Representation of Near-Semirings and Approximation of Their Categories

K. V. Krishna
Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781 039, India.
E-mail: kv.krishna@member.ams.org

N. Chatterjee
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi - 110 016, India.
E-mail: niladri@maths.iitd.ac.in

AMS Mathematics Subject Classification (2000): 16Y60, 16Y30, 16Y99

Abstract. This work observes that $S$-semigroups are essentially the representations of near-semirings to proceed to establish categorical representation of near-semirings. Further, this work addresses some approximations to find a suitable category in which a given near-semiring is primitive.

Keywords: Near-semiring; $S$-Semigroup; Representation; Category.

1. Introduction

An algebraic structure $(S, +, \cdot)$ is said to be a near-semiring if

1. $(S, +)$ is semigroup with identity 0,
2. $(S, \cdot)$ is semigroup,
3. $(x + y)z = xz + yz$ for all $x, y, z \in S$, and
4. $0x = 0$ for all $x \in S$.

The standard examples of near-semirings are typically of the form $\mathcal{M}(\Gamma)$, the set of all mappings on a semigroup $(\Gamma, +)$ with identity zero, with respect to pointwise addition and composition of mappings, and certain subsets of this set.
Two important subsets of $\mathcal{M}(\Gamma)$ are the set of constant mappings, and the set of mappings which fix zero. In fact, these two sets are subnear-semirings of $\mathcal{M}(\Gamma)$ in the usual sense. In an arbitrary near-semiring $S$, these substructures can be defined as constant part $S_c = \{ s \in S \mid s0 = s \}$ and zero-symmetric part $S_0 = \{ s \in S \mid s0 = 0 \}$. A near-semiring $S$ is said to be a zero-symmetric near-semiring if $S = S_0$ ($S = S_c$, respectively). Another example of near-semiring that generalizes $\mathcal{M}(\Gamma)$ is: let $\Sigma \subseteq \text{End}(\Gamma)$, the set of endomorphisms on $\Gamma$, and define $\mathcal{M}_\Sigma(\Gamma) = \{ f : \Gamma \to \Gamma \mid f\alpha = \alpha f, \forall \alpha \in \Sigma \}$. Then $\mathcal{M}_\Sigma(\Gamma)$ is a near-semiring. Indeed, $\mathcal{M}(\Gamma) = \mathcal{M}_{\{\text{id}_\Gamma\}}(\Gamma)$.

A semigroup $(\Gamma, +)$ with zero $o$ is said to be an $S$-semigroup if there exists a composition $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \to \Gamma$ such that

1. $(x + y)\gamma = x\gamma + y\gamma$,
2. $(xy)\gamma = x(y\gamma)$, and
3. $0\gamma = o$, for all $x, y \in S, \gamma \in \Gamma$.

It is clear that $\Gamma$ is an $S$-semigroup with $S = \mathcal{M}(\Gamma)$. Also, the semigroup $(S, +)$ of a near-semiring $(S, +, \cdot)$ is an $S$-semigroup.

For further details on near-semirings or $S$-semigroups one may refer [6, 8, 10, 11]. In what follows $S$ always denotes a near-semiring, and an additive semigroup with zero is simply referred as semigroup.

In this work we first observe that the notion of $S$-semigroup gives an algebraic representation of near-semirings which further helps us to establish a categorical representation. This enables one to make use of the special properties of the near-semirings, and provides a practical approach to the problem of classifying certain classes of near-semirings. We also made an attempt to approximate categories in which a given arbitrary near-semiring is primitive, as an extension of the work of Holcombe [3] and that of Clay [2] for near-rings.

2. Representations

Let $\Gamma, \Gamma'$ be two $S$-semigroups. A mapping $f : \Gamma \to \Gamma'$ is said to be an $S$-homomorphism if $f(x + y) = f(x) + f(y)$; $f(ax) = af(x)$ for all $a \in S$ and all $x, y \in \Gamma$. Near-semiring homomorphism can be defined in usual way.

*In the literature, zero-symmetric near-semirings are often referred as seminearrings [4, 6, 7, 10].
Following Jacobson [5], we define a representation of a near-semiring $S$ as a homomorphism of $S$ into the near-semiring of mappings of some semigroup with zero.

Let us recall the following embedding theorem from [4] before going to observe that $S$-semigroups are precisely the representations of near-semirings.

**Embedding Theorem.** For every near-semiring $S$ there exists a semigroup $\Gamma$, such that $S$ can be embedded in $\mathfrak{M}(\Gamma)$.

