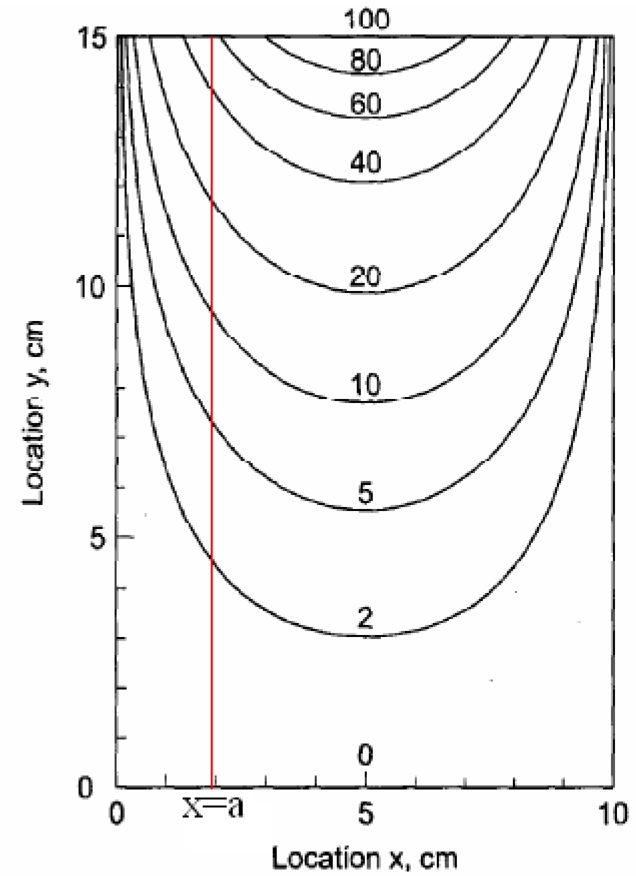
THE HEAT CONDUCTION PROBLEM





Partial Derivatives

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Find the first partial derivatives of the function.

13. $f(x, y) = 3x - 2y^4$

14. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

15. $z = xe^{3y}$

16. $z = y \ln x$

17. $f(x, y) = \dfrac{x - y}{x + y}$

18. $f(x, y) = x^y$

19. $w = \sin \alpha \cos \beta$

20. $f(s, t) = st^2/(s^2 + t^2)$

21. $f(r, s) = r \ln(r^2 + s^2)$

22. $f(x, t) = \arctan(x\sqrt{t})$

23. $u = te^{w/t}$

24. $f(x, y) = \int_y^x \cos(t^2)\, dt$

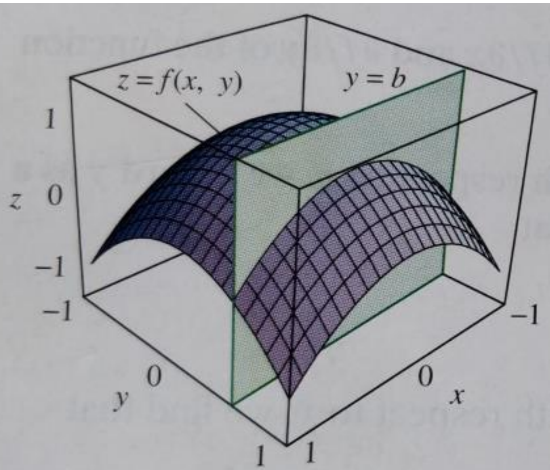


FIGURE 13.4.1 A vertical plane parallel to the xz-plane intersects the surface $z = f(x, y)$ in an x-curve.

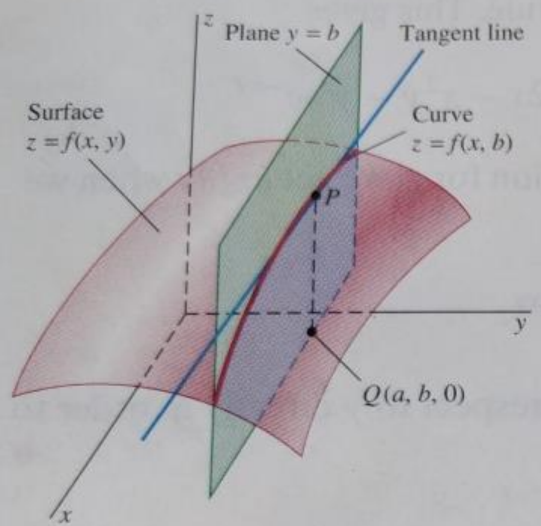


FIGURE 13.4.2 An x-curve and its tangent line at P.

