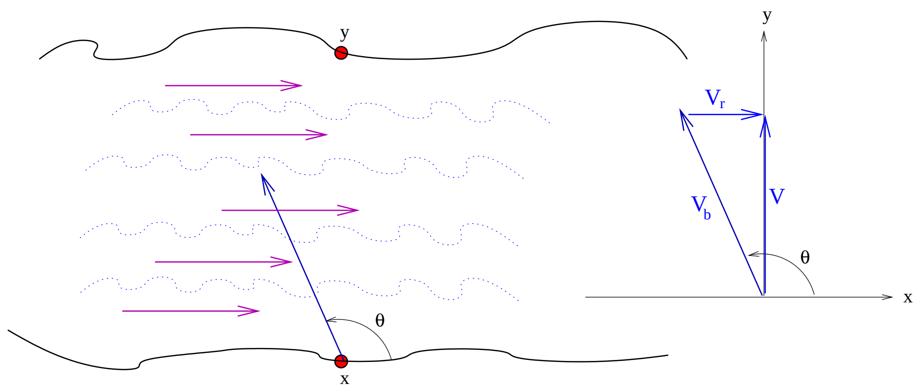
DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic year 2021 - 2022
MA 101 Mathematics I
Problem Sheet 1: Revision of vectors, equations of lines and planes, vector differentiation, limits and continuities of functions of several variables.

Instructors: Dr. J. C. Kalita and Dr. S. Upadhyay

1. 

A man wants to paddle his boat across a river from point X to the point Y on the opposite shore directly across from X. If he can paddle the boat at the rate of 5 kilometers per hour and the current in the river is 3 kilometers per hour, in what direction θ should he steer his boat in order to go straight across the river? Also what is his resultant speed across the river?

2. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

3. (a) Find a point at which the given lines intersect:
$$\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\,\langle -1, 1, 2\rangle$$
$$\mathbf{r}_2(s) = \langle 2, 0, 2\rangle + s\,\langle -1, 1, 0\rangle\,.$$
   (b) Find an equation of the plane that contains these lines.

4. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

5. Find a point on the curve $\mathbf{r} = 4\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j} + 3t\mathbf{k}$ at a distance 10π units along the curve from the origin in the direction of increasing arc length.

6. Reparametrize the curve
$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$
with respect to the arc length measured from the point $(1, 0)$ in the direction of increasing t. Express the parametrization in its simplest form. What can you conclude about the curve?

7. At what point does the curve $y = e^x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

8. Find the unit tangent vector, unit normal vector and the binormal vector at the given point.

   (a) $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t \right\rangle, \ \left(1, \frac{2}{3}, 1\right)$

   (b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle, \ (1, 0, 0)$.

9. The helix $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

10. (a) A particle moves with constant speed along a curve in space. Show that its velocity and acceleration vectors are always perpendicular.

    (b) Let $\mathbf{r}(t) = (2t^3 + 3)\mathbf{i} + (\ln t)\mathbf{j} + 3\mathbf{k}$ be the position vector of a moving particle at time $t > 0$. Find the time(s) at which velocity and acceleration vectors are perpendicular.

11. For each of the following find the limit if it exists, or show that the limit does not exist.

    (a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

    (b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

    (c) $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$

    (d) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

    (e) $\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2 + y^2)}{x^2 + y^2}$

    (f) $\lim_{(x,y)\to(4,\pi)} x^2 \sin\left(\frac{y}{x}\right)$

    (g) $\lim_{(x,y)\to(0,1)} f(x, y)$, where

$$f(x, y) = \begin{cases} \frac{x+y-1}{\sqrt{x}-\sqrt{1-y}} & \text{if} \quad x + y \neq 1 \\ 0 & \text{if} \quad x + y = 1. \end{cases}$$

12. Examine whether

$$f(x, y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

is a continuous function.

Extra Questions

1. Show that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ represents the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

2. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k_1} = \frac{\mathbf{v_2} \times \mathbf{v_3}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}, \quad \mathbf{k_2} = \frac{\mathbf{v_3} \times \mathbf{v_1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})} \quad \text{and} \quad \mathbf{k_3} = \frac{\mathbf{v_1} \times \mathbf{v_2}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}.$$

   Show that

   (a) $\mathbf{k}_i$ is perpendicular to to $\mathbf{v}_j$ if $i \neq j$.

   (b) $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

   (c) $\mathbf{k_1} \cdot (\mathbf{k_2} \times \mathbf{k_3}) = \dfrac{\mathbf{1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}$.

3. Given the vectors $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ and $\mathbf{b} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$, verify that

$$|\mathbf{a} \times \mathbf{b}|^{\mathbf{2}} = |\mathbf{a}|^{\mathbf{2}}|\mathbf{b}|^{\mathbf{2}} - (\mathbf{a} \cdot \mathbf{b})^{\mathbf{2}},$$

   by computing each side in terms of the components of $\mathbf{a}$ and $\mathbf{b}$.

4. If $\mathbf{u}(t) = \mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$ and $\mathbf{v}(t) = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$,

   (a) Find $D_t[\mathbf{u}(t) \cdot \mathbf{v}(t)]$

   (b) Find $D_t[\mathbf{u}(t) \times \mathbf{v}(t)]$

   (c) Verify that $\lim\limits_{t \to \pi}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim\limits_{t \to \pi}[\mathbf{u}(t)] \cdot \lim\limits_{t \to \pi}[\mathbf{v}(t)]$ and $\lim\limits_{t \to \pi}[\mathbf{u}(t) \times \mathbf{v}(t)] = \lim\limits_{t \to \pi}[\mathbf{u}(t)] \times \lim\limits_{t \to \pi}[\mathbf{v}(t)]$.

5. Show that the curvature of a plane parametric curve $x = f(t),\ y = g(t)$ is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{\frac{3}{2}}}$$

   where the dots indicate the derivatives with respect to t.