From this theorem one can ascertain that every near-semiring can be embedded into a near-semiring with unity.

Let $a \mapsto \bar{a}$ be a representation of $S$ that acts on a semigroup $\Gamma$. Define a composition from $S \times \Gamma$ to $\Gamma$ by $ax = \bar{a}(x)$, for $x \in \Gamma$ and $a \in S$, so that $\Gamma$ is an $S$-semigroup. Hence, every representation of a near-semiring $S$ determines an $S$-semigroup.

On the other hand, every $S$-semigroup $\Gamma$ determines a representation of the near-semiring $S$. Indeed, for $a \in S$, define a mapping $a_S$ on $\Gamma$ by $a_S(x) = ax$ for all $x \in \Gamma$. Then $\tau : S \rightarrow \mathfrak{M}(\Gamma)$ given by $\tau(a) = a_S$ is a near-semiring homomorphism. Hence $\tau$ is a representation of $S$.

This discussion can be summarized as follows.

**Theorem 2.1.** The concepts of $S$-semigroup and representation of a near-semiring $S$ are equivalent.

In the following we obtain a representation of near-semirings in a more general way using the theory of categories. Let $\mathcal{C}$ be a category; write $X \in \mathcal{C}$ to indicate that $X$ is an object of $\mathcal{C}$. For any $X, Y \in \mathcal{C}$, the set of morphisms in $\mathcal{C}$ from $X$ to $Y$ is written as $[X, Y]_{\mathcal{C}}$. The category of sets and mappings is denoted by $\mathcal{S}$; $\mathcal{S}$ denotes the category of semigroups and their homomorphisms. The category of $S$-semigroups and $S$-homomorphisms for a fixed near-semiring $S$ will be denoted by $\mathcal{S}_S$. The contravariant representable functor $h_X : \mathcal{C} \rightarrow \mathcal{S}$ is given by $h_X(Y) = [Y, X]_{\mathcal{C}}, h_X(u) = vu$ for any $v : Z \rightarrow X$, where $u : Y \rightarrow Z$. The forgetful functor from $\mathcal{S}$ to $\mathcal{F}$ will be denoted by $\rho : \mathcal{S} \rightarrow \mathcal{F}$, and it is $\bar{\rho} : \mathcal{S}_S \rightarrow \mathcal{F}$. For other terminology and fundamental concepts of category theory that are used in the rest of the paper, one may refer [1, 9].

An object $X \in \mathcal{C}$ is said to be a semigroup object in $\mathcal{C}$ if and only if there exists a functor $\sigma : \mathcal{C} \rightarrow \mathcal{S}$ such that the following functor diagram commutes.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{h_X} & \mathcal{F} \\
\sigma \downarrow & & \downarrow \bar{\rho} \\
\mathcal{S} & \xrightarrow{\rho} & \mathcal{F}
\end{array}
\]

It is more practical to deal with morphisms rather than functors in some
circumstances. For that purpose, Lemma 2.2 is formulated in the similar lines of a theorem for group objects (cf. Theorem 4.1 of [1]).

**Lemma 2.2.** Let $\mathcal{C}$ be a category with finite products and a final object $e$, and let $X \in \mathcal{C}$. Let $\eta$ be the unique element of $[X, e]_{\mathcal{C}}$. $X$ is a semigroup object in $\mathcal{C}$ if and only if there exist morphisms $m \in [X \times X, X]_{\mathcal{C}}$, and $\varepsilon \in [e, X]_{\mathcal{C}}$ such that the diagrams

\[
\begin{align*}
X \times X \times X &\xrightarrow{m \times 1_X} X \times X \\
1_X \times m &\downarrow \\
X \times X &\xrightarrow{m} X
\end{align*}
\quad
\begin{align*}
X \times X &\xrightarrow{(\varepsilon \eta) \times 1_X} X \times X \\
\Delta &\downarrow \\
X &\xrightarrow{1_X} X
\end{align*}
\]

are commutative, where $\Delta$ is the ‘diagonal’ morphism.

**Theorem 2.3.** Let $(X, \sigma)$ be a semigroup object in $\mathcal{C}$, a category with finite products and a final object. Then $[X, X]_{\mathcal{C}}$ is a near-semiring, say $S$, and for any $Y \in \mathcal{C}$, $[Y, X]_{\mathcal{C}}$ is an $S$-semigroup.

