Coupled ψ-v and immersed interface method for incompressible viscous flows

The unsteady Navier-Stokes (N-S) equations for incompressible viscous flows for a fluid in two-dimensions (2D) can be written as $\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} - \frac{1}{\mathrm{Re}}\nabla^2\vec{u} = -\nabla p$ where $\vec{u} = (u, v)$ is the velocity vector, t is the time, p the pressure and Re is the Reynolds number given by $\mathrm{Re} = Lu_0/\upsilon$ with L being some characteristic length, u_0 some characteristic velocity and υ the kinematic viscosity of the fluid.

Though this formulation accurately represents the fluid flow phenomena, the direct numerical solution of these highly non-linear equations traditionally have been difficult to obtain due to the pressure term. To overcome such difficulties an alternative formulation using vorticity (ω) and stream function (ψ) is in use for quite some time. Although the ω-ψ formulation is quite popular, vorticity on the boundaries are generally unspecified and one has to carry out a variety of numerical approximations in order to specify the boundary values of vorticity.

In 2005, we developed a second and then a fourth order compact finite difference scheme [1,2] for the steady-state N-S equations in the biharmonic form which was later extended to its unsteady counterpart in Cartesian coordinates [3] given by $\nabla^4\psi - \mathrm{Re}\,u\left[\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial x\partial y^2}\right] - \mathrm{Re}\,v\left[\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right] = \mathrm{Re}\frac{\partial}{\partial t}(\nabla^2\psi)$. This formulation, known as the streamfunction-velocity (ψ-v) formulation is second order accurate in both time and space, implicit and unconditionally stable. Recently it has been extended to curvilinear coordinates [4] where we had tremendous success in computing flow in many complex fluid flow situations including both external and internal flows. In figure 1, we display a few samples of the flows computed through the approach in [4].

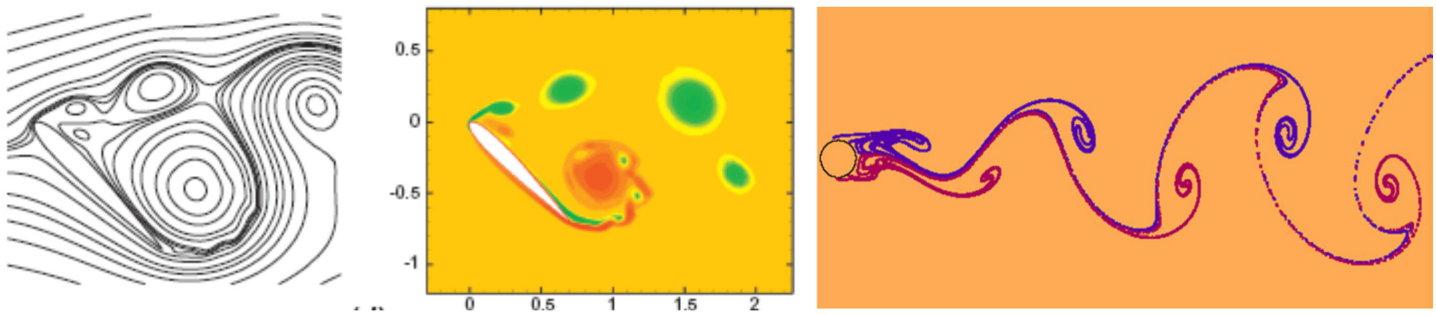

Fig. 1: Streamlines (left)) and vorticity (middle) contours for flow past a NACA 0012 aerofoil with angle of attack 45 degrees for Re=3000, and streaklines for the flow past a stationary circular cylinder for Re=80.

Note that in the aforesaid approach, the physical and the computational planes were different. For physical domains other than rectangular, we used a transformation mapping from the physical curvilinear plane to the computational rectangular plane. Figure 2 (a) and (b) respectively show the grids used for the flow past a circular cylinder and an aerofoil with angle of attack in the physical domain, and in figure 2(c), we show the grid used in actual computational plane. However, in almost all practical fluid flow problems, the physical domain may be of irregular shape and not necessarily be such that one can use transformation from the physical to the computational plane. To overcome this problem using Cartesian grid in the finite difference framework, we plan to utilize the Immersed Interface Method (IIM) developed by Li and Leveque [5, 6, 7], where the physical and computational planes are the same (see figure 2(d)). Of late, the IIM and its variants have become very popular because of its ability to efficiently solve interface problems and problems on irregular domains. This method is based on uniform or adaptive grids or triangulation in Cartesian, polar or spherical coordinates. Standard finite difference or finite element methods are used away from the interface or boundaries. The finite difference or finite element schemes are modified locally near or on the interfaces or the boundaries according to the interface relations so that high order accuracy can be maintained in the entire domain. We have already developed an HOC (High Order Compact) scheme coupled with IIM for flow across circular interfaces [8].

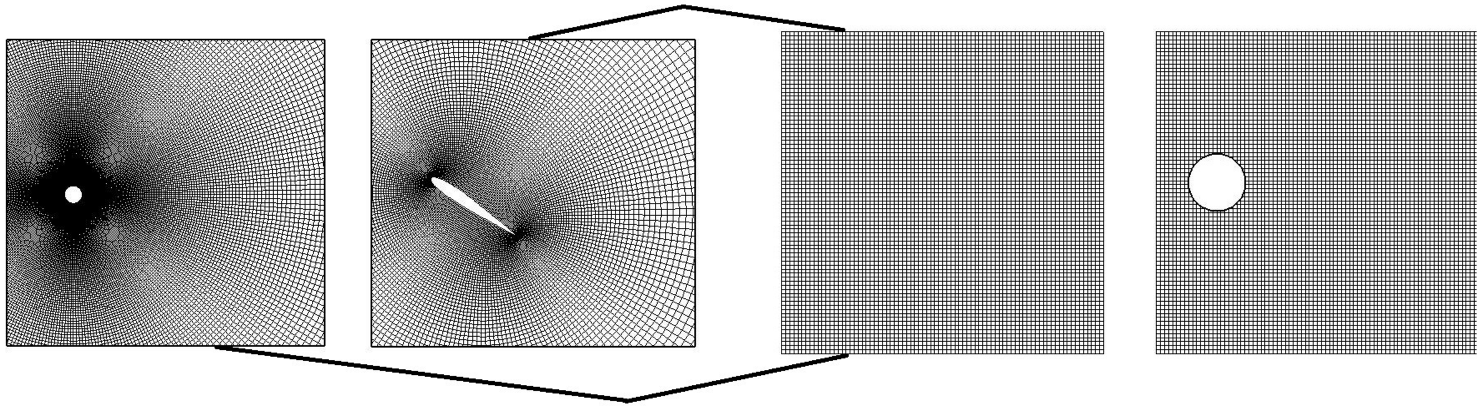

Fig. 2(a-b) Physical planes for Circular cylinder and Aerofoil with grids 2(c) Computational plane 2(d) Physical and comp. plane in IIM

In the proposed research we plan plane to solve the N-S equations in irregular domains in the ψ-v or pure streamfunction formulation through IIM on Cartesian grids as shown in Fig 2(d). Till date the implementation of the IIM to the N-S equations in 2D has been limited to the primitive variable and the ω-ψ formulations only [7]. Once accomplished, to the best of our knowledge, this will be the first time that the IIM has been clubbed with the ψ-v formulation of the N-S equations in irregular domains. This is likely to drastically reduce the computational time as the core of the computation involves only one variable ψ. Specifically we would be interested in problems of droplet formation and deformation [9].

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