CONSTRUCTION OF $\mathbb{R}$

1. Motivation

We are used to thinking of real numbers as successive approximations. For example, we write

$$\pi = 3.14159\ldots$$

to mean that π is a real number which, accurate to 5 decimal places, equals the above string. To be precise, this means that $|\pi - 3.14159| < \frac{1}{2} \times 10^{-5}$. But this does not tell us what π *is*. If we want a more accurate approximation, we can calculate one; to 10 decimal places, we have $\pi = 3.1415926536\ldots$ Continuing, we will develop a **sequence of rational approximations to** π. One such sequence is

$$3, 3.1, 3.14, 3.142, 3.1416, 3.14159, \ldots$$

But this is not the only sequence of rational numbers that approximate π closer and closer (far from it!). For example, here is another (important) one:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{104348}{33215}, \frac{1043835}{332263}\ldots$$

(This sequence, which represents the *continued fractions* approximations to π [you should read about continued fractions for fun!], gives accuracies of $1, 3, 5, 6, 9, 9$ decimal digits, as you can readily check.) More's the point, here is another (somewhat arbitrary) rational approximating sequence to π:

$$0, 0, 157, -45, -10, 3.14159, 0, 3.14, 3.1415, 3.151592, 3.14159265, \ldots$$

While here we start with a string of numbers that really have nothing to do with π, the sequence eventually settles down to approximate π closer and closer (this time by an additional 2 decimal places each step). The point is, just what the sequence does for *any initial segment* of time is irrelevant; all that matters is what happens to the *tail* of the sequence.

We are going to use the above insights to actually give a *construction* of the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$. The idea is, a real number *is* a sequence of rational approximations. But we have to be careful since, as we saw above, very different sequences of rational numbers can equally well approximate the same real number. To take care of this ambiguity, we will have to define real numbers as **sets of rational approximating sequences, all with the same tail behaviour**. To make this precise, the following section contains precise definitions and some theorems which we will need to set out the formal mathematical construction of $\mathbb{R}$ (as envisioned by Bolzano and Cauchy in the early 19th century).

2. Cauchy Sequences

We have already been working with sequences, but to be sure we're on the same page:

Definition 1. *A* **sequence** *of rational numbers (aka a* **rational sequence***) is a function from the natural numbers $\mathbb{N}$ into the rational numbers $\mathbb{Q}$. That is, it is an assignment of a rational number to each natural number. We usually denote such a function by $n \mapsto a_n$, so the terms in the sequence are written $\{a_1, a_2, a_3, \ldots\}$ To refer to the whole sequence, we will write $(a_n)_{n=1}^{\infty}$, or for the sake of brevity simply (a_n).*

We are interested primarily in the tail behaviour of a sequence. There is a *wide* variety of different behaviours a sequence may have as we follow it along. The following are two important, related behaviours which we will need now, and which will also be important to us all term. The first is *tending to* 0. Put simply, a sequences tends to 0 if the tail gets (and stays) arbitrarily small. That is: given any small positive rational number ω, eventually the terms in the sequence are all $< \omega$ in absolute value.

Definition 2. *Let (a_n) be a rational sequence. Say that (a_n)* **tends to** 0 *if, given any small $\omega > 0$, there is a natural number $N = N_\omega$ such that, after N (i.e. for all $n > N$), $|a_n| \leq \omega$. We often denote this symbolically by $a_n \to 0$.*

A typical example given is the sequence $a_n = 1/n$ (though zillions of others will do, of course). Given any small rational $\omega > 0$, if we want $1/N < \omega$, we simply have to choose $N > 1/\omega$. Here we are directly using the fact that $\mathbb{Q}$ *is an Archimedian field*. Note, also, that in this case, since $1/n < 1/N$ whenever $n > N$, we will have that $|a_n| = 1/n < 1/N < \omega$. Thus, we have proves that the sequence $(1/n)$ tends to 0.

We can also, at this point, define what it means for a sequence to tend to any given rational number q: (a_n) tends to q if the sequence $(a_n - q)$ tends to 0. (For example: the sequence with terms $a_n = n/(n+1)$ tends to 1, since $a_n - 1 = -1/(n+1)$ which, following an argument like the one above, tends to 0.) We should note that another word used to describe a sequence tending to some number is **convergence**: we say that a $1/n$ *converges* to 0.

