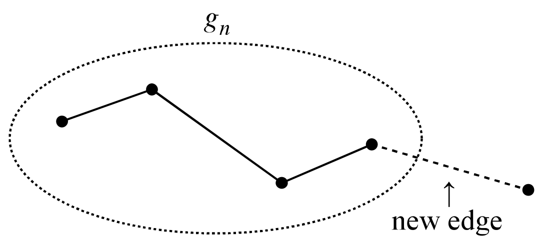
Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n + 1$. Let G_n be any two-ended graph with n edges. By the induction assumption, G_n is a line graph. Now suppose that we create a two-ended graph G_{n+1} by adding one more edge to G_n. This can be done in only one way: the new edge must join one of the two endpoints of G_n to a new vertex; otherwise, G_{n+1} would not be two-ended.



Clearly, G_{n+1} is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n + 1$ edges, which completes the proof by induction.

■

Problem 12.11.
If G is any simple graph, then a graph isomorphism from G to the same graph G is called a *graph automorphism*[11]. As a simple example, the identity function $\text{id} : V(G) \to V(G)$ is always a graph automorphism.

(a) If D is the *Dürer graph* pictured in Figure 12.25, briefly describe a graph automorphism of D that is *not* the identity function.

(b) Define a relation R on $V(G)$ by declaring that $v \; R \; w$ precisely when there exists a graph automorphism f of G with $f(v) = w$. In the special case of the Dürer graph, prove that $1 \; R \; 10$.

Hint: Try to map $1, 2, 3$ to $10, 11, 12$, respectively. Where must the other vertices go?

(c) In the Dürer graph, prove that $\text{NOT}(1 \; R \; 4)$.

Hint: Length 3 cycles.

(d) Prove carefully that for any simple graph G (not necessarily the Dürer graph), the relation R defined above is an equivalence relation.

[11] So-named because "auto" means "self", so an automorphism is a "self-isomorphism."

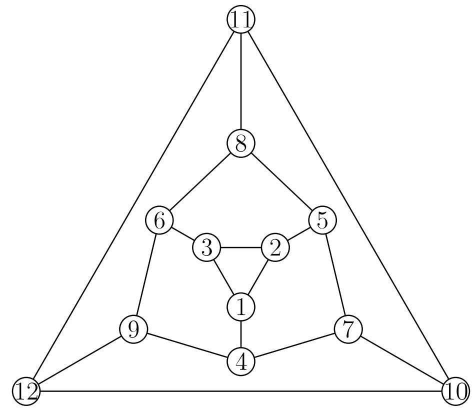


Figure 12.25 The Dürer graph, D.

Hint: If f and g are graph automorphisms, prove that $g \circ f$ is, too.

(e) Because R is an equivalence relation, it partitions the vertices into equivalence classes.[12] What are these equivalence classes for the Dürer graph? How do you know?

Hint: There are only two classes.

Problems for Section 12.5

Practice Problems

Problem 12.12.
Let B be a bipartite graph with vertex sets $L(B), R(B)$. Explain why the sum of the degrees of the vertices in $L(B)$ equals the sum of the degrees of the vertices in $R(B)$.

Class Problems

Problem 12.13.
A certain Institute of Technology has a lot of student clubs; these are loosely overseen by the Student Association. Each eligible club would like to delegate one of its members to appeal to the Dean for funding, but the Dean will not allow a student to

[12] Nodes in the same equivalence class can be thought of, informally, as having the "same role" in the graph, since you can move one to the other through an isomorphism.

be the delegate of more than one club. Fortunately, the Association VP took Math for Computer Science and recognizes a matching problem when she sees one.

(a) Explain how to model the delegate selection problem as a bipartite matching problem. (This is a *modeling problem*; we aren't looking for a description of an algorithm to solve the problem.)

(b) The VP's records show that no student is a member of more than 9 clubs. The VP also knows that to be eligible for support from the Dean's office, a club must have at least 13 members. That's enough for her to guarantee there is a proper delegate selection. Explain. (If only the VP had taken an *Algorithms* class, she could even have found a delegate selection without much effort.)

Problem 12.14.
A simple graph is called *regular* when every vertex has the same degree. Call a graph *balanced* when it is regular and is also a bipartite graph with the same number of left and right vertices.