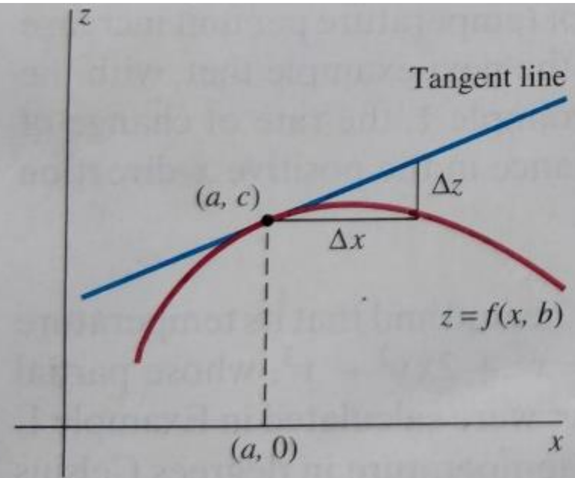


FIGURE 13.4.3 Projection into the xz-plane of the x-curve through $P(a, b, c)$ and its tangent line.

EXAMPLE 5 Suppose that the graph $z = 5xy\exp(-x^2 - 2y^2)$ in Fig. 13.4.7 represents a terrain featuring two peaks (hills, actually) and two pits. With all distances measured in miles, z is the altitude above the point (x, y) at sea level in the xy-plane. For instance, the height of the pictured point P is $z(-1, -1) = 5e^{-3} \approx 0.2489$ (mi), about 1314 ft above sea level. We ask at what rate we climb if, starting at the point $P(-1, -1, 0.2489)$, we head either due east (the positive x-direction) or due north (the positive y-direction). If we calculate the two partial derivatives of $z(x, y)$, we get

$$\frac{\partial z}{\partial x} = 5y(1 - 2x^2)\exp(-x^2 - 2y^2) \quad \text{and} \quad \frac{\partial z}{\partial y} = 5x(1 - 4y^2)\exp(-x^2 - 2y^2).$$

(You should check this.) Substituting $x = y = -1$ now gives

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,-1)} = 5e^{-3} \approx 0.2489 \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(-1,-1)} = 15e^{-3} \approx 0.7468.$$

The units here are in miles per mile—that is, the ratio of rise to run in vertical miles per horizontal mile. So if we head east, we start climbing at an angle of

$$\alpha = \tan^{-1}(0.2489) \approx 0.2440 \quad \text{(rad)},$$

about 13.98°. (See Fig. 13.4.8.) But if we head north, then we start climbing at an angle of

$$\beta = \tan^{-1}(0.7468) \approx 0.6415 \quad \text{(rad)},$$

approximately 36.75°. (See Fig. 13.4.9.) Do these results appear to be consistent with Fig. 13.4.7? ◆

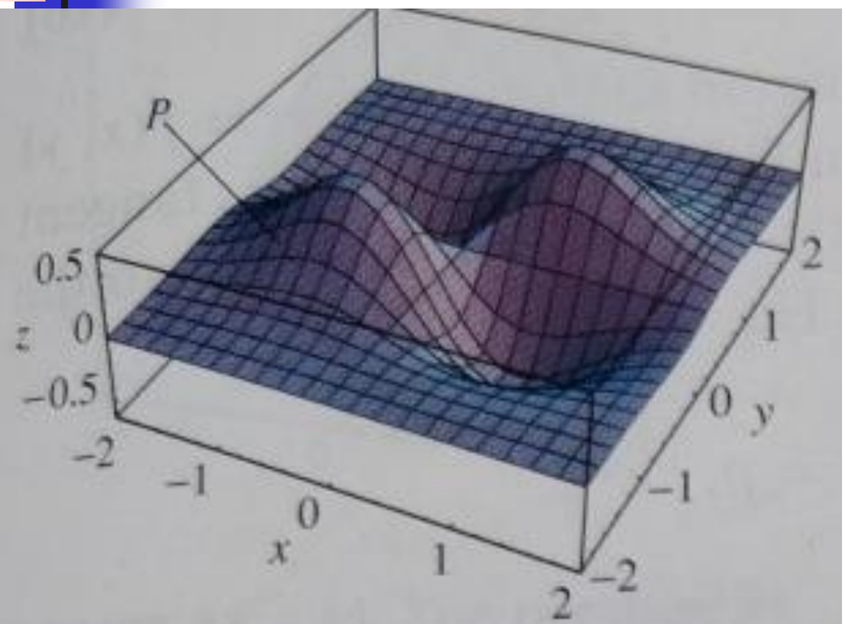


FIGURE 13.4.7 The graph $z = 5xy\exp(-x^2 - 2y^2)$.