The next definition involves something closely related to convergence, but which is subtly different. Consider, for example, a sequence of rational approximations to π, such as $3, 3.1, 3.14, 3.142, 3.1416, 3.14159, \ldots$ This sequence is *not* converging to any rational number. But it *looks* like it's tending to something – the terms are getting closer and closer to each other (very quickly, in fact – by a factor of 10 at each successive step). The sequence is then said to be a **Cauchy sequence**.

Definition 3. *Let (a_n) be a rational sequence. We call (a_n) a* **Cauchy sequence** *if the difference between its terms tends to 0. To be precise: given any small rational number ω, there is a natural number $N = N_\omega$ such that for any $m, n > N$, $|a_n - a_m| < \omega$.*

The following theorem tells us that the notion of convergence is stronger than Cauchy-ness.

Theorem 4. *If (a_n) is a convergent rational sequence (that is, $a_n \to q$ for some rational number q), then (a_n) is a Cauchy sequence.*

Proof. We know that $a_n \to q$. Here is a ubiquitous trick: instead of using ω in the definition, start with an arbitrary small $\omega > 0$, and then choose N so that $|a_n - q| < \omega/2$ when $n > N$. Then if $m, n > N$, we have

$$|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$

This shows that (a_n) is a Cauchy sequence. □

Our intuition tells us that, if the terms get closer and closer to each other, they must be getting closer and closer to *something*. It is this intuition which motivated Cauchy to use such sequences to define the real numbers: $\mathbb{R}$ is a *completion* of $\mathbb{Q}$. The sequence

$3, 3.1, 3.14, 3.142, 3.1416, 3.14159, \ldots$ is Cauchy but does not converge to a rational number; hence, there must be some *irrational number* to which it converges, and we will complete the rationals by adding it (and all the other numbers to which Cauchy sequences look like they're tending).

The next theorem shows that, even though a Cauchy sequence may not converge, it certainly can't get arbitrarily large.

Theorem 5. *If (a_n) is a Cauchy sequence, then it is* **bounded***; that is, there is some large number M such that $|a_n| \leq M$ for all n.*

Proof. Since (a_n) is Cauchy, setting $\omega = 1$ we know that there is some N such that $|a_m - a_n| < 1$ whenever $m, n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as

$$a_{N+1} - 1 < a_n < a_{N+1} + 1.$$

This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list:

$$\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}.$$

Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then (as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$. Hence, M is a bound for the sequence. $\square$

We will use Theorem 5 in Section 4 to verify the field axioms for our newly-minted real numbers.

3. Equivalence Relations

We are now almost ready to discuss Cauchy's construction of the real number system. We will define a real number (almost) as: *a Cauchy sequence of rational numbers*. However, as discussed in Section 1, we need to deal with the ambiguity presented by different sequences converging to the same number. Hence, we will need to group Cauchy sequences into sets, all of which have the same tail behaviour, and we will define a real number to be such a *set of Cauchy sequences*.

The question is, how do we properly define the sets? This is a case where we have a large collection of objects, and we want to divide them into groups by similar behaviour. This situation comes up *extremely often* in mathematics. One method would be to label a collection of bins, and then toss all the elements into the different bins accordingly. The problem with this is that it can often be very difficult to come up with the labels: where does one even begin? In our case, we would need to label the bins explicitly with all the different possible tail behaviours of sequences, and that is an arduous task indeed. In general, there is a procedure for grouping objects which does not require labelling. Instead, we divide our objects into *equivalence classes*.

The important idea is the following: suppose we have already grouped the objects into bins. For example, we group all human beings into families. Then we notice the following very simple facts about families: first, any person is related to him- or herself, automatically. Now, if Sally is related to Tom, then Tom is related to Sally (we say that relation is *symmetric*). And finally, there is *transitivity*, which is an important way of recognizing relation: if Sally is related to Tom, and Tom is related to Rachel, then Sally is also related to Rachel.

These three properties of family relation are, in fact, all that is necessary to group a set into smaller, non-intersecting sets (as we'll see below). So we begin by abstracting the notion of family relation to **equivalence relation**.

Definition 6. *Let S be a set of (mathematical) objects. A* relation *among pairs of elements of S is said to be an* **equivalence relation** *if the following three properties hold:*

Reflexivity: for any $s \in S$, s is related to s.
Symmetry: for any $s, t \in S$, if s is related to t then t is related to s.
Transitivity: for any $s, t, r \in S$, if s is related to t and t is related to r, then s is related to r.