Prove that if G is a balanced graph, then the edges of G can be partitioned into blocks such that each block is a perfect matching.

For example, if G is a balanced graph with $2k$ vertices each of degree j, then the edges of G can be partitioned into j blocks, where each block consists of k edges, each of which is a perfect matching.

Exam Problems

Problem 12.15.
Marvel is staging 4 test screenings of *Avengers:* ∞ *War* exclusively for a random selection of MIT students![13] For scheduling purposes, each of the selected students will specify which of the four screenings don't conflict with their schedule–every student is available for at least two out of the four screenings. However, each screening has only 20 available seats, not all of which need to be filled each time. Marvel is thus faced with a difficult scheduling problem: how do they make sure each of the chosen students is able to find a seat at a screening? They've recruited you to help solve this dilemma.

(a) Describe how to model this situation as a matching problem. Be sure to specify what the vertices/edges should be and briefly describe how a matching would determine seat assignments for each student in a screening for which they are available. (This is a *modeling problem*; we aren't looking for a description of an algorithm to

[13] Sadly this isn't actually happening, as far as we know.

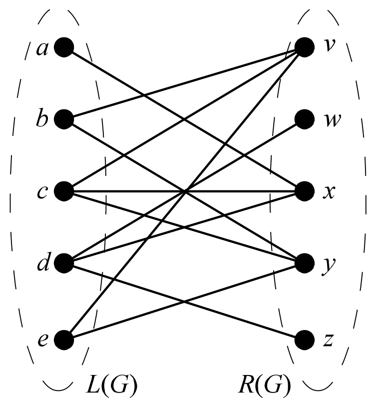


Figure 12.26 Bipartite graph G.

Homework Problems

Problem 12.19.

A *Latin square* is $n \times n$ array whose entries are the number $1, \ldots, n$. These entries satisfy two constraints: every row contains all n integers in some order, and also every column contains all n integers in some order. Latin squares come up frequently in the design of scientific experiments for reasons illustrated by a little story in a footnote.[14]

[14] At Guinness brewery in the eary 1900’s, W. S. Gosset (a chemist) and E. S. Beavan (a “maltster”) were trying to improve the barley used to make the brew. The brewery used different varieties of barley according to price and availability, and their agricultural consultants suggested a different fertilizer mix and best planting month for each variety.

Somewhat sceptical about paying high prices for customized fertilizer, Gosset and Beavan planned a season long test of the influence of fertilizer and planting month on barley yields. For as many months as there were varieties of barley, they would plant one sample of each variety using a different one of the fertilizers. So every month, they would have all the barley varieties planted and all the fertilizers used, which would give them a way to judge the overall quality of that planting month. But they also wanted to judge the fertilizers, so they wanted each fertilizer to be used on each variety during the course of the season. Now they had a little mathematical problem, which we can abstract as follows.

Suppose there are n barley varieties and an equal number of recommended fertilizers. Form an $n \times n$ array with a column for each fertilizer and a row for each planting month. We want to fill in the entries of this array with the integers $1,\ldots,n$ numbering the barley varieties, so that every row contains all n integers in some order (so every month each variety is planted and each fertilizer is used), and also every column contains all n integers (so each fertilizer is used on all the varieties over the course of the growing season).

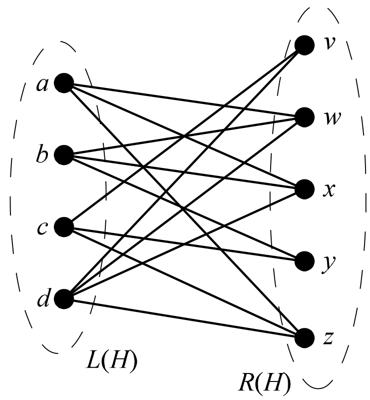


Figure 12.27 Bipartite Graph H.

For example, here is a 4×4 Latin square:

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

(a) Here are three rows of what could be part of a 5×5 Latin square:

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Fill in the last two rows to extend this “Latin rectangle” to a complete Latin square.

(b) Show that filling in the next row of an $n \times n$ Latin rectangle is equivalent to finding a matching in some $2n$-vertex bipartite graph.

(c) Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.