**Proof.** Let $S = [X, X]_{\mathcal{C}}$ and $a, b \in S$. By Lemma 2.2, there is a semigroup structure $(S, +)$ defined by $a + b = m\{a, b\}$, where $\{a, b\}$ is the unique morphism making the following diagram (1) commutative with $p_1, p_2 : X \times X \rightarrow X$ canonical projections, and $m$ is obtained as in Lemma 2.2.

\[
\begin{align*}
X &\xleftarrow{p_1} X \times X \xrightarrow{p_2} X \\
&\xleftarrow{a} X \\
&\xrightarrow{b} X
\end{align*}
\quad
\begin{align*}
X &\xleftarrow{p_1} X \times X \xrightarrow{p_2} X \\
&\xleftarrow{a} X \\
&\xrightarrow{b} X
\end{align*}
\]

Clearly, $S$ is a semigroup under the composition of morphisms in $\mathcal{C}$. Right distributivity follows from the commutative diagram (2), so that $S$ is a near-semiring.

Again, since $(X, \sigma)$ is a semigroup object in $\mathcal{C}$, for any $Y \in \mathcal{C}$, $\Gamma = [Y, X]_{\mathcal{C}}$ is a semigroup, where addition $+$ on $\Gamma$ is given by $\alpha + \beta = m\{\alpha, \beta\}$ for $\alpha, \beta \in \Gamma$ and $\{\alpha, \beta\}$ is the unique morphism, such that the following diagram commutes.

\[
\begin{align*}
X &\xleftarrow{p_1} X \times X \xrightarrow{p_2} X \\
&\xleftarrow{\alpha} Y \\
&\xrightarrow{\beta} X
\end{align*}
\]
Define an action from $S \times \Gamma$ to $\Gamma$ by $(a, \alpha) \mapsto a\alpha$, for $a \in S$ and $\alpha \in \Gamma$. By a similar argument to the first part of this proof we see that
\[(a + b)\alpha = a\alpha + b\alpha, \text{ and } (ab)\alpha = a(b\alpha)\]
for all $a, b \in S$ and $\alpha \in \Gamma$, so that $\Gamma$ is an $S$-semigroup.

Given the situation of Theorem 2.3, we call $[X, X]_\mathcal{C}$ the endomorphism near-semiring of $X$ in $\mathcal{C}$.

**Remark 2.4.** Let $(X, \sigma)$ be a semigroup object in a category $\mathcal{C}$ with finite products and final object. If $S = [X, X]_\mathcal{C}$ then there is a contravariant functor $\mu_X : \mathcal{C} \rightarrow S_S$ such that the following diagram commutes,

\[\begin{array}{ccc}
\mathcal{C} & \xrightarrow{b_X} & \mathcal{F} \\
\downarrow{\mu_X} & & \downarrow{\bar{\rho}} \\
S_S & \xrightarrow{\bar{\rho}} & \mathcal{F}
\end{array}\]

where $\bar{\rho}$ is the forgetful functor from $S_S$ to $\mathcal{F}$.

**Example 2.5.** In the category of sets and mappings $\mathcal{F}$, semigroup objects are just semigroups. Then the endomorphism near-semiring of a semigroup $\Gamma$ in $\mathcal{C}$ is the set of all mappings of $\Gamma$ into itself.

**Example 2.6.** In the category $\mathcal{F}^*$ of pointed sets, let us consider the zero of semigroup objects $\Gamma^*$ as distinguished element. The endomorphism near-semiring of $\Gamma^*$ is the set of zero-preserving maps of $\Gamma^*$ into itself. This near-semiring is zero-symmetric.

**Example 2.7.** Let $\Sigma$ be a semigroup. The category $\mathcal{S}_\Sigma$, of $\Sigma$-sets, has objects as pairs $(X, m)$, where $X$ is a set and $m : \Sigma \times X \rightarrow X$ is a mapping with the property that $m(\alpha\beta, x) = m(\alpha, m(\beta, x))$ for all $x \in X$ and $\alpha, \beta \in \Sigma$. A morphism $f : (X_1, m_1) \rightarrow (X_2, m_2)$ is a mapping $f : X_1 \rightarrow X_2$, such that the following diagram commutes.

\[\begin{array}{ccc}
\Sigma \times X_1 & \xrightarrow{m_1} & X_1 \\
\downarrow{1_\Sigma \times f} & & \downarrow{f} \\
\Sigma \times X_2 & \xrightarrow{m_2} & X_2
\end{array}\]