The following proposition goes most of the way to showing that an equivalence relation divides a set into bins.

Theorem 7. *Let S be a set, with an equivalence relation on pairs of elements. For $s \in S$, denote by $[s]$ the set of all elements in S that are related to s. Then for any $s, t \in S$, either $[s] = [t]$ or $[s]$ and $[t]$ are disjoint.*

Proof. Let s, t be any two elements of s. Let us assume that there is some element r which is in both $[s]$ and $[t]$; in other words, r is related to s and r is related to t. Now, let a be any element in $[s]$. Since a is related to s and s is related to r (by symmetry), by transitivity a is related to r. But r is also related to t, so (again by transitivity) a is related to t. This means that $a \in [t]$. We have thus demonstrated that $[s] \subseteq [t]$. But the other inclusion follows by almost exactly the same argument: if $b \in [t]$, then b is related to t, and since t is related to r (by symmetry), b is related to r by transitivity. But r is related to s, so by transitivity b is related to s. Hence, $b \in [s]$, and so $[t] \subseteq [s]$. Hence, $[t] = [s]$ if they have even one element in their intersection. $\square$

The sets $[s]$ for $s \in S$ are called the **equivalence classes**, and they are the bins.

The astute observer will notice that the above proof never used the reflexive property of the relation. In fact, the result holds perfectly well for relations that are symmetric and transitive but not reflexive. The only trouble is that without reflexivity, an element may not be related to *anything*. If so, that element won't be in any equivalence class – it won't get sorted into a bin at all. But reflexivity saves the day.

Corollary 8. *If S is a set with an equivalence relation on pairs of elements, then the equivalence classes are non-empty disjoint sets whose union is all of S.*

Proof. We have already seen, in Proposition 7 that different equivalence classes are disjoint. Now, by reflexivity, for each $s \in S$, $s \in [s]$, and so each equivalence class is nonempty (containing at least one element). Also, since each $s \in S$ is in an equivalence class, the union of all classes is the whole set S. $\square$

To summarize: any equivalence relation will divide a set into bins, without the need to explicitly label them. Moreover, as discussed above, if we already have a set divided into bins, then there is an equivalence relation lurking about: we declare to elements to be related if they are in the same bin. So equivalence relations are just another way of dividing sets into small pieces; any such division may be accomplished with an equivalence relation.

4. CAUCHY'S CONSTRUCTION OF $\mathbb{R}$

We now stand ready to give Cauchy's construction of $\mathbb{R}$. The real numbers will be constructed as *equivalence classes of Cauchy sequences*. Let $\mathscr{C}_\mathbb{Q}$ denote the set of all Cauchy sequences of rational numbers. We must define an equivalence relation on $\mathscr{C}_\mathbb{Q}$.

Definition 9. *Let (a_n) and (b_n) be in $\mathscr{C}_\mathbb{Q}$. Say they are equivalent (i.e. related) if $a_n - b_n \to 0$; i.e. if the sequence $(a_n - b_n)$ tends to 0.*

The point of this definition is precisely what we had in mind for the equivalence classes to represent real numbers. To say that $a_n - b_n \to 0$ means that the tails of the two sequences get, and stay, arbitrarily close to each other: *the two sequences have the same tail behaviour*. But to be precise, we need to make sure that, indeed, Definition 9 gives an equivalence relation.

Proposition 10. *Definition 9 yields an equivalence relation on $\mathscr{C}_\mathbb{Q}$.*

Proof. we need to show that this relation is reflexive, symmetric, and transitive.

- **Reflexive:** $a_n - a_n = 0$, and the sequence all of whose terms are 0 clearly converges to 0. So (a_n) is related to (a_n).
- **Symmetric:** Suppose (a_n) is related to (b_n), so $a_n - b_n \to 0$. But $b_n - a_n = -(a_n - b_n)$, and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 2, it follows that $b_n - a_n \to 0$ as well. Hence, (b_n) is related to (a_n).
- **Transitive:** Here we will use the $\omega/2$ trick we applied to prove Theorem 4. Suppose (a_n) is related to (b_n), and (b_n) is related to (c_n). This means that $a_n - b_n \to 0$ and $b_n - c_n \to 0$. To be fully precise, let us fix a small $\omega > 0$; then there exists an N such that for all $n > N$, $|a_n - b_n| < \omega/2$; also, there exists an M such that for all $n > M$, $|b_n - c_n| < \omega/2$. Well, then, as long as n is bigger than both N and M, we have that
$$|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \frac{\omega}{2} + \frac{\omega}{2} = \omega.$$
So, choosing L equal to the max of N, M, we see that given $\omega > 0$ we can always choose L so that for $n > L$, $|a_n - c_n| < \omega$. This means that $a_n - c_n \to 0$ – i.e. (a_n) is related to (c_n).

