Monoids of non-halting programs with tests

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Abstract. In order to study the axiomatization of the `if-then-else` construct over possibly non-halting programs and tests, the notion of *C-sets* was introduced in the literature by considering the tests from an abstract C-algebra. This paper extends the notion of C-sets to C-monoids which include the composition of programs as well as composition of programs with tests. For the class of C-monoids where the C-algebras are adas a canonical representation in terms of functional C-monoids is obtained.

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Introduction

The algebraic properties of the program construct `if-then-else` have been studied in great detail under various contexts. For example, in [9, 18, 22], the authors investigated on axiom schema for determination of the semantic equivalence between the conditional expressions. The authors in [3, 6, 19] studied complete proof systems for various versions of `if-then-else`. While a transformational characterization of `if-then-else` was given in [17], an axiomatization of equality test algebras was considered in [10, 21]. In [1, 23], `if-then-else` was studied as an action of Boolean algebra on a set. Due to their close relation with program features, functions have been canonical models for studies on algebraic semantics of programs.

In [13] Kennison defined comparison algebras as those equipped with a quaternary operation $C(s,t,u,v)$ satisfying certain identities modelling the equality test. He also showed that such algebras are simple if and only if C is the direct comparison operation C_0 given by $C_0(s,t,u,v)$ taking value u if $s=t$ and v otherwise. This was extended by Stokes in [24] to semigroups and monoids. He showed that every comparison semigroup (monoid) is embeddable in the comparison semigroup (monoid) $\mathcal{T}(X)$ of all total functions $X \to X$, for some set X. He also obtained a similar result in terms of partial functions $X \to X$. In [11] Jackson and Stokes gave a complete axiomatization of `if-then-else` over halting programs and tests. They also modelled composition of functions and of functions with predicates and called this object a B-monoid and further showed that the more natural setting of only considering composition of functions would not admit a finite axiomatization. They proved that every B-monoid is embeddable in a functional B-monoid comprising total functions and halting tests and thus achieved a Cayley-type theorem for the class of B-monoids. The work listed above predominantly considered the case where the tests are halting and drawn from a Boolean algebra. A natural interest is to study non-halting tests and programs.

There are multiple studies (e.g., see [4, 8, 14, 15]) on extending two-valued Boolean logic to three-valued logic. However McCarthy's logic (cf. [18]) is distinct in that it models the short-circuit evaluation exhibited by programming languages that evaluate expressions in sequential order, from left to right. In [7] Guzmán and Squier gave a complete axiomatization of McCarthy's three-valued logic and called the corresponding algebra a C-algebra, or the algebra of conditional logic. While studying `if-then-else` algebras in [16], Manes defined an *ada* (Algebra of Disjoint Alternatives) which is essentially a C-algebra equipped with an oracle for the halting problem.

Jackson and Stokes in [12] studied the algebraic theory of computable functions, which can be viewed as possibly non-halting programs, together with composition, `if-then-else` and `while-do`. In this work they assumed that the tests form a Boolean algebra. Further, they demonstrated how an algebra of non-halting tests could be constructed from Boolean tests in their setting. Jackson and Stokes proposed an alternative approach by considering an abstract collection of non-halting tests and posed the following problem:

Characterize the algebras of computable functions associated with an abstract C-algebra of non-halting tests.

The authors in [20] have approached the problem by adopting the approach of Jackson and Stokes in [11]. The notion of a *C-set* was introduced through which a complete axiomatization for `if-then-else` over a class of possibly non-halting programs and tests, where tests are drawn from an ada, was provided.

In this paper, following the approach of Jackson and Stokes in [11], we extend the notion of C-sets to include composition of possibly non-halting programs and of these programs with possibly non-halting tests. This object is termed a *C-monoid* and we establish our main result, Theorem 3.1, stating that the C-monoid can be represented in the standard model. This is a Cayley-type theorem as in [2].

1. Preliminaries

In this section we present the necessary background material. First we recall the concept of a C-algebra introduced by Guzmán and Squier [7].