The endomorphism near-semiring of $X$, a semigroup object in $\mathcal{S}_\Sigma$, is the set of mappings $f$ of $X$ into itself, such that $f(m(\alpha, x)) = m(\alpha, f(x))$ for all $x \in X$ and $\alpha \in \Sigma$. This example generalizes near-semirings of the form $\mathcal{M}_\Sigma(X)$ and $S$-semigroups.
Example 2.8. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories with $\mathcal{C}_2$ a subcategory of $\mathcal{C}_1$. The category $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is defined to have objects $f : B \rightarrow A$, where $B \in \mathcal{C}_2$, $A \in \mathcal{C}_1$ and $f \in [B, A]_{\mathcal{C}_1}$. A morphism from $f : B \rightarrow A$ to $g : C \rightarrow D$ is a pair $(a, b)$ such that $bf = ga$, where $a \in [B, C]_{\mathcal{C}_2}$, $b \in [A, D]_{\mathcal{C}_1}$, i.e. a morphism from $f$ to $g$ can be given by a commutative diagram as below.

$$
\begin{array}{ccc}
B & \xrightarrow{a} & C \\
\downarrow{f} & & \downarrow{g} \\
A & \xrightarrow{b} & D
\end{array}
$$

Further, let $\mathcal{C}_3$ be a subcategory of $\mathcal{C}_1$ and define $\langle \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ to have objects $(f, f')$, where $f : B \rightarrow A$, $f' : C \rightarrow A$ and $A \in \mathcal{C}_1$, $B \in \mathcal{C}_2$, $C \in \mathcal{C}_3$, $f \in [B, A]_{\mathcal{C}_1}$, $f' \in [C, A]_{\mathcal{C}_1}$; and morphisms from $(f, f')$ to $(g, g')$ are the commutative diagrams,

$$
\begin{array}{ccc}
B & \xrightarrow{f} & A & \xleftarrow{f'} & C \\
\downarrow{g} & & \downarrow{g'} & & \\
B_1 & & A_1 & & C_1
\end{array}
$$

where $A_1 \in \mathcal{C}_1$, $B_1 \in \mathcal{C}_2$, $C_1 \in \mathcal{C}_3$, $g \in [B_1, A_1]_{\mathcal{C}_1}$, $g' \in [C_1, A_1]_{\mathcal{C}_1}$. A natural extension of these ideas give a category $\langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k \rangle$, where $\mathcal{C}_j$ is a subcategory of $\mathcal{C}_1$ for $j = 2, 3, \ldots, k$. In a more general case, suppose $F : \mathcal{C} \rightarrow \mathcal{C}'$ is an embedding functor, we can construct the category $\langle \mathcal{C}', F(\mathcal{C}) \rangle$.

As an example of the construction, let $\Sigma$ be a semigroup and $\Sigma'$ be a subsemigroup of $\Sigma$. There exists an embedding $F : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_{\Sigma'}$, where $\mathcal{F}_\Sigma$ (and $\mathcal{F}_{\Sigma'}$) is pointed $\Sigma$-sets ($\Sigma'$-sets respectively). Let $X$ be a semigroup object of $\mathcal{F}_\Sigma$ and $Y$ a subsemigroup of $X$ which is also a semigroup object of $\mathcal{F}_{\Sigma'}$. Then the endomorphism near-semiring of $(F(Y) \subseteq X)$ is the set of all mappings $f : X \rightarrow X$, such that $f(Y) \subseteq Y$, $f(m(x')) = m(x, f(x))$, $f(m(x)) = m(x, f(y))$ for all $x \in X, y \in Y, x' \in \Sigma'$, and $\alpha \in \Sigma$. These near-semirings are examples of an important class of near-semirings which deserves study in its own right.

Several other examples come in the same line. So far it is observed how the near-semirings arise in essentially the same way as endomorphism sets of semigroup objects in particular categories.

Let $S$ be a near-semiring and $\mathcal{C}$ be a category with finite products. An object $X$ is said to be an $S$-semigroup object in $\mathcal{C}$ if and only if there exist

1. a functor $\sigma$ such that $(X, \sigma)$ is a semigroup object in $\mathcal{C}$, and
2. a near-semiring homomorphism $\tau : S \rightarrow [X, X]_{\mathcal{C}}$.

In this case, $(X, \sigma, \tau)$ denotes an $S$-semigroup object in $\mathcal{C}$. An $S$-semigroup object $(X, \sigma, \tau)$ is said to be faithful in $\mathcal{C}$ if and only if $\tau$ is one-one.