$\square$

So, we really have an equivalence relation, and so by Corollary 8, the set $\mathscr{C}_\mathbb{Q}$ is partitioned into disjoint subsets (equivalence classes). Now, hold your breath… here it comes…

Definition 11. *The* **real numbers** $\mathbb{R}$ *are the equivalence classes $[(a_n)]$ of Cauchy sequences of rational numbers, as per Definition 9. That is,* each such equivalence class is a real number.

Great! Now we know what real numbers are. Except… do we? We have a formal construction, which produces a set of objects. But the real numbers we know (and perhaps love) have lots of structure to them, as spelled out by the field axioms, order axioms, and least upper bound property, as spelled out in §1.5 – 1.19 in [1]. In fact, we need to work harder in order to recognize these equivalence classes as real numbers. In fact, we are used to $\mathbb{Q}$ being a subset of $\mathbb{R}$, but as we've constructed $\mathbb{R}$, it's hard to see $\mathbb{Q}$. Let's squint a little.

Definition 12. *Given any rational number q, define a real number $\tilde{q}$ to be the equivalence class of the sequence $(q, q, q, q, \ldots)$ consisting entirely of q.*

So we view $\mathbb{Q}$ as being inside $\mathbb{R}$ by thinking of each rational number q as its associated equivalence class $\tilde{q}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Okay, so we know where the rationals sit. But what about the algebraic structure of $\mathbb{R}$? Before we can verify field axioms, we need to decide what it means to add two equivalence classes of Cauchy sequences; to multiply them; what 1 and 0 are. Before we can verify the order axioms, we need to know what it means to say $[(a_n)] < [(b_n)]$ in the first place!

Definition 13. *Let $s, t \in \mathbb{R}$, so there are Cauchy sequences $(a_n), (b_n)$ of rational numbers with $s = [(a_n)]$ and $t = [(b_n)]$.*

(a) *Define $s + t$ to be the equivalence class of the sequence $(a_n + b_n)$.*
(b) *Define $s \cdot t$ to be the equivalence class of the sequence $(a_n \cdot b_n)$.*

Okay, so now we have all the main players, and all we need to do is verify the field axioms (to see that $\mathbb{R}$ is a field), right? Well, not quite. You see, there's a catch when you define things using equivalence relations. If you ever wish to define some operation on equivalence classes, you can't just willy-nilly define it for one member of the class and expect it to make sense for all members of the class. If we are thinking in terms of family relation, one could consider applying the property "is friendly with a Montague" to Juliet, which is surely true (she was *very* friendly with Romeo). However, this property most certainly does not extend to her father, head of the Capulets, who were practically at war with the Montagues!

We wish to define the real number $[(a_n)] + [(b_n)]$ to be simply $[(a_n + b_n)]$, à la Definition 13(a). In mathematical terms, the Montagues vs. Capulets problem is just this: the real number $[(a_n)]$ is represented by a zillion other sequences, like $[(c_n)]$ (for example, with $(a_n) = (1, 1, 1, 1, \ldots)$ and $(c_n) = (0.9, 0.99, 0.999, 0.999, \ldots)$). Similarly, $[(b_n)] = [(d_n)]$ for a zillion other (d_n)s. So, what makes us think that, just because $[(a_n)] = [(c_n)]$ and $[(b_n)] = [(d_n)]$ that $[(a_n + b_n)] = [(c_n + d_n)]$? In fact, there's no a priori reason we should think this is the case. We need to *prove* that this works, in order to show that the method of addition we've laid out is *well-defined*.

Proposition 14. *The operations $+, \cdot$ in Definition 13(a),(b) are well defined.*

Proof. Suppose that $[(a_n)] = [(c_n)]$ and $[(b_n)] = [(d_n)]$. Thus means that $a_n - c_n \to 0$ and $b_n - d_n \to 0$. Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n)$. Now, using the familiar $\omega/2$ trick, you can construct a proof that this tends to 0, and so $[(a_n + b_n)] = [(c_n + d_n)]$.