Definition 1.1. A *C-algebra* is an algebra $\langle M, \vee, \wedge, \neg \rangle$ of type $(2, 2, 1)$, which satisfies the following axioms for all $\alpha, \beta, \gamma \in M$:

$$\neg\neg\alpha = \alpha \tag{1.1}$$
$$\neg(\alpha \wedge \beta) = \neg\alpha \vee \neg\beta \tag{1.2}$$
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \tag{1.3}$$
$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \tag{1.4}$$
$$(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta \wedge \gamma) \tag{1.5}$$
$$\alpha \vee (\alpha \wedge \beta) = \alpha \tag{1.6}$$
$$(\alpha \wedge \beta) \vee (\beta \wedge \alpha) = (\beta \wedge \alpha) \vee (\alpha \wedge \beta) \tag{1.7}$$

It is easy to see that every Boolean algebra is a C-algebra. Let $\mathfrak{Z}$ denote the C-algebra with the universe $\{T, F, U\}$ and the following operations.

$\neg$	
T	F
F	T
U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

In fact, the C-algebra $\mathfrak{Z}$ is the McCarthy's three-valued logic.

In view of the fact that the class of C-algebras is a variety, for any set X, $\mathfrak{Z}^X$ is a C-algebra with the operations defined pointwise. Here, the set of all functions from a set X to a set Y is denoted by Y^X. Guzmán and Squier in [7] showed that elements of $\mathfrak{Z}^X$ along with the C-algebra operations may be viewed in terms of *pairs of sets*. This is a pair (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$. For any element $\alpha \in \mathfrak{Z}^X$, associate the pair of sets $(\alpha^{-1}(T), \alpha^{-1}(F))$. Conversely, for any pair of sets (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$ associate the function α where $\alpha(x) = T$ if $x \in A$, $\alpha(x) = F$ if $x \in B$ and $\alpha(x) = U$ otherwise. With this correlation, the operations can be expressed as follows:

$$\neg(A_1, A_2) = (A_2, A_1)$$
$$(A_1, A_2) \wedge (B_1, B_2) = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$$
$$(A_1, A_2) \vee (B_1, B_2) = ((A_1 \cup (A_2 \cap B_1), A_2 \cap B_2)$$

Notation 1.2. We use M to denote an arbitrary C-algebra. By a C-algebra with T, F, U we mean a C-algebra with nullary operations T, F, U, where T is the (unique) left-identity (and right-identity) for $\wedge$, F is the (unique) left-identity (and right-identity) for $\vee$ and U is the (unique) fixed point for $\neg$. Note that U is also a left-zero for both $\wedge$ and $\vee$ while F is a left-zero for $\wedge$.

We now recall the definition of ada (algebra of disjoint alternatives) introduced by Manes in [16] .

Definition 1.3. An *ada* is a C-algebra M with T, F, U equipped with an additional unary operation $(\ )^{\downarrow}$, which is an oracle for the halting problem, subject to the following equations for all $\alpha, \beta \in M$:

$$F^{\downarrow} = F \tag{1.8}$$
$$U^{\downarrow} = F \tag{1.9}$$
$$T^{\downarrow} = T \tag{1.10}$$
$$\alpha \wedge \beta^{\downarrow} = \alpha \wedge (\alpha \wedge \beta)^{\downarrow} \tag{1.11}$$
$$\alpha^{\downarrow} \vee \neg(\alpha^{\downarrow}) = T \tag{1.12}$$
$$\alpha = \alpha^{\downarrow} \vee \alpha \tag{1.13}$$

The C-algebra $\mathfrak{Z}$ with the unary operation $(\ )^{\downarrow}$ defined by (1.8), (1.9) and (1.10) forms an ada. This ada will also be denoted by $\mathfrak{Z}$. One may easily resolve the notation overloading – whether $\mathfrak{Z}$ is a C-algebra or an ada – depending on the context. In [16] Manes showed that the ada $\mathfrak{Z}$ is the only subdirectly irreducible ada. For any set X, $\mathfrak{Z}^X$ is an ada with operations defined pointwise. Note that the ada $\mathfrak{Z}$ is also simple.

We use the following notations related to sets and equivalence relations.

Notation 1.4.

