DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic Year 2021-2022
MA 101 Mathematics I
Problem Sheet 3: Critical points, maxima and minima, Lagrange's multipliers, volume of solids of revolution,etc
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1. Find the local maximum and minimum values and saddle point(s) of the functions:
   (a) $f(x, y) = x^2 + y^2 + x^2y + 4$
   (b) $f(x, y) = 4xy - x^4 - y^4$
   (c) $f(x, y) = \sin x \cosh y$
   (d) $f(x, y) = x + 2y + \dfrac{4}{x} - y^2$.

2. Find the absolute maximum and minimum values of $f(x, y) = 4xy^2 - x^2y^2 - xy^3$ on the set D where D is the closed triangular region in the xy-plane with vertices $(0, 0)$, $(0, 6)$ and $(6, 0)$.

3. For the following functions the origin is a critical point; determine whether $f(\mathbf{0})$ is a local minimum value, a local maximum value or neither
   (a) $f(x, y, z) = 5x^2 + 4y^2 + 7z^2 + 4xy + 2z \sin x + 6y \sin z$
   (b) $f(w, x, y, z) = wx + 2xy + 3yz - w^2 - 2x^2 - 3y^2 - 4z^2$.

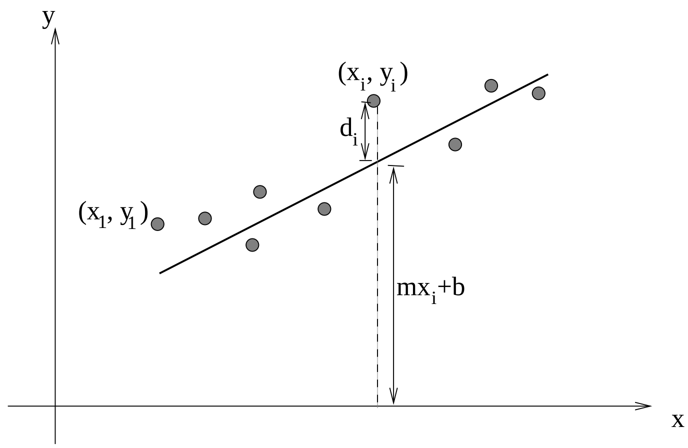
4. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

5. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$.

6. The plane $x + y + z = 12$ cuts the paraboloid $z = x^2 + y^2$ in an ellipse. Find the highest and lowest points on this ellipse.

7. Use Langrange multipliers to find the minimum and maximum values of the functions subject to the given constraint(s)
   (a) $f(x, y) = 4x + 6y$; $x^2 + y^2 = 13$
   (b) $f(x_1, x_2, ..., x_n) = x_1 + x_2 + \cdots + x_n$; $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$
   (c) $f(x, y) = e^{-xy}$; $x^2 + 4y^2 \leq 1$.

   Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately for some values of m and b. The scientist performs an experiment and collects data in the form of points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), and then plots these points. The points don't exactly lie on a straight line, so the scientist wants to find constants m and b such that the line $y = mx + b$ "fits" the points as well as possible. Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least**

8. 

squares determines m and b so as to minimize $\sum_{i=1}^{n} d_i^2$, the sum of the squares of these deviations. Show that according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

$$m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i.$$

9. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

   (a) $y = x^2$, $y^2 = x$; about x-axis

   (b) $y^2 = x$, $x = 2y$; about y-axis

   (c) $y = x$, $y = x^2$; about the line $x = -1$.

10. Find the volume of the wedge that is cut from a circular cylinder with unit radius and unit height by a plane that passes through a diameter of the base of the cylinder and through a point on the circumference of its top.

11. Prove that the length of one arch of the sine curve $y = \sin x$ is equal to half the circumference of the ellipse $2x^2 + y^2 = 2$.

12. Find the total length of the asteroid given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and then find the area of the surface generated by revolving the asteroid around the y-axis.