Multiplication is a little trickier; this is where we will use Theorem 5. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $[(a_n)] = [(c_n)]$ and $[(b_n)] = [(d_n)]$; we wish to show that $[(a_n \cdot b_n)] = [(c_n \cdot d_n)]$, or, in other words, that $a_n \cdot b_n - c_n \cdot d_n \to 0$. Well, we add and subtract one of the other cross terms, say $b_n \cdot c_n$:

$$\begin{aligned} a_n \cdot b_n - c_n \cdot d_n &= a_n \cdot b_n + (b_n \cdot c_n - b_n \cdot c_n) - c_n \cdot d_n \\ &= (a_n \cdot b_n - b_n \cdot c_n) + (b_n \cdot c_n - c_n \cdot d_n) \\ &= b_n \cdot (a_n - c_n) + c_n \cdot (b_n - d_n). \end{aligned}$$

Hence, we have $|a_n \cdot b_n - c_n \cdot d_n| \leq |b_n| \cdot |a_n - c_n| + |c_n| \cdot |b_n - d_n|$. Now, from Theorem 5, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all n. Taking some number R (for example $= M + L$) which is bigger than both, we have

$$|a_n \cdot b_n - c_n \cdot d_n| \leq |b_n| \cdot |a_n - c_n| + |c_n| \cdot |b_n - d_n| \leq R(|a_n - c_n| + |b_n - d_n|).$$

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\omega/2$ trick (actually, this time we'll want to use $\omega/2R$, I suggest you work out the details), we see that $a_n \cdot b_n - c_n \cdot d_n \to 0$. Whew! □

So, now that we have well-defined operations, it behooves us to prove that $\mathbb{R}$, equipped with them, is a field. This is a long and laborious task, and we shall not really work through it here; the interested reader should be in a position to prove the field axioms him- or herself! We will work out one slightly trickier example here, just to give a flavour.

Theorem 15. *Given any real number $s \neq 0$, there is a real number t such that $s \cdot t = 1$.*

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $(0, 0, 0, 0, \ldots)$. In other words, $s = [(a_n)]$ where $a_n - 0$ does *not* converge to 0. From this, we are to deduce the existence of a real number $t = [(b_n)]$ such that $s \cdot t = [(a_n \cdot b_n)]$ is the same equivalence class as $[(1, 1, 1, 1, \ldots)]$. Doing so is actually an easy consequence of the fact that nonzero rational numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. (a_n) does not tend to 0), there's no reason any number of the terms in (a_n) can't equal 0. However, it turns out that *eventually*, $a_n \neq 0$. That is,

Lemma 16. *If (a_n) is a Cauchy sequence which does not tend to 0, then there is an N such that, for $n > N$, $a_n \neq 0$.*

The proof of Lemma 16 is left to you. We will now use it to complete the proof of Theorem 15.

Let N be such that $a_n \neq 0$ for $n > N$. Define a sequence b_n of rational numbers as follows: for $n \leq N$, $b_n = 0$, and for $n > N$, $b_n = 1/a_n$; $(b_n) = (0, 0, \ldots, 0, 1/a_{N+1}, 1/a_{N+2}, \ldots)$. (This makes sense since, for $n > N$, a_n is a nonzero rational number, so $1/a_n$ exists.) Then $a_n \cdot b_n$ is equal to $a_n \cdot 0 = 0$ for $n \leq N$, and equals $a_n \cdot b_n = a_n \cdot 1/a_n = 1$ for $n > N$. Well, then, if we look at the sequence $(1, 1, 1, 1, \ldots)$, we have $(1, 1, 1, 1, \ldots) - (a_n \cdot b_n)$ is the sequence which is $1 - 0 = 0$ for $n \leq N$ and equals $1 - 1 = 0$ for $n > N$. Since this sequence is eventually equal to 0, it converges to 0, and so $[(a_n \cdot b_n)] = [(1, 1, 1, 1, \ldots)] = 1 \in \mathbb{R}$. This shows that $t = [(b_n)]$ is a multiplicative inverse to $s = [(a_n)]$.

□

Our choice of (b_n) to start with 0s was arbitrary; we could have chosen it to equal 17 until after the Nth stage, or chosen each term randomly from $\mathbb{Q}$; the only important choice is that $b_n = 1/a_n$ for $n > N$.