(1) Let X be a set and $\perp \notin X$. The pointed set $X \cup \{\perp\}$ with base point $\perp$ is denoted by $X_{\perp}$.
(2) While the set of all functions $X \to X$ is denoted by $\mathcal{T}(X)$, the set of all functions on $X_{\perp}$ which fix $\perp$ is denoted by $\mathcal{T}_o(X_{\perp})$, i.e., $\mathcal{T}_o(X_{\perp}) = \{f \in \mathcal{T}(X_{\perp}) \ : \ f(\perp) = \perp\}$.
(3) Under an equivalence relation σ on a set A, the equivalence class of an element $p \in A$ will be denoted by $\overline{p}^{\sigma}$. Within a given context, if there is no ambiguity, we may simply denote the equivalence class by $\overline{p}$.

In order to axiomatize `if-then-else` over possibly non-halting programmes and tests, in [20], Panicker et al. considered the tests from a C-algebra and introduced the notion of C-sets. We now recall the notion of a C-set.

Definition 1.5. Let $S_{\perp}$ be a pointed set with base point $\perp$ and M be a C-algebra with T, F, U. The pair $(S_{\perp}, M)$ equipped with an action

$$\_\,[\_\,,\_] : M \times S_{\perp} \times S_{\perp} \to S_{\perp}$$

is called a *C-set* if it satisfies the following axioms for all $\alpha, \beta \in M$ and $s, t, u, v \in S_\perp$:

$$U[s,t] = \perp \qquad (U\text{-axiom}) \quad (1.14)$$

$$F[s,t] = t \qquad (F\text{-axiom}) \quad (1.15)$$

$$(\neg\alpha)[s,t] = \alpha[t,s] \qquad (\neg\text{-axiom}) \quad (1.16)$$

$$\alpha[\alpha[s,t],u] = \alpha[s,u] \qquad (\text{positive redundancy}) \quad (1.17)$$

$$\alpha[s,\alpha[t,u]] = \alpha[s,u] \qquad (\text{negative redundancy}) \quad (1.18)$$

$$(\alpha \wedge \beta)[s,t] = \alpha[\beta[s,t],t] \qquad (\wedge\text{-axiom}) \quad (1.19)$$

$$\alpha[\beta[s,t],\beta[u,v]] = \beta[\alpha[s,u],\alpha[t,v]] \qquad (\text{premise interchange}) \quad (1.20)$$

$$\alpha[s,t] = \alpha[t,t] \Rightarrow (\alpha \wedge \beta)[s,t] = (\alpha \wedge \beta)[t,t] \qquad (\wedge\text{-compatibility}) \quad (1.21)$$

Let M be a C-algebra with T, F, U treated as a pointed set with base point U. The pair (M, M) is a C-set under the following action for all $\alpha, \beta, \gamma \in M$:

$$\alpha[\beta, \gamma] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma).$$

We denote the action of the C-set (M, M) by $\_ \llbracket \_ \,, \_ \rrbracket$. In [20], Panicker et al. showed that the axiomatization is complete for the class of C-sets $(S_\perp, M)$ when M is an ada. In that connection, they obtained some properties of C-sets. Amongst, in Proposition 1.6 below, we list certain properties related to congruences which are useful in the present work. Viewing C-sets as two-sorted algebras, a *congruence* of a C-set is a pair (σ, τ), where σ is an equivalence relation on $S_\perp$ and τ is a congruence on the ada M such that $(s,t),(u,v) \in \sigma$ and $(\alpha, \beta) \in \tau$ imply that $(\alpha[s,u], \beta[t,v]) \in \sigma$.

Proposition 1.6 ([20])**.** *Let $(S_\perp, M)$ be a C-set where M is an ada. For each maximal congruence θ on M, let E_θ be the relation on $S_\perp$ given by*

$$E_\theta = \{(s,t) \in S_\perp \times S_\perp : \ \beta[s,t] = \beta[t,t] \ \text{ for some } \beta \in \overline{T}^\theta\}.$$

Then we have the following properties:

(i) *For $\alpha \in M$ and $s, t \in S_\perp$, if $(\alpha, \beta) \in \theta$ then according to $\beta = T, F$ or U, we have $(\alpha[s,t], s) \in E_\theta, (\alpha[s,t], t) \in E_\theta$ or $(\alpha[s,t], \perp) \in E_\theta$, respectively.*