Problem 12.20.
Take a regular deck of 52 cards. Each card has a suit and a value. The suit is one of four possibilities: heart, diamond, club, spade. The value is one of 13 possibilities, $A, 2, 3, \ldots, 10, J, Q, K$. There is exactly one card for each of the 4×13 possible combinations of suit and value.

Ask your friend to lay the cards out into a grid with 4 rows and 13 columns. They can fill the cards in any way they'd like. In this problem you will show that you can always pick out 13 cards, one from each column of the grid, so that you wind up with cards of all 13 possible values.

(a) Explain how to model this trick as a bipartite matching problem between the 13 column vertices and the 13 value vertices. Is the graph necessarily degree-constrained?

(b) Show that any n columns must contain at least n different values and prove that a matching must exist.

Problem 12.21.
Scholars through the ages have identified *twenty* fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly *eight* of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select *one* additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Hint: Look for a matching in an appropriately defined bipartite graph. Be sure to clearly specify your (left and right) vertices and edges.

Problem 12.22.
Suppose n teams play in a round-robin tournament. Each day, each team will play a match with another team. Over a period of $n - 1$ days, every team plays every other team exactly once. There are no ties. Show that for each day we can select a winning team, without selecting the same team twice.[15]

Hint: Define a bipartite graph G with $L(G)$ the set of days and $R(G)$ the set of teams. For any set D of days, there may, or may not, have been a team that lost on all of those days.

[15] Based on 2012 Putnam Exam problem B3.

- R3: Goose, Merlin
- R4: Slider, Stinger, Cougar
- R5: Slider, Jester, Viper
- R6: Jester, Merlin
- R7: Jester, Stinger
- R8: Goose, Merlin, Viper

Two recitations can not be held in the same 90-minute time slot if some staff member is assigned to both recitations. The problem is to determine the minimum number of time slots required to complete all the recitations.

(a) Recast this problem as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

(b) Show a coloring of this graph using the fewest possible colors; explain why no fewer colors will work. What schedule of recitations does this imply?

Problem 12.27.

This problem generalizes the result proved Theorem 12.6.3 that any graph with maximum degree at most w is $(w+1)$-colorable.

A simple graph G is said to have *width* w iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most w vertices that precede it in the sequence. If the degree of every vertex is at most w, then the graph obviously has width at most w—just list the vertices in any order.

(a) Prove that every graph with width at most w is $(w+1)$-colorable.

(b) Describe a 2-colorable graph with minimum width n.

(c) Prove that the average degree of a graph of width w is at most $2w$.

(d) Describe an example of a graph with 100 vertices, width 3, but *average* degree more than 5.

Problem 12.28.

A sequence of vertices of a graph has *width* w iff each vertex is adjacent to at most w vertices that precede it in the sequence. A simple graph G has width w if there is a width-w sequence of all its vertices.

(a) Explain why the width of a graph must be at least the minimum degree of its vertices.

(b) Prove that if a finite graph has width w, then there is a width-w sequence of all its vertices that ends with a minimum degree vertex.

(c) Describe a simple algorithm to find the minimum width of a graph.

Problem 12.29.
Let G be a simple graph whose vertex degrees are all $\leq k$. Prove by induction on number of vertices that if every connected component of G has a vertex of degree strictly less than k, then G is k-colorable.

Problem 12.30.
A basic example of a simple graph with chromatic number n is the complete graph on n vertices, that is $\chi(K_n) = n$. This implies that any graph with K_n as a subgraph must have chromatic number at least n. It's a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no *triangle*—length three cycle—and hence no subgraph isomorphic to K_n for $n \geq 3$. Namely, let G be the 11-vertex graph of Figure 12.28. The reader can verify that G is triangle-free.

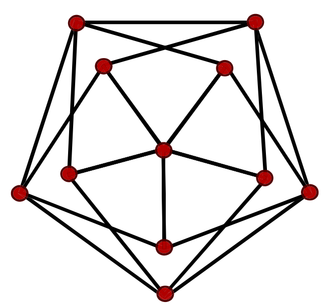


Figure 12.28 Graph G with no triangles and $\chi(G) = 4$.

(a) Show that G is 4-colorable.

(b) Prove that G can't be colored with 3 colors.