If $\mathcal{C}$ is the category of sets and mappings then a semigroup object in $\mathcal{C}$ is simply a semigroup and the concept of an $S$-semigroup in $\mathcal{C}$ coincides with the
natural definition of an $S$-semigroup. Therefore, $S$-semigroups are special cases of the concept of $S$-semigroups in a category $\mathcal{C}$.

**Theorem 2.9.** Let $S$ be a near-semiring and let $\mathcal{C}$ be a category with finite products and final object. Then for $X \in \mathcal{C}$, $X$ is an $S$-semigroup object in $\mathcal{C}$ if and only if there exists a contravariant functor $\lambda : \mathcal{C} \rightarrow S_S$ such that the following diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{h_X} & \mathcal{I} \\
\lambda \downarrow & & \downarrow \rho \\
S_S & \xrightarrow{\bar{\rho}} & S_S
\end{array}
\]

is commutative, where $\bar{\rho} : S_S \rightarrow S$ is the forgetful functor.

**Proof.** Suppose $(X, \sigma, \tau)$ is an $S$-semigroup object in $\mathcal{C}$. Let $Y \in \mathcal{C}$, and write $\Gamma = [Y, X]_{\mathcal{C}}$. Consider the structure of an $S$-semigroup to $\Gamma$ as follows: define $s \cdot \gamma = \tau(s)\gamma$ for $\gamma \in \Gamma, s \in S$. Thus there exists a functor $\lambda : \mathcal{C} \rightarrow S_S$, such that $\lambda(Y)$ is the $S$-semigroup $\Gamma = [Y, X]_{\mathcal{C}}$.

Conversely, suppose $\lambda$ exists, and that $\rho^* : S_S \rightarrow S$ and $\rho : S \rightarrow \mathcal{I}$ are forgetful functors. Then $(X, \rho^* \circ \lambda)$ is a semigroup object in $\mathcal{C}$. A near-semiring homomorphism $\tau : S \rightarrow [X, X]_{\mathcal{C}}$ is defined as follows. Since $\lambda(Y)$ is an $S$-semigroup, one may construct a near-semiring homomorphism $\bar{\tau} : S \rightarrow [h_X(Y), h_X(Y)]_{\mathcal{I}}$

for any $Y \in \mathcal{C}$. For each $s \in S$, the homomorphism $\bar{\tau}(s)$ induces a natural transformation $T_s : h_X \rightarrow h_X$. As a consequence of Yoneda lemma, one may find a unique morphism $g_s \in [X, X]_{\mathcal{C}}$ in natural correspondence with $T_s$. Now define $\tau : S \rightarrow [X, X]_{\mathcal{C}}$ by $\tau(s) = g_s$ for all $s \in S$. This gives the required near-semiring homomorphism. 

**Remark 2.10.** It is possible to define an $S$-homomorphism between $S$-semigroups in the same category $\mathcal{C}$. For instance, given a near-semiring $S$ and a category $\mathcal{C}$ with finite products and final object, let $(X, \sigma, \tau), (Y, \sigma', \tau')$ be $S$-semigroups in $\mathcal{C}$. A morphism $f : X \rightarrow Y$ in $\mathcal{C}$ is an $S$-homomorphism in $\mathcal{C}$ if and only if

- for all $s \in S$ the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{\tau(s)} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\tau'(s)} & Y
\end{array}
\]

- there exists a natural transformation $\xi : \sigma \rightarrow \sigma'$ such that the induced natural transformation $T_\xi : h_X \rightarrow h_Y$ corresponds via the Yoneda lemma to the morphism $f : X \rightarrow Y$ in $\mathcal{C}$. 

Thus, one can define a category of $S$-semigroups and $S$-homomorphisms in a category with finite products.

3. Approximation Theorems

First we formulate the notions: transparent $S$-subsemigroups, minimality and primitivity in categories for near-semirings as an extension of those parallel notions for near-rings given by Holcombe [3]. Then we proceed to approximate categories in which the given near-semiring is primitive. Unless otherwise stated, in the following $\mathcal{C}$ is a category with finite products and a final object, also there exists a forgetful functor $U : \mathcal{C} \rightarrow \mathcal{S}$.

Suppose $(X, \sigma, \tau)$ and $(Y, \sigma', \tau')$ are $S$-semigroups in $\mathcal{C}$ and $u : Y \rightarrow X$ is an $S$-homomorphism in $\mathcal{C}$. We call $(Y, u)$ an $S$-subsemigroup of $X$ if and only if $u$ is a monomorphism in $\mathcal{C}$, and $U(u)$ is an inclusion in $\mathcal{S}$. An $S$-subsemigroup $(Y, u)$ of $X$ is called transparent if and only if $[Y, X]_{\mathcal{C}} = \{ uf | f \in [Y, Y]_{\mathcal{C}} \}$, i.e. any morphism in $[Y, X]_{\mathcal{C}}$ can be decomposed into the composition of a morphism in the near-semiring $[Y, Y]_{\mathcal{C}}$ with $u$.