(ii) *The pair (E_θ, θ) is a C-set congruence.*

(iii) *For the C-set (M, M) the equivalence E_θ on M, denoted by E_{θ_M}, is a subset of θ.*

(iv) *$\bigcap_\theta E_\theta = \Delta_{S_\perp}$, where θ ranges over all maximal congruences on M.*

(v) *The intersection of all maximal congruences on M is trivial, that is $\bigcap \theta = \Delta_M$ where θ ranges over all maximal congruences on M.*

For more details on C-sets one may refer to [20].

2. C-monoids

We now include the case where the composition of two elements of the base set and of an element with a predicate is allowed. Our motivating example is $(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X)$, where $\mathcal{T}_o(X_\perp)$ is considered to be a monoid with zero by equipping it with composition of functions. The composition will be written from left to right, i.e., $(f \cdot g)(x) = g(f(x))$. The monoid identity in $\mathcal{T}_o(X_\perp)$ is the identity function $id_{X_\perp}$ and the zero element is $\zeta_\perp$, the constant function taking the value $\perp$. We also include composition of functions with predicates via the natural interpretation given by the following for all $f \in \mathcal{T}_o(X_\perp)$ and $\alpha \in \mathfrak{Z}^X$:

$$(f \circ \alpha)(x) = \begin{cases} T, & \text{if } \alpha(f(x)) = T; \\ F, & \text{if } \alpha(f(x)) = F; \\ U, & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that if the composition takes value T or F at some point $x \in X_\perp$ then as $\alpha \in \mathfrak{Z}^X$ this implies that $f(x) \neq \perp$.

With this example in mind we define a C-monoid as follows.

Definition 2.1. Let $(S_\perp, \cdot)$ be a monoid with identity element 1 and zero element $\perp$ where $\perp \cdot s = \perp = s \cdot \perp$. Let M be a C-algebra and $(S_\perp, M)$ be a C-set with $\perp$ as the base point of the pointed set $S_\perp$. The pair $(S_\perp, M)$ equipped with a function

$$\circ \colon S_\perp \times M \to M$$

is said to be a *C-monoid* if it satisfies the following axioms for all $s, t, r, u \in S_\perp$ and $\alpha, \beta \in M$:

$$\perp \circ \alpha = U \qquad (\perp\text{-}\circ\text{-axiom}) \tag{2.2}$$

$$t \circ U = U \qquad (U\text{-}\circ\text{-axiom}) \tag{2.3}$$

$$1 \circ \alpha = \alpha \qquad (1\text{-}\circ\text{-axiom}) \tag{2.4}$$

$$s \circ (\neg\alpha) = \neg(s \circ \alpha) \qquad (\neg\text{-}\circ\text{-axiom}) \tag{2.5}$$

$$s \circ (\alpha \wedge \beta) = (s \circ \alpha) \wedge (s \circ \beta) \qquad (\wedge\text{-}\circ\text{-axiom}) \tag{2.6}$$

$$(s \cdot t) \circ \alpha = s \circ (t \circ \alpha) \qquad (\text{semigroup action}) \tag{2.7}$$

$$\alpha[s, t] \cdot u = \alpha[s \cdot u, t \cdot u] \qquad (\text{right composition}) \tag{2.8}$$

$$r \cdot \alpha[s, t] = (r \circ \alpha)[r \cdot s, r \cdot t] \qquad (\text{left composition}) \tag{2.9}$$

$$\alpha[s, t] \circ \beta = \alpha[\![s \circ \beta, t \circ \beta]\!] \qquad (\circ\text{-interchange}) \tag{2.10}$$

The following are examples of C-monoids.

