Problem 15.6. (a) How many of the billion numbers in the integer interval $[1..10^9]$ contain the digit 1? (*Hint:* How many don't?)

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected, and 15-bit strings with exactly 6 ones.

Problem 15.7.

(a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n. \tag{15.11}$$

That is

$$\mathcal{S}_{n,k} ::= \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid (15.11) \text{ is true}\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

(b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \{(y_1, y_2, \ldots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Problem 15.8.
An n-vertex *numbered tree* is a tree whose vertex set is $[1..n]$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers in $[1..n]$ obtained by the following recursive process:[6]

> If there are more than two vertices left, write down the *father* of the largest leaf, delete this *leaf*, and continue this process on the resulting smaller tree. If there are only two vertices left, then stop—the code is complete.

For example, the codes of a couple of numbered trees are shown in the Figure 15.7.

[6]The necessarily unique node adjacent to a leaf is called its *father*.

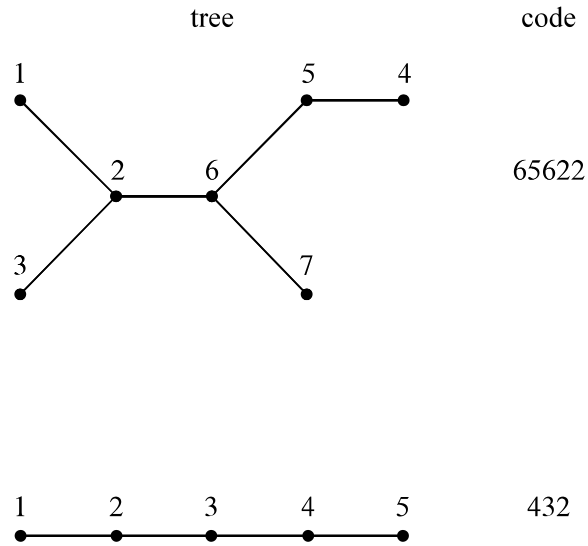

Figure 15.7

(a) Describe a procedure for reconstructing a numbered tree from its code.

(b) Conclude there is a bijection between the n-vertex numbered trees and sequences $(n-2)$ integers in $[1..n]$. State how many n-vertex numbered trees there are.

Problem 15.9.
Let X and Y be finite sets.

(a) How many binary relations from X to Y are there?

(b) Define a bijection between the set $[X \to Y]$ of all total functions from X to Y and the set $Y^{|X|}$. (Recall Y^n is the Cartesian product of Y with itself n times.) Based on that, what is $|[X \to Y]|$?

(c) Using the previous part, how many *functions*, not necessarily total, are there from X to Y? How does the fraction of functions vs. total functions grow as the size of X grows? Is it $O(1)$, $O(|X|)$, $O(2^{|X|})$,...?

(d) Show a bijection between the powerset $\text{pow}(X)$ and the set $[X \to \{0,1\}]$ of 0-1-valued total functions on X.

(e) Let X be a set of size n and B_X be the set of all bijections from X to X.

Problem 15.17.
A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22,369,621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ can't be an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Problem 15.18.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Problem 15.19. **(a)** There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

(b) How many nonnegative integer solutions are there for the following equality?

$$x_1 + x_2 + \cdots + x_m = k. \tag{15.13}$$

(c) How many nonnegative integer solutions are there for the following inequality?

$$x_1 + x_2 + \cdots + x_m \leq k. \tag{15.14}$$

(d) How many length m weakly increasing sequences of nonnegative integers $\leq k$ are there?

Homework Problems

Problem 15.20.
This problem is about binary relations on the set of integers in the interval $[1..n]$ and digraphs whose vertex set is $[1..n]$.

(a) How many digraphs are there?

(b) How many simple graphs are there?

(c) How many asymmetric binary relations are there?

(d) How many linear strict partial orders are there?

Problem 15.21.
Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels `a`, `e`, `i`, `o`, `u` appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) In how many different ways can $2n$ students be paired up?

(d) Two n-digit sequences of digits $0,1,\ldots,9$ are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences `03088929` and `00238899` are the same type. How many types of n-digit sequences are there?

Problem 15.29.
Let p be a **prime number**.

(a) Explain why the multinomial coefficient

$$\binom{p}{k_1, k_2, \ldots, k_n}$$

is divisible by p if all the k_i's are nonnegative integers less than p.

(b) Conclude from part (a) that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}. \tag{15.15}$$

(Do not prove this using Fermat's "little" Theorem. The point of this problem is to offer an independent proof of Fermat's theorem.)

(c) Explain how (15.15) immediately proves Fermat's Little Theorem 9.10.8:

$$n^{p-1} \equiv 1 \pmod{p}$$

when n is not a multiple of p.

Homework Problems

Problem 15.30.
The *degree sequence* of a simple graph is the weakly decreasing sequence of degrees of its vertices. For example, the degree sequence for the 5-vertex numbered tree pictured in the Figure 15.7 in Problem 15.8 is $(2, 2, 2, 1, 1)$ and for the 7-vertex tree it is $(3, 3, 2, 1, 1, 1, 1)$.

We're interested in counting how many numbered trees there are with a given degree sequence. We'll do this using the bijection defined in Problem 15.8 between n-vertex numbered trees and length $n-2$ code words whose characters are integers between 1 and n.

The *occurrence number* for a character in a word is the number of times that the character occurs in the word. For example, in the word `65622`, the occurrence number for `6` is two, and the occurrence number for `5` is one. The *occurrence sequence* of a word is the weakly decreasing sequence of occurrence numbers of characters in the word. The occurrence sequence for this word is $(2, 2, 1)$ because it has two occurrences of each of the characters `6` and `2`, and one occurrence of `5`.

(a) There is a simple relationship between the degree sequence of an n-vertex numbered tree and the occurrence sequence of its code. Describe this relationship and explain why it holds. Conclude that counting n-vertex numbered trees with a

(a) Prove that there exist a nonnegative integer n such that

$$3^n \equiv 1 pmod\, 10^{2014}.$$

Hint: Use pigeonhole principle or Euler's theorem.

(b) Conclude that there exist a natural number n such that 3^n has at least 2013 consecutive zeros.

Problem 15.41. **(a)** Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an n-card deck with the number of 4-card sequences.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall's Theorem and degree-constrained (12.5.5) graphs.

Problem 15.42.
The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Problem 15.43.
Suppose $2n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 4n\}$. Using the Pigeonhole Principle, show that there must be two selected numbers whose difference is 2. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

Problem 15.44.
Let

$$k_1, k_2, \ldots, k_{101}$$

be a sequence of 101 integers. A sequence

$$k_{m+1}, k_{m+2}, \ldots, k_n$$