Following proofs that are generally easier than this one, we can show that $\mathbb{R}, +, \cdot$ is a field. Now, we want $\mathbb{R}$ to be an *ordered field*, which means there is an order relation $<$ on $\mathbb{R}$ which respects the field operations, as spelled out in §1.5 and 1.17 of [1]. Like before, having newly constructed these "real" numbers, we must define what it means for one to be less than another. We will take our intuition from rational approximating sequences again. How do we know that $\pi > 3.141592$? Well, 3 isn't greater than 3.141592, nor is 3.14.

3.142 is, as is 3.1416, but 3.14159 is not. However, eventually, after a certain stage in this approximating sequence to π (once we get past the 6th decimal digit), we find that *all the terms in the sequence after some point* are greater than 3.141592.

Definition 17. *Let $s \in \mathbb{R}$. Say that s is* positive *if $s \neq 0$, and if $s = [(a_n)]$ for some Cauchy sequence such that for some N, $a_n > 0$ for all $n > N$. Given two real numbers s, t, say that $s > t$ if $s - t$ is positive.*

Again, to be fully rigorous we would have to show that this definition is well-defined (that if any one Cauchy sequence in the equivalence class of s eventually has only positive terms, then any other Cauchy sequence in the same equivalence class eventually has only positive terms). This is an exercise similar to other proofs in these notes and on your homework; the interested reader can fill in the details.

Now, we can verify all of the order axioms hold for $\mathbb{R}$. As with the field axioms, this is a laborious task and is best done by each person in private. We will provide a proof of one of the axioms here, just for flavour.

Theorem 18. *Let s, t be real numbers such that $s > t$, and let $r \in \mathbb{R}$. Then $s + r > t + r$.*

Proof. Let $s = [(a_n)]$, $t = [(b_n)]$, and $r = [(c_n)]$. Since $s > t$, i.e. $s - t > 0$, we know that there is an N such that, for $n > N$, $a_n - b_n > 0$. So $a_n > b_n$ for $n > N$. Now, adding c_n to both sides of this inequality (as we know we can do for rational numbers), we have $a_n + c_n > b_n + c_n$ for $n > N$, or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N$. By Definition 17, this means that $s + r = [(a_n + c_n)] > [(b_n + c_n)] = t + r$. $\square$

Proofs of the other order axioms follow suit.

So, we have constructed an ordered field $\mathbb{R}$. In particular, now that we know when a real number is positive, we can define the absolute value of a real number just is we do in the rational case. This means that we can apply Definitions 2 and 3 to sequences of real numbers as well. We will do so in what follows.

The thing that really distinguished $\mathbb{R}$ from $\mathbb{Q}$ is the *least upper bound property*. To complete our discussion, we must show that this crazy set $\mathbb{R}$ of equivalence classes of Cauchy sequences really satisfies the least upper bound property! This takes a little work, but it is worth doing since the proof we're about to lay out is a prototype for many similar theorems you'll see in years to come.

Before getting directly to this proof, let us pause to prove that $\mathbb{R}$ is an Archimedean field. Of course, once we've proven that $\mathbb{R}$ has the least upper bound property, we will know it's Archimedean by a theorem proved in the first week of class; but we can actually use the Archimedean property to help prove the least upper bound property in this case.

Theorem 19. *$\mathbb{R}$ has the Archimedean property.*

Proof. Let $s, t > 0$ be real numbers. We need to find a natural number m so that $m \cdot s > t$. First, recall that, by m in this context, we mean $[(m, m, m, m, \ldots)]$. So, letting $s = [(a_n)]$ and $t = [(b_n)]$, what we need to show is that there exists m with

$$[(m, m, m, m, \ldots)] \cdot [(a_1, a_2, a_3, a_4, \ldots)] = [(ma_1, ma_2, ma_3, ma_4, \ldots)] > [(b_1, b_2, b_3, b_4, \ldots)].$$

Now, to say that $[(ma_n)] > [(b_n)]$, or $[(ma_n - b_n)]$ is positive, is, by Definition 17, just to say that there is N such that $ma_n - b_n > 0$ for all $n > N$. So, what we hope to show is:

$$\text{There exist } m, N \in \mathbb{N} \text{ so that } ma_n > b_n \text{ for all } n > N.$$

To produce a contradiction, we assume this is not the case; assume that

$$\text{for every } m \text{ and } N\text{, there exists an } n > N \text{ so that } ma_n \le b_n. \qquad (*)$$

Now, since (b_n) is a Cauchy sequence, by Theorem 5 it is bounded – there is a rational number M such that $b_n \le M$ for all n. Now, by the Archimedean property *for the rational numbers*, given any small rational number $\omega > 0$ there is an m such that $M/m < \omega/2$. Fix such an m. Then if $ma_n \le b_n$, we have $a_n \le b_n/m \le M/m < \omega/2$.