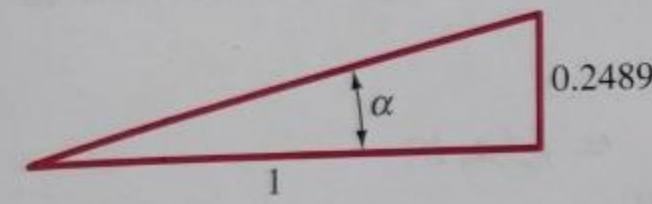


FIGURE 13.4.8 The angle of climb in the *x*-direction.

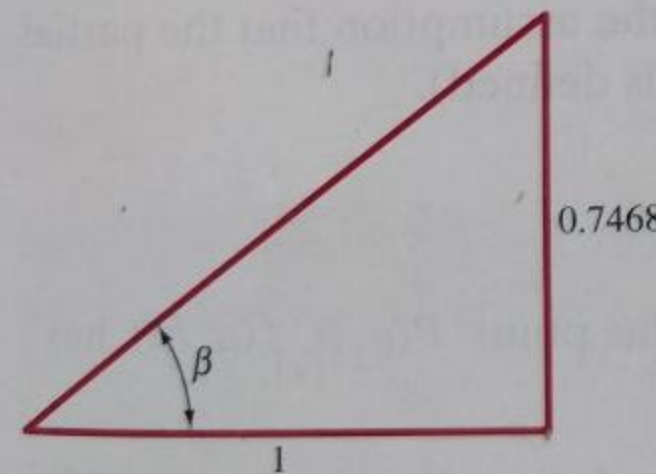


FIGURE 13.4.9 The angle of climb in the *y*-direction.

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \partial w / \partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\mathrm{x} + h\hat{\mathrm{e}}_{\mathbf{i}}) - f(\mathrm{x})}{h}$$

Higher Derivatives

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

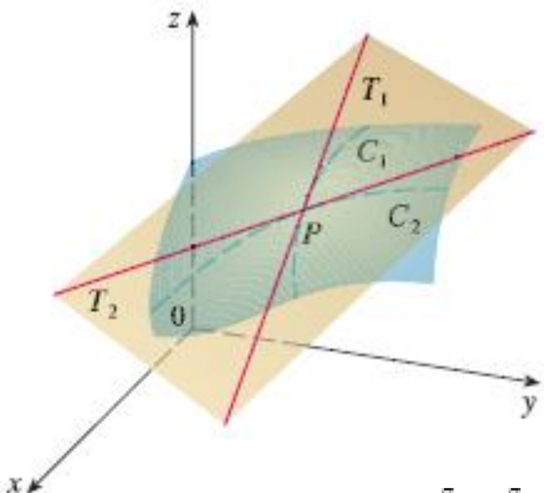
$$f_{xy}(a, b) = f_{yx}(a, b)$$

Tangent Planes

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

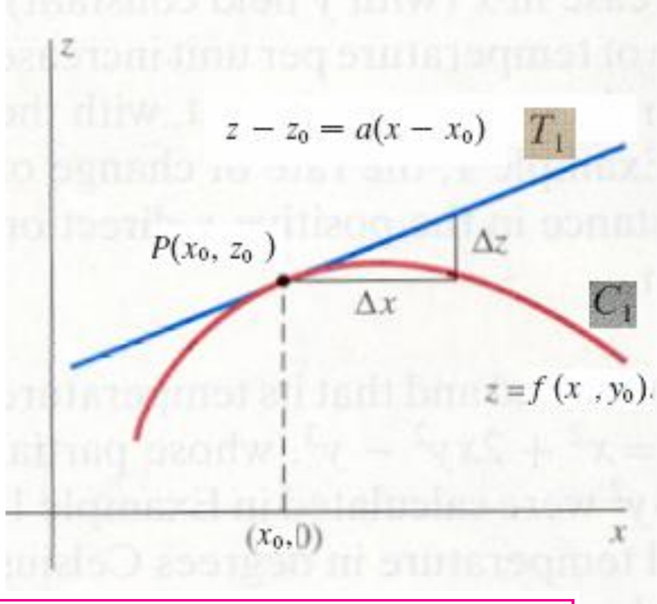
By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form