Let $X$ be an $S$-semigroup in $\mathcal{C}$ and $f \in [K, X]_{\mathcal{C}}$ a monomorphism, for $K \in \mathcal{C}$. We call $(K, f)$ is a generator of $X$ if and only if $U(f)$ is a set inclusion, and for every $a \in [K, X]_{\mathcal{C}}$, there exists $s_a \in S$ such that $\tau(s_a)f = a$. An $S$-semigroup $X$ in $\mathcal{C}$ is called $\mathcal{C}$-minimal if and only if given a nontrivial monomorphism $f \in [K, X]_{\mathcal{C}}$ with $U(f)$ a set inclusion, either $(K, f)$ is a generator of $X$, or there exists a transparent $S$-subsemigroup $(Y, u)$ of $X$ such that $f$ factors through $u$ in the following way: there exists $f' \in [K, Y]_{\mathcal{C}}$ such that $U(f')$ is a set inclusion and $f = uf'$. Further, a near-semiring $S$ is said to be $\mathcal{C}$-primitive for some $\mathcal{C}$ if there exists a $\mathcal{C}$-minimal $S$-semigroup $X$ in $\mathcal{C}$ which is faithful.

Naturally there may exist near-semirings which are not $\mathcal{C}$-primitive for any $\mathcal{C}$. Though finding a suitable category $\mathcal{C}$ such that given a near-semiring $S$ is $\mathcal{C}$-primitive is difficult, it is often possible to find a category $\mathcal{C}$ over which $S$ can be represented in a useful way. For example there may be representations of $X$ over $\mathcal{C}$ such that $\tau : S \rightarrow [X, X]_{\mathcal{C}}$ is one-one. Now replace $\mathcal{C}$ by other categories so that the representations are preserved, and at the same time to make the homomorphism $\tau$ nearer to being an isomorphism, which is clearly a desirable objective.

Let $(X, \tau)$ be an $S$-semigroup in $\mathcal{C}$, where $\tau : S \rightarrow [X, X]_{\mathcal{C}}$ the near-semiring homomorphism. Suppose $G = Aut_{S}(X)$, the group of all invertible $S$-homomorphisms in $\mathcal{C}$. Construct a category $\mathcal{C}_G$ in which objects are the pairs $(A, \alpha)$, where $A \in \mathcal{C}$ and $\alpha : G \rightarrow [A, A]_{\mathcal{C}}$ is a semigroup homomorphism. A morphism $\xi$ of $\mathcal{C}_G$, say $(A, \alpha) \xrightarrow{\xi} (B, \beta)$, is a morphism $\xi \in [A, B]_{\mathcal{C}}$ such that $\beta(g)\xi = \xi\alpha(g)$ for all $g \in G$. 
Remark 3.1. \( \mathcal{C}_G \) is a category with finite products and final object. Moreover, there exists a forgetful functor \( \mathcal{U}_G : \mathcal{C}_G \to \mathcal{S} \).

The objects \( X \in \mathcal{C} \) may be equipped with the structure of an object \( X_G \in \mathcal{C}_G \) by defining \( X_G = (X, \text{id}_X) \).

Remark 3.2. \((X_G, \tau_G)\) is an \( S \)-semigroup in \( \mathcal{C}_G \), where \( \tau_G : S \to [X_G, X_G]_{\mathcal{C}_G} \) is defined by \( \tau_G(s) = \tau(s) \forall s \in S \). Moreover, if \( \tau \) is one-one then \( \tau_G \) is one-one.

Theorem 3.3. If \( X \) is \( \mathcal{C} \)-minimal then \( X_G \) is \( \mathcal{C}_G \)-minimal.

Proof. Let \( f_G \in [K_G, X_G]_{\mathcal{C}_G} \) be a monomorphism and \( \mathcal{U}_G(f_G) \) be a set inclusion. On forgetting the \( G \)-structure, we obtain a monomorphism \( f \in [K, X]_{\mathcal{C}} \). Since \( X \) is \( \mathcal{C} \)-minimal, there are two cases.