Example 2.2. Recall from [20] that the pair $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$ equipped with the action (2.11) for all $f, g \in \mathcal{T}_o(X_\perp)$ and $\alpha \in \mathfrak{Z}^X$ is a C-set. Note that $\mathcal{T}_o(X_\perp)$ is treated as a pointed set with base point $\zeta_\perp$.

$$\alpha[f, g](x) = \begin{cases} f(x), & \text{if } \alpha(x) = T; \\ g(x), & \text{if } \alpha(x) = F; \\ \perp, & \text{otherwise.} \end{cases} \tag{2.11}$$

The C-set $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$ equipped with the operation $\circ$ given in (2.1) and with $\mathcal{T}_o(X_\perp)$ treated as a monoid with zero is in fact a C-monoid. For verification of axioms (2.2) – (2.10) refer to A.1 in the Appendix. Such C-monoids will be called *functional C-monoids.*

Example 2.3. Let $S_\perp$ be a non-trivial monoid with identity 1 and zero $\perp$ and no non-zero zero-divisors, i.e., $s \cdot t = \perp \Rightarrow s = \perp$ or $t = \perp$. Then $S_\perp^X$ is also a monoid with zero for any set X with operations defined pointwise. For $f, g \in S_\perp^X$ define $(f \cdot g)(x) = f(x) \cdot g(x)$. The identity of $S_\perp^X$ is the constant function ζ_1 taking the value 1. The zero and base point of $S_\perp^X$ is the constant function $\zeta_\perp$ taking the value $\perp$. Recall from [20] that the pair $\big(S_\perp^X, \mathfrak{Z}^X\big)$ is a C-set under action (2.11). In fact it is also a C-monoid with $\circ$ defined as follows for all $f \in S_\perp^X$ and $\alpha \in \mathfrak{Z}^X$:

$$(f \circ \alpha)(x) = \begin{cases} \alpha(x), & \text{if } f(x) \neq \perp; \\ U, & \text{otherwise.} \end{cases}$$

For verification of axioms (2.2) – (2.10) refer to A.2 in the Appendix.

Example 2.4. Let $S_\perp$ be a non-trivial monoid with zero and no non-zero zero-divisors, i.e., $s \cdot t = \perp \Rightarrow s = \perp$ or $t = \perp$. In [20] the authors showed that for any pointed set $S_\perp$ with base point $\perp$, the pair $(S_\perp, \mathfrak{Z})$ is a (basic) C-set with respect to the following action for all $a, b \in S_\perp$ and $\alpha \in \mathfrak{Z}$:

$$\alpha[a, b] = \begin{cases} a, & \text{if } \alpha = T; \\ b, & \text{if } \alpha = F; \\ \perp, & \text{if } \alpha = U. \end{cases}$$

This basic C-set $(S_\perp, \mathfrak{Z})$ equipped with $\circ \colon S_\perp \times \mathfrak{Z} \to \mathfrak{Z}$ defined below for all $s \in S_\perp$ and $\alpha \in \mathfrak{Z}$ is a C-monoid.

$$s \circ \alpha = \begin{cases} \alpha, & \text{if } s \neq \perp; \\ U, & \text{if } s = \perp. \end{cases}$$

For verification of axioms (2.2) – (2.10) refer to A.3 in the Appendix.

3. Representation of a class of C-monoids

In this section we obtain a Cayley-type theorem for a class of C-monoids as stated in the following main theorem.

Theorem 3.1. *Every C-monoid $(S_\perp, M)$ where M is an ada is embeddable in the C-monoid $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$ for some set X. Moreover, if both $S_\perp$ and M are finite then so is X.*

Sketch of the proof. For each maximal congruence θ of M, we consider the C-set congruence (E_θ, θ) of $(S_\perp, M)$. Corresponding to each such congruence, we construct a homomorphism of C-monoids from $(S_\perp, M)$ to the functional C-monoid over the set $S_\perp/E_\theta$. This collection of homomorphisms has the property that every distinct pair of elements from each component of the

C-monoid will be separated by some homomorphism from this collection. We then set X to be the disjoint union of $S_\perp/E_\theta$'s excluding the equivalence class $\overline{\perp}^{E_\theta}$. We complete the proof by constructing a monomorphism – by pasting together each of the individual homomorphisms from the collection defined earlier – from the C-monoid $(S_\perp, M)$ to the functional C-monoid over the pointed set $X_\perp$ with a new base point $\perp$.

