DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic Year 2021-2022
MA 101 Mathematics I
Problem Sheet 4: Multiple integrals and applications, Green's Theorem, Stokes Theorem and Divergence Theorem.
Instructors: Dr. J. C. Kalita and Dr. S.Upadhyay

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1. Evaluate the integrals:
   (a) $\int_0^3 \int_0^1 \sqrt{x+y}\,dxdy$
   (b) $\iint_R \frac{xy^2}{x^2+1}dA,\ R = \{(x,y)|\ 0 \le x \le 1,\ -3 \le y \le 3\}$
   (c) $\iint_R x\sin(x+y)dA,\ R = \left[0, \frac{\pi}{6}\right] \times \left[0, \frac{\pi}{3}\right].$
2. Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ and above the rectangle $\{(x,y)|\ -1 \le x \le 1,\ 1 \le y \le 4\}$.
3. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the square $R = [-1,1] \times [-2,2]$.
4. Evaluate the iterated integrals:
   (a) $\int_0^1 \int_y^{e^y} \sqrt{x}\,dxdy$
   (b) $\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} e^{\sin\theta}drd\theta.$
5. Evaluate the double integrals:
   (a) $\iint_D \frac{2y}{x^2+1}dA,\ R = \{(x,y)|\ 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$
   (b) $\iint_D x\cos y\,dA$, D is bounded by $y = 0, y = x^2, x = 1$
   (c) $\iint_D y^3 dA$, D is the triangular region with vertices (0,2), (1,1) and (3,2).
6. Find the volume of the given solids.
   (a) Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$
   (b) Under the surface $z = xy$ and above the triangle with vertices (1,1), (4,1) and (1,2)
   (c) Bounded by the cylinder $x^2 + z^2 = 9$ and the planes $x = 0, y = 0, z = 0, x + 2y = 2$ in the first octant
   (d) Bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.
7. Get an upper bound and lower bound of each of the integrals given below by using the result that if m, M are such that $m \le f(x,y) \le M$ for all $(x,y) \in D$ then $m \times A(D) \le \iint_D f(x,y)dA \le M \times A(D)$, where $A(D)$ is the area of D.
   (a) $\iint_D \sqrt{x^3+y^3}dA,\ D = [0,1] \times [0,1]$
   (b) $\iint_D e^{x^2+y^2}dA$, D being the disk with center at origin and radius 0.5.
8. Using polar coordinates, find:
   (a) $\iint_R xy\,dA$, where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

(d) The volume inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

(e) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx$

(f) $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dxdy.$

9. Find the mass, center of mass and the moments of inertia I_x, I_y and I_o of the lamina D bounded by the parabola $x = y^2$ and the line $y = x - 2$; the density is $\rho(x, y) = 3$.

10. Find the area of the surface:

(a) The part of the plane $3x + 2y + z = 6$ that lies in the first octant

(b) The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

(c) The part of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$

(d) The part cut from the paraboloid $z = r^2$ by the cylinder $r = 1$.

11. Evaluate the following triple integrals.

(a) $\displaystyle\iiint_E 2x\,dV$, where $E = \left\{(x, y, z)|\ 0 \le y \le 2,\ 0 \le x \le \sqrt{4 - y^2},\ 0 \le z \le y\right\}$

(b) $\displaystyle\iiint_E x\,dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

12. Use triple integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + 3y + 6z = 12$.

13. Use cylindrical or spherical coordinates whichever is appropriate, to:

(a) Evaluate $\displaystyle\iiint_E \sqrt{x^2 + y^2}\,dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$

(b) Evaluate $\displaystyle\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$

(c) Evaluate $\displaystyle\iiint_E \sqrt{x^2 + y^2 + z^2}\,dV$, where E is bounded below by the cone $\phi = \frac{\pi}{6}$ and above by the sphere $\rho = 2$.

14. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

15. Show that $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}e^{-(x^2+y^2+z^2)}\,dxdydz = 2\pi.$

16. Use the given transformation to evaluate the integral:

(a) $\displaystyle\iint_R (3x + 4y)\,dA$, where R is the region bounded by the lines $y = x$, $y = x - 2$, $y = -2x$ and $y = 3 - 2x$; $x = \dfrac{1}{3}(u + v)$, $y = \dfrac{1}{3}(v - 2u)$

(b) $\displaystyle\iint_R xy\,dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $x = \frac{u}{v}$, $y = v$.

17. Evaluate the following integrals by making appropriate change of variables:

(a) $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$ and $(0, 1)$

(b) $\displaystyle\iint_R \frac{1}{(x^2 + y^2)^2}\,dxdy$, where R is the first quadrant region bounded by the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 6x$ and the circles $x^2 + y^2 = 2y$, $x^2 + y^2 = 8y$.

18. Let R be the solid ellipsoid with constant density δ and boundary surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Use appropriate transformations to show that the mass M of R is $\dfrac{4}{3}\pi\delta abc$.