Now, (a_n) is a Cauchy sequence, and so there exists N so that for $n, k > N$, $|a_n - a_k| < \omega/2$. By Asumption ($*$), we also have an $n > N$ such that $ma_n \le b_n$, which means that $a_n < \omega/2$. But then for every $k > N$, we have that $a_k - a_n < \omega/2$, so $a_k < a_n + \omega/2 < \omega/2 + \omega/2 = \epsilon$. Hence, $a_k < \omega$ for all $k > N$. This proves that $a_k \to 0$, which (by Definition 17) contradicts the fact that $[(a_n)] = s > 0$.

□

It will be handy to have one more Theorem about how the rationals and reals compare before we proceed. This theorem is known as **the density of $\mathbb{Q}$ in $\mathbb{R}$**, and it follows almost immediately from the construction or $\mathbb{R}$ from $\mathbb{Q}$.

Theorem 20. *Given any real number r, and any small (rational) number $\omega > 0$, there is a rational number q such that $|r - q| < \omega$.*

Proof. The real number r is represented by a Cauchy sequence $(a_1, a_2, a_3, \ldots)$. Since this sequence is Cauchy, given ω, there is N so that for all $m, n > N$, $|a_n - a_m| < \omega$. Picking some fixed $\ell > N$, we can take the rational number q given by $q = [(a_\ell, a_\ell, a_\ell, \ldots)]$. Then we have $r - q = [(a_n - a_\ell)_{n=1}^\infty]$, and $q - r = [(a_\ell - a_n)_{n=1}^\infty]$. Now, since $\ell > N$, we see that for $n > N$, $a_n - a_\ell < \omega$ and $a_\ell - a_n < \omega$, which means (by Definition 17) that $r - q < \omega$ and $q - r < \omega$; hence, $|r - q| < \omega$. □

We really shouldn't be surprised by this proof; we defined real numbers to be (essentially) approximating sequences of rationals, so we should expect to be able to approximate reals by rationals!

And now…

Theorem 21. *$\mathbb{R}$ has the least upper bound property.*

The proof of the theorem is fairly complicated, and we will divide it into several smaller Lemmas. First, Let $S \subset \mathbb{R}$ be a nonempty subset, and let M be an upper bound for S. We are going to construct two *sequences of real numbers*, (u_n) and (ℓ_n). First, since S is nonempty, there is some element $s_0 \in S$. Now, we go through the following inductive procedure to produce numbers $u_0, u_1, u_2, \ldots$ and $\ell_1, \ell_2, \ell_3, \ldots$

- Set $u_0 = M$ and $\ell_0 = s_0$.
- Suppose that we have already defined u_n and ℓ_n. Consider the number $m_n = (u_n + \ell_n)/2$, the average between u_n and ℓ_n.
  - If m_n is an upper bound for S, define $u_{n+1} = m_n$ and $\ell_{n+1} = \ell_n$.
  - If m_n is not an upper bound for S, define $u_{n+1} = u_n$ and $\ell_{n+1} = m_n$.

Since $s_0 < M$, it is easy to prove by induction that (u_n) is a non-increasing sequence $(u_{n+1} \leq u_n)$ and (ℓ_n) is a non-decreasing sequence $(\ell_{n+1} \geq \ell_n)$. This gives us

Lemma 22. *(u_n) and (ℓ_n) are Cauchy sequences of real numbers.*

Proof. Note that each $\ell_n \leq M$ for all n. Since (ℓ_n) is non-decreasing, it follows that (ℓ_n) is Cauchy (this is left for you to show; consider the proof of Theorem 3.14 in [1]). For (u_n), we have $u_n \geq s_0$ for all n, and so $-u_n \leq -s_0$. Since (u_n) is non-increasing, $(-u_n)$ is non-decreasing, and so as above, $(-u_n)$ is Cauchy. It is easy to verify that, therefore, (u_n) is Cauchy. □

A Cauchy sequence of rational numbers looks like it must be tending to something, but may not (as discussed above). This doesn't happen in $\mathbb{R}$; in fact, that is the whole point of the construction. To bring this home, the following Lemma shows that (u_n) does tend to a real number.