The proof of Theorem 3.1 will be developed through various subsections. First in Subsection 3.1, we study some properties of maximal congruences of adas. We then present a collection of homomorphisms which separate every distinct pair of elements from each component of $(S_\perp, M)$ in Subsection 3.2. In Subsection 3.3, we construct the required functional C-monoid and establish an embedding from $(S_\perp, M)$. Finally, we consolidate the proof in Subsection 3.4.

In what follows $(S_\perp, M)$ is a C-monoid with M as an ada. Let θ be a maximal congruence on M and E_θ be the equivalence on $S_\perp$ as defined in Proposition 1.6 so that the pair (E_θ, θ) is a congruence on $(S_\perp, M)$. We denote the quotient set $S_\perp/E_\theta$ by $S_{\theta_\perp}$ and use S_θ to denote the set $S_{\theta_\perp} \setminus \{\overline{\perp}^{E_\theta}\}$. Further, we use q, s, t, u, v to denote elements of $S_\perp$ and α, β, γ to denote elements of the ada M.

3.1. Properties of maximal congruences

The following properties are useful in proving the main theorem.

Proposition 3.2. *No two elements of $\{T, F, U\}$ are related under θ. That is, $(T, F) \notin \theta$, $(T, U) \notin \theta$ and $(F, U) \notin \theta$.*

Proof. If $(T, F) \in \theta$ then we show that $\theta = M \times M$; contradicting the maximality of θ. Suppose $(T, F) \in \theta$ and let $\alpha, \beta \in M$. Then $(T, F), (\alpha, \alpha) \in \theta \Rightarrow (T \wedge \alpha, F \wedge \alpha) \in \theta$ that is $(\alpha, F) \in \theta$. Similarly $(\beta, F) \in \theta$ and so using the symmetry and transitivity of θ we have $(\alpha, \beta) \in \theta$ and consequently $\theta = M \times M$. The proof of $(T, U) \notin \theta$ follows along similar lines. Finally since $(F, U) \in \theta \Leftrightarrow (T, U) \in \theta$, the result follows. □

Proposition 3.3. *For each $q \in S_\perp$, we have*

(i) $(q \circ T)[q, \perp] = q$.
(ii) $(q \circ T, F) \notin \theta$.
(iii) $(q \circ T, U) \in \theta \Leftrightarrow (q, \perp) \in E_\theta$.
(iv) $(q \circ T, T) \in \theta \Leftrightarrow (q \circ F, F) \in \theta \Leftrightarrow (q, \perp) \notin E_\theta$.
(v) $(s, t) \in E_\theta \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta$ *for all* $\alpha \in M$.
(vi) $(1, \perp) \notin E_\theta$.

Proof.

(i) Using (2.9) we have $q = q \cdot 1 = q \cdot T[1, \perp] = (q \circ T)[q \cdot 1, q \cdot \perp] = (q \circ T)[q, \perp]$.
(ii) We prove the result by contradiction. Suppose $(q \circ T, F) \in \theta$. Using the fact that θ is a congruence on M and (2.5) we have $(q \circ T, F) \in \theta \Rightarrow (\neg(q \circ T), \neg F) \in \theta \Rightarrow (q \circ (\neg T), \neg F) \in \theta \Rightarrow (q \circ F, T) \in \theta$. Similarly using the fact that θ is a congruence, (2.6) and (2.5) we have

$((q\circ F)\vee(q\circ T),(T\vee F))\in\theta\Rightarrow(q\circ(F\vee T),(T\vee F))\in\theta\Rightarrow(q\circ T,T)\in\theta$. Thus we have $(q\circ T,F)\in\theta$ and $(q\circ T,T)\in\theta$. From the symmetry and transitivity of θ it follows that $(T,F)\in\theta$, a contradiction by Proposition 3.2. The result follows.

(iii) ($\Rightarrow$:) Let $(q\circ T,U)\in\theta$. Using Proposition 1.6(i) we can say that for any choice of $s,t\in S_\perp$ we have $((q\circ T)[s,t],\perp)\in E_\theta$. On choosing $s=q,t=\perp$ and using Proposition 3.3(i) we have $((q\circ T)[q,\perp],\perp)\in E_\theta$ that is $(q,\perp)\in E_\theta$ as desired.