Consider the case when \((K, f)\) is generator of \( X \) in \( \mathcal{C} \). Suppose \( a_G \in [K_G, X_G]_{\mathcal{C}_G} \), and consider the corresponding morphism \( a \in [K, X]_{\mathcal{C}} \). There exists \( s_a \in S \) such that \( \tau(s_a) f = a \). Since \( \tau_G(s_a) = \tau(s_a) \) we have \( \tau_G(s_a) f = a \) and so \( \tau_G(s_a) f_G = a_G \). Hence \((K_G, f_G)\) is a generator of \( X_G \) in \( \mathcal{C}_G \).

On the other hand, suppose \((Y, u)\) is a transparent \( S \)-subsemigroup of \( X \) in \( \mathcal{C} \) and \( f' \in [K, Y]_{\mathcal{C}} \) is such that \( \mathcal{U}(f') \) is a set inclusion and \( f = uf' \). We turn \( Y \) into an object of \( \mathcal{C}_G \) as follows. Let \( g \in G \), then \( gu \in [Y, X]_{\mathcal{C}} \) and hence, by transparency of \( Y \), \( gu = uf_g \) for some unique \( f_g \in [Y, Y]_{\mathcal{C}} \). Define a mapping

\[
\beta : G \to [Y, Y]_{\mathcal{C}}
\]

by \( \beta(g) = f_g \) for all \( g \in G \). For \( g, g' \in G \) we have \( \beta(gg') = f_{gg'} \) and \( uf_{gg'} = gg'u = guf_{g'} = uf_gf_{g'} \), so that \( \beta \) is a semigroup homomorphism and hence \((Y, \beta)\) is an object of \( \mathcal{C}_G \). Now we shall prove that \((Y, \beta)\) is transparent \( S \)-subsemigroup of \( X_G \) in \( \mathcal{C}_G \). Let \((Y, \beta) \xrightarrow{u} X_G \) be any morphism in \( \mathcal{C}_G \). Then for each \( g \in G \), the following left side diagram is commutative. Since \( Y \) is transparent \( S \)-subsemigroup of \( X \) in \( \mathcal{C} \) we have \( \eta = uf \), for some \( f \in [Y, Y]_{\mathcal{C}} \), so that the outer square of the following right side diagram is equals to left side diagram and hence commutes.

\[
\begin{array}{ccc}
Y & \xrightarrow{\eta} & X \\
\beta(g) \downarrow & & \downarrow \beta(g) \\
Y & \xrightarrow{g} & X
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Y & \xrightarrow{f} & Y \\
\eta \downarrow & & \downarrow \eta \\
Y & \xrightarrow{u} & X
\end{array}
\]

Note that the right hand square of the right side diagram also commutes and hence, because \( u \) is a monomorphism, the left hand square of right diagram commutes, i.e. \( \beta(g)f = f\beta(g) \) for all \( g \in G \), so that \((Y, \beta) \xrightarrow{f} (Y, \beta)\) is a morphism of \( \mathcal{C}_G \). Thus \((Y, \beta)\) is transparent in \( X_G \) in the category \( \mathcal{C}_G \). Finally,
the diagram

\[ \begin{align*}
K_G & \xrightarrow{f_G} X_G \\
\downarrow f' & \downarrow u \\
(Y, \beta) & 
\end{align*} \]

commutes in \( \mathcal{C}_G \) from similar considerations. Hence \( X_G \) is \( \mathcal{C}_G \)-minimal.

Though the results are valid with the semigroup structure of \([X,X]_\varphi\) in place of the group \( G \), by choosing the group \( G \) we could further narrow down the category to \( \mathcal{C}_G \). As we can embed \([X_G,X_G]_{\mathcal{C}_G}\) in \([X,X]_{\varphi}\), Theorem 3.3 gives us an approximation theorem without disturbing the special nature of the representation \( X \) of \( S \).

If \( X \) has any \( G \)-closed \( S \)-subsemigroup we can produce a better approximation to \( S \). Here, an \( S \)-subsemigroup \((Y, u)\) of \( X \) is referred as \( G \)-closed if and only if given \( g \in G \) there exists a unique \( f \in [Y,Y]_\varphi \) such that \( gu = uf \).

**Remark 3.4.** Since \( u \) is monomorphism, if \((Y, u)\) is transparent in \( \mathcal{C} \) then \((Y, u)\) is \( G \)-closed.

**Lemma 3.5.** Let \((Y, u)\) be a \( G \)-closed \( S \)-subsemigroup of \( X \) in \( \mathcal{C} \). Define \( G' = \text{Aut}_{S/\ker\tau'}(Y) \), where \( \tau' : S \rightarrow [Y,Y]_\varphi \) is the \( S \)-semigroup structure near-semiring homomorphism. There is an embedding functor \( F : \mathcal{C}_{G'} \rightarrow \mathcal{C}_G \).