Lemma 23. *There is a real number u such that $u_n \to u$.*

Proof. Fix a term u_n in the sequence (u_n). By Theorem 20, there is a rational number q_n such that $|u_n - q_n| < 1/n$. Consider the sequence $(q_1, q_2, q_3, \ldots)$ of rational numbers. We will show this sequence is Cauchy. Fix $\omega > 0$. By the Archimedean property, choose N so that $1/N < \omega/3$. We know, since (u_n) is Cauchy, that there is an M such that for $n, m > M$, $|u_n - u_m| < \omega/3$. Then, so long as $n, m > \max\{N, M\}$, we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \frac{\omega}{3} + \frac{\omega}{3} + \frac{\omega}{3} = \omega.$$

Thus, (q_n) is a Cauchy sequence of rational numbers, and so it represents a real number $u = [(q_n)]$. We must show that $u_n - u \to 0$, but this is practically built into the definition of u. To be precise, letting $\tilde{q}_n$ be the real number $[(q_n, q_n, q_n, \ldots)]$, we see immediately that $\tilde{q}_n - u \to 0$ (this is precisely equivalent to the statement that (q_n) is Cauchy). But $u_n - \tilde{q}_n < 1/n$ by construction; the reader may readily verify the assertion that if a sequence $\tilde{q}_n \to u$ and $u_n - \tilde{q}_n \to 0$, then $u_n \to u$. □

So (u_n), a non-increasing sequence of upper bounds for S, tends to a real number u. As you've guessed, u is the least upper bound of our set S. To prove this, we need one more lemma.

Lemma 24. *$\ell_n \to u$.*

Proof. First, note in the first case above, we have that

$$u_{n+1} - \ell_{n+1} = m_n - \ell_n = \frac{u_n + \ell_n}{2} - \ell_n = \frac{u_n - \ell_n}{2}.$$

In the second case, we also have

$$u_{n+1} - \ell_{n+1} = u_n - m_n = u_n - \frac{u_n + \ell_n}{2} = \frac{u_n - \ell_n}{2}.$$

Now, this means that $u_1 - \ell_1 = \frac{1}{2}(M - s)$, and so $u_2 - \ell_2 = \frac{1}{2}(u_1 - \ell_1) = (\frac{1}{2})^2(L - s)$, and in general (as you can easily prove by induction), $u_n - \ell_n = 2^{-n}(L - s)$. Since $L > s$ so $L - s > 0$, and since $2^{-n} < 1/n$, by the Archimedean property for $\mathbb{R}$, we have for any $\omega > 0$ that $2^{-n}(L - s) < \omega$ for all sufficiently large n. Thus, $u_n - \ell_n < 2^{-n}(L - s) < \omega$ as well, and so $u_n - \ell_n \to 0$. Again, the reader should verify that, since $u_n \to u$, we have $\ell_n \to u$ as well. □

Proof of Theorem 21. First, we show that u is an upper bound. Well, suppose it is not, so that $u < s$ for some $s \in S$. Then $\omega \equiv s - u$ is > 0, and since $u_n \to u$ and is non-increasing, there must be an n so that $u_n - u < \omega$, meaning that $u_n < u + \omega = u + (s - u) = s$. Since u_n is an upper bound for S, however, this is a contradiction. Hence, u is an upper bound for S.

Now, we also know that, for each n, ℓ_n is not an upper bound, meaning that for each n, there is an $s_n \in S$ so that $\ell_n \leq s_n$. Lemma 24 tells us that $\ell_n \to u$, and since the sequence (ℓ_n) is non-decreasing, this means that for each $\omega > 0$, there is an N so that for $n > N$, $\ell_n > u - \omega$. Hence, for $n > N$, $s_n \geq \ell_n > u - \omega$ as well. In particular, for each $\omega > 0$, there is an element $s \in S$ such that $s > u - \omega$. This means that no number smaller than u can be an upper bound for S. Hence, u is the least upper bound for S: $\sup s$ exists. Hurrah! $\square$

This concludes our construction of these unreal Réal numbers. Remember: they are only real (in the vernacular sense) inasumch as they are well-approximated by rational numbers. In any case, we can now rest assured that the declaration of Theorem 1.19 in [1] (the existence of the real numbers) is valid; there really does exist (abstractly) an ordered-field with the least upper bound property. Good times!

REFERENCES

[1] Rudin, W.: *Principles of Mathematical Analysis.* Third Edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York – Aukland – Düsseldorf, 1976.