(:$\Leftarrow$) First note that, for $\alpha\in M$,

$$\alpha[\perp,\perp]=\perp \tag{3.1}$$

(cf. [20, Proposition 2.8(1)]). Now assume that $(q,\perp)\in E_\theta$. Then there exists $\beta\in\overline{T}^\theta$ such that $\beta[q,\perp]=\beta[\perp,\perp]$. However, by (3.1), we have $\beta[q,\perp]=\perp$. Thus, $\beta[q,\perp]\circ T=\perp\circ T$ so that $\beta[\![q\circ T,\perp\circ T]\!]=U$ (using (2.2) and (2.10)). Consequently, using (3.1) on (M,M), we have $\beta[\![q\circ T,U]\!]=\beta[\![U,U]\!]$. Hence $(q\circ T,U)\in E_{\theta_M}$ and so from Proposition 1.6(iii), $(q\circ T,U)\in\theta$.

Thus $(q\circ T,U)\in\theta\Leftrightarrow(q,\perp)\in E_\theta$.

(iv) We first show that $(q\circ T,T)\in\theta\Leftrightarrow(q\circ F,F)\in\theta$ by making use of the substitution property of the congruence θ with respect to $\neg$, the fact that $\neg$ is an involution and (2.5). Thus $(q\circ T,T)\in\theta\Leftrightarrow(\neg(q\circ T),\neg T)\in\theta\Leftrightarrow(q\circ(\neg T),\neg T)\in\theta\Leftrightarrow(q\circ F,F)\in\theta$. Using Proposition 3.2, Proposition 3.3(ii) and Proposition 3.3(iii) we show the equivalence $(q\circ T,T)\in\theta\Leftrightarrow(q,\perp)\notin E_\theta$. We have $(q\circ T,T)\in\theta\Rightarrow(q\circ T,U)\notin\theta\Rightarrow(q,\perp)\notin E_\theta$. Conversely $(q,\perp)\notin E_\theta\Rightarrow(q\circ T,U)\notin\theta$. Using Proposition 3.3(ii) it follows that $(q\circ T,F)\notin\theta$. Since θ is a maximal congruence the only remaining possibility is that $(q\circ T,T)\in\theta$ which completes the proof.

(v) Consider $(s,t)\in E_\theta$ and $\alpha\in M$. Then there exists $\beta\in\overline{T}^\theta$ such that $\beta[s,t]=\beta[t,t]$. Thus $\beta[s,t]\circ\alpha=\beta[t,t]\circ\alpha$. Using (2.10) we have $\beta[\![s\circ\alpha,t\circ\alpha]\!]=\beta[\![t\circ\alpha,t\circ\alpha]\!]$ from which it follows that $(s\circ\alpha,t\circ\alpha)\in E_{\theta_M}\subseteq\theta$ by Proposition 1.6(iii).

(vi) Suppose that $(1,\perp)\in E_\theta$. Using Proposition 3.3(v), (2.4), (2.2) we have $(1,\perp)\in E_\theta\Rightarrow(1\circ T,\perp\circ T)\in\theta$ and so $(T,U)\in\theta$ a contradiction by Proposition 3.2.

$\square$

3.2. A class of homomorphisms separating pairs of elements

For each maximal congruence θ on M, in this subsection, we present homomorphisms $\phi_\theta\colon S_\perp\to\mathcal{T}_o(S_{\theta_\perp})$ and $\rho_\theta\colon M\to\mathfrak{Z}^{S_\theta}$. Then we establish that $(\phi_\theta,\rho_\theta)$ is a homomorphism from $(S_\perp,M)$ to the functional C-monoid $(\mathcal{T}_o(S_{\theta_\perp}),\mathfrak{Z}^{S_\theta})$. Further, we ascertain that every pair of elements in $S_\perp$ (or M) are separated by some ϕ_θ (or ρ_θ).