**Proof.** \( F \) can be obtained by defining a semigroup monomorphism \( \theta : G \rightarrow G' \). For that, let \( g \in G \); then \( g \in [X,X]_\varphi \), \( g \) is invertible and \( \tau(s)g = g\tau(s) \) for any \( s \in S \). Also, since \((Y, u)\) is \( G \)-closed there is unique \( f \in [Y,Y]_\varphi \) such that \( gu = uf \). Define \( \theta : G \rightarrow G' \) by setting \( \theta(g) = f \), for \( g \in G \). We shall ascertain that \( f \in G' \). Since \( \theta(1) \) is the identity morphism on \( Y \), it follows that \( \theta(g) \) is invertible. To show that \( f\tau'(\bar{s}) = \tau'(\bar{s})f \) for all \( \bar{s} \in S/\ker\tau' \), we have to prove that \( f\tau'(s) = \tau'(s)f \) for all \( s \in S \). Since \( u \) is \( S \)-homomorphism in \( \mathcal{C} \), we have:

\[
uf\tau'(s) = gu\tau'(s) = g\tau(s)u = \tau(s)gu = \tau(s)uf = u\tau'(s)f
\]

and thus \( f\tau'(s) = \tau'(s)f \) for all \( s \in S \). It is easy to see that \( \theta \) is a semigroup monomorphism, as desired.

Let \((A, \alpha) \in \mathcal{C}_G\), so that \( \alpha : G' \rightarrow [A,A]_\varphi \) is a semigroup homomorphism. Set \( F((A,\alpha)) = (A,\alpha\theta) \) so that \( F((A,\alpha)) \in \mathcal{C}_G \), and \( F \) is an embedding functor.

Consider the category \( \mathcal{D} = \langle \mathcal{C}_G, F(\mathcal{C}_{G'}) \rangle \) (cf. Example 2.8 for notation). Note that this is a category with finite products, final object, and there exists a forgetful functor. The object \( X_G \in \mathcal{C}_G \) can naturally be equipped with the
structure, \( X_\ast \), defined to be \((F(Y) \subseteq X_G)\), and the near-semiring homomorphism \( \tau : S \rightarrow [X_\ast, X_\ast]\) is defined by \( \tau(s) = \tau(s) \) for all \( s \in S \). Thus \( X_\ast \) is an \( S \)-semigroup object of \( \mathcal{D} \). If \( X \) is faithful in \( \mathcal{C} \) then \( X_\ast \) is faithful in \( \mathcal{D} \). Further, if \( X_G \) is \( \mathcal{C}_G \)-minimal, then in a similar way to that of Theorem 3.3, one can finalize that \( X_\ast \) is \( \mathcal{D} \)-minimal.

This can be summarized as the second approximation theorem as follows:

**Theorem 3.6.** The object \( X_\ast \) of \( \mathcal{D} \) is an \( S \)-semigroup object and if \( X \) is faithful then \( X_\ast \) is faithful. Moreover, if \( X \) is \( \mathcal{C} \)-minimal then \( X_\ast \) is \( \mathcal{D} \)-minimal.

Further, if \( X \) has \( G \)-closed \( S \)-subsemigroups in \( \mathcal{C} \) then each of which gives an approximation theorem in the following way.

**Theorem 3.7.** Let \((Y_i, u_i)\) be \( G \)-closed \( S \)-subsemigroups of \( X \) for \( i = 1, 2, \ldots, k \) and \( G_i = \text{Aut}_{S/\tau_i}(Y_i) \) for each \( i = 1, 2, \ldots, k \). Let \( F_i : \mathcal{C}_{G_i} \rightarrow \mathcal{C}_{G_i} \) be an appropriate embedding functor for each \( i = 1, 2, \ldots, k \). Consider the category \( \mathcal{D}_k = \langle \mathcal{C}_{G_i}, F_1(\mathcal{C}_{G_1}), F_2(\mathcal{C}_{G_2}), \ldots, F_k(\mathcal{C}_{G_k}) \rangle \).

If \( X \) is a \( \mathcal{C} \)-minimal, faithful \( S \)-semigroup in \( \mathcal{C} \), then \( X \) can be given the structure of a \( \mathcal{D}_k \)-minimal faithful \( S \)-semigroup in \( \mathcal{D}_k \).

**References**