1 $$z - z_0 = a(x - x_0) + b(y - y_0)$$

$$z - z_0 = a(x - x_0) \qquad \text{where } \ y = y_0$$

$$a = f_x(x_0, y_0).$$



2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Normal

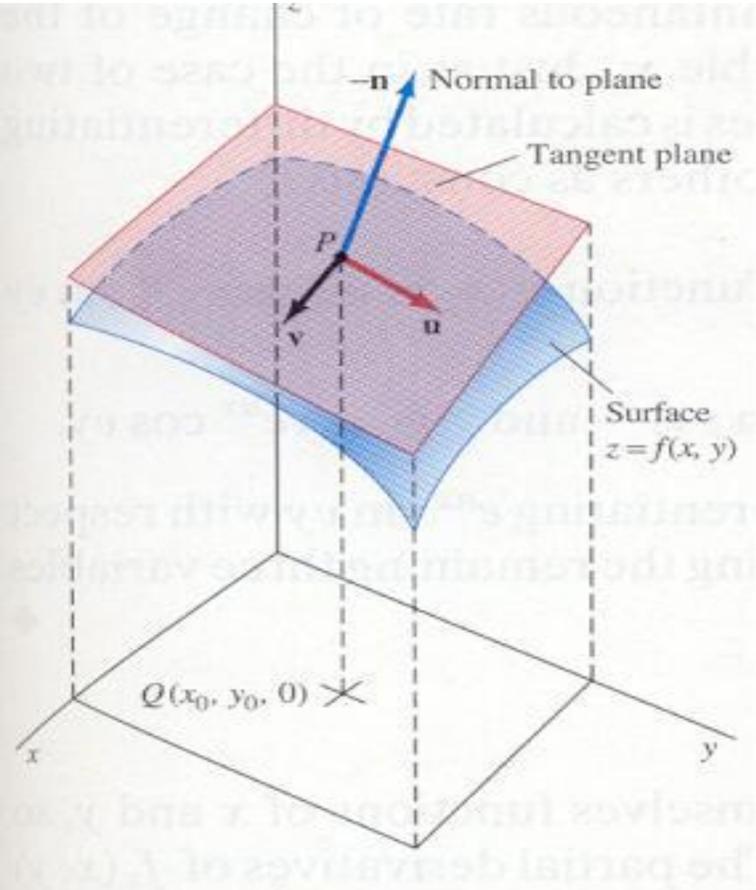


FIGURE 13.4.10 The surface $z = f(x, y)$, its tangent plane at $P(x_0, y_0, z_0)$, and the vector $-\mathbf{n}$ normal to both at P.

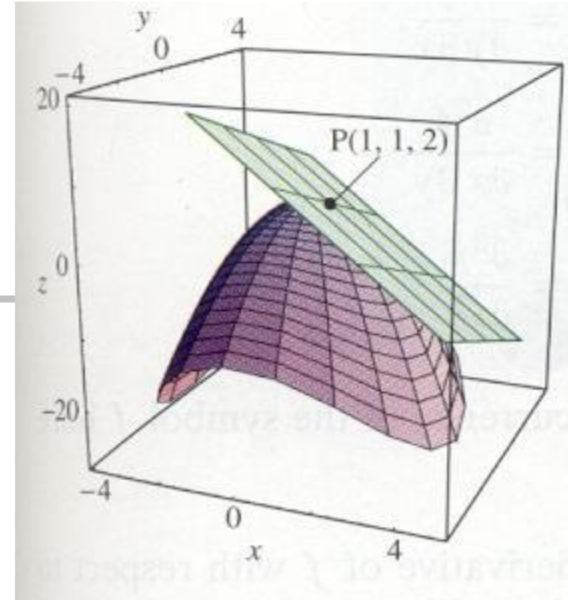


FIGURE 13.4.11 The paraboloid and tangent plane of Example 6.

Example 6: Write an equation of the tangent plane to the Paraboloid $z = 5 - 2x^2 - y^2$ at P(1, 1, 2)