Problems for Section 4.4

Practice Problems

Problem 4.15.
The *inverse* R^{-1} of a binary relation R from A to B is the relation from B to A defined by:

$$b \; R^{-1} \; a \quad \text{iff} \quad a \; R \; b.$$

In other words, you get the diagram for R^{-1} from R by "reversing the arrows" in the diagram describing R. Now many of the relational properties of R correspond to different properties of R^{-1}. For example, R is *total* iff R^{-1} is a *surjection*.

Fill in the remaining entries is this table:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	
an injection	
a bijection	

Hint: Explain what's going on in terms of "arrows" from A to B in the diagram for R.

Problem 4.16.
Describe a total injective function [$= 1$ out], [≤ 1 in,] from $\mathbb{R} \to \mathbb{R}$ that is not a bijection.

Problem 4.17.
For a binary relation $R : A \to B$, some properties of R can be determined from just the arrows of R, that is, from graph(R), and others require knowing if there are elements in the domain A or the codomain B that don't show up in graph(R). For each of the following possible properties of R, indicate whether it is always determined by

1. graph(R) alone,
2. graph(R) and A alone,
3. graph(R) and B alone,

4. all three parts of R.

Properties:

(a) surjective

(b) injective

(c) total

(d) function

(e) bijection

Problem 4.18.
For each of the following real-valued functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

(a) $x \to x + 2$

(b) $x \to 2x$

(c) $x \to x^2$

(d) $x \to x^3$

(e) $x \to \sin x$

(f) $x \to x \sin x$

(g) $x \to e^x$

Problem 4.19.
Let $f : A \to B$ and $g : B \to C$ be functions and $h : A \to C$ be their composition, namely, $h(a) ::= g(f(a))$ for all $a \in A$.

(a) Prove that if f and g are surjections, then so is h.

(b) Prove that if f and g are bijections, then so is h.

(c) If f is a bijection, then so is f^{-1}.

Problem 4.20.
Give an example of a relation R that is a total injective function from a set A to itself but is not a bijection.

Class Problems

Problem 4.21. **(a)** Prove that if A surj B and B surj C, then A surj C.

(b) Explain why A surj B iff B inj A.

(c) Conclude from (a) and (b) that if A inj B and B inj C, then A inj C.

(d) According to Definition 4.5.2, A inj B requires a total injective *relation*. Explain why A inj B iff there is a total injective *function* from A to B.

Problem 4.22.
Five basic properties of binary relations $R : A \to B$ are:

1. R is a surjection [$\geq$ 1 in]
2. R is an injection [$\leq$ 1 in]
3. R is a function [$\geq$ 1 out]
4. R is total [$\geq$ 1 out]
5. R is empty [$=$ 0 out]

Below are some assertions about R. For each assertion, indicate all the properties above that the relation R must have. For example, the first assertion impllies that R is a total surjection. Variables $a, a_1, \ldots$ range over A and $b, b_1, \ldots$ range over B.

(a) $\forall a \, \forall b. \; a \; R \; b$.

(b) $\text{NOT}(\forall a \, \forall b. \; a \; R \; b)$.

(c) $\forall a \, \forall b. \; \text{NOT}(a \; R \; b)$.

(d) $\forall a \, \exists b. \; a \; R \; b$.

(e) $\forall b \, \exists a. \; a \; R \; b$.

(f) R is a bijection.

(g) $\forall a \, \exists b_1 \; a \; R \; b_1 \bigwedge \forall b. \; a \; R \; b \text{ IMPLIES } b = b_1$.

3. $x \neq y$ IMPLIES $f(x) \neq f(y)$
4. $f(x) = f(y)$ IMPLIES $x = y$
5. NOT$[\exists x \exists y (x \neq y$ AND $f(x) = f(y))$
6. NOT$[\exists z \forall x (f(x) \neq z)$
7. $\exists g \forall x (g(f(x)) = x)$
8. $\exists g \forall x (f(g(x)) = x)$

Problem 4.32.
Prove that if relation $R : A \to B$ is a total injection, [≥ 1 out], [≤ 1 in], then

$$R^{-1} \circ R = \text{Id}_A,$$

where Id_A is the identity function on A.
(A simple argument in terms of "arrows" will do the job.)

Problem 4.33.
Let $R : A \to B$ be a binary relation.
(a) Prove that R is a function iff $R \circ R^{-1} \subseteq \text{Id}_B$.
Write similar containment formulas involving $R^{-1} \circ R$, $R \circ R^{-1}$, Id_a, Id_B equivalent to the assertion that R has each of the following properties. No proof is required.

(b) total.

(c) a surjection.

(d) a injection.

Problem 4.34.
Let $R : A \to B$ and $S : B \to C$ be binary relations such that $S \circ R$ is a bijection and $|A| = 2$.

Give an example of such R, S where neither R nor S is a function. Indicate exactly which properties—total, surjection, function, and injection—your examples of R and S have.

Hint: Let $|B| = 4$.

Problem 4.35.
The set $\{1, 2, 3\}^{\omega}$ consists of the **infinite** sequences of the digits 1,2, and 3, and likewise $\{4, 5\}^{\omega}$ is the set of infinite sequences of the digits 4,5. For example

$$\begin{aligned} &\texttt{123123123...} &&\in \{1, 2, 3\}^{\omega}, \\ &\texttt{222222222222...} &&\in \{1, 2, 3\}^{\omega}, \\ &\texttt{4554445554444...} &&\in \{4, 5\}^{\omega}. \end{aligned}$$

(a) Give an example of a total injective function

$$f : \{1, 2, 3\}^{\omega} \to \{4, 5\}^{\omega}.$$

(b) Give an example of a bijection $g : (\{1, 2, 3\}^{\omega} \times \{1, 2, 3\}^{\omega}) \to \{1, 2, 3\}^{\omega}$.

(c) Explain why there is a bijection between $\{1, 2, 3\}^{\omega} \times \{1, 2, 3\}^{\omega}$ and $\{4, 5\}^{\omega}$. (You need not explicitly define the bijection.)

Problems for Section 4.5

Practice Problems

Problem 4.36.
Assume $f : A \to B$ is total function, and A is finite. Replace the $\star$ with one of $\leq, =, \geq$ to produce the *strongest* correct version of the following statements:

(a) $|f(A)| \star |B|$.

(b) If f is a surjection, then $|A| \star |B|$.

(c) If f is a surjection, then $|f(A)| \star |B|$.

(d) If f is an injection, then $|f(A)| \star |A|$.

(e) If f is a bijection, then $|A| \star |B|$.

Class Problems

Problem 4.37.
Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.