Proposition 3.4. *The function $\phi_\theta\colon S_\perp\to\mathcal{T}_o(S_{\theta_\perp})$ given by $\phi_\theta(s)=\psi_\theta^s$, where $\psi_\theta^s(\overline{t}^{E_\theta})=\overline{t\cdot s}^{E_\theta}$, is a monoid homomorphism that maps the zero (and base point) of $S_\perp$ to that of $\mathcal{T}_o(S_{\theta_\perp})$, that is $\perp\mapsto\zeta_{\overline{\perp}}$.*

Proof. *Claim: ϕ_θ is well-defined.* It suffices to show that ψ_θ^s is well-defined and that $\psi_\theta^s \in \mathcal{T}_o(S_{\theta_\perp})$, that is $\psi_\theta^s(\overline{\perp}) = \overline{\perp}$. In order to show the well-definedness of ψ_θ^s we consider $\overline{u} = \overline{t}$ that is $(u,t) \in E_\theta$. Then there exists $\beta \in \overline{T}^\theta$ such that $\beta[u,t] = \beta[t,t]$. Consequently

$$\begin{aligned}
\beta[u \cdot s, t \cdot s] &= \beta[u,t] \cdot s && \text{from (2.8)}\\
&= \beta[t,t] \cdot s\\
&= \beta[t \cdot s, t \cdot s] && \text{from (2.8)}
\end{aligned}$$

Thus $(u \cdot s, t \cdot s) \in E_\theta$ and so $\psi_\theta^s(\overline{u}) = \psi_\theta^s(\overline{t})$. Also $\psi_\theta^s(\overline{\perp}) = \overline{\perp \cdot s} = \overline{\perp}$. Thus $\psi_\theta^s \in \mathcal{T}_o(S_{\theta_\perp})$.

Claim: $\phi_\theta(\perp) = \zeta_{\overline{\perp}}$. We have $\phi_\theta(\perp) = \psi_\theta^\perp$ where $\psi_\theta^\perp(\overline{t}) = \overline{t \cdot \perp} = \overline{\perp}$. Thus $\phi_\theta(\perp) = \zeta_{\overline{\perp}}$.

Claim: $\phi_\theta(1) = id_{S_{\theta_\perp}}$. We have $\phi_\theta(1) = \psi_\theta^1$ where $\psi_\theta^1(\overline{t}) = \overline{t \cdot 1} = \overline{t}$.

Claim: ϕ_θ is a semigroup homomorphism. Consider $\phi_\theta(s \cdot t) = \psi_\theta^{s \cdot t}$ where $\psi_\theta^{s\cdot t}(\overline{u}) = \overline{u \cdot (s \cdot t)} = \overline{(u \cdot s) \cdot t} = \psi_\theta^t(\overline{u \cdot s}) = \psi_\theta^t(\psi_\theta^s(\overline{u})) = (\psi_\theta^s \cdot \psi_\theta^t)(\overline{u})$. Thus $\phi_\theta(s \cdot t) = \phi_\theta(s) \cdot \phi_\theta(t)$. □

Proposition 3.5. *The function $\rho_\theta \colon M \to \mathfrak{Z}^{S_\theta}$ given by*

$$\rho_\theta(\alpha) = \begin{cases} (S_\theta, \emptyset), & \textit{if } \alpha = T;\\ (\emptyset, S_\theta), & \textit{if } \alpha = F;\\ (A_\theta^\alpha, B_\theta^\alpha), & \textit{otherwise} \end{cases}$$

where $A_\theta^\alpha = \{\overline{t}^{E_\theta} : t \circ \alpha \in \overline{T}^\theta\}$ and $B_\theta^\alpha = \{\overline{t}^{E_\theta} : t \circ \alpha \in \overline{F}^\theta\}$, is a homomorphism of C-algebras with T, F, U.

Proof. *Claim: ρ_θ is well-defined.* If $\alpha \in \{T, F\}$ then the proof is obvious. If $\alpha \notin \{T, F\}$ then we show that $A_\theta^\alpha \cap B_\theta^\alpha = \emptyset$ and that $A_\theta^\alpha, B_\theta^\alpha \subseteq S_\theta$, that is, $\overline{\perp} \notin A_\theta^\alpha \cup B_\theta^\alpha$. Let $\overline{t} \in A_\theta^\alpha \cap B_\theta^\alpha$. Then $t \circ \alpha \in \overline{T}^\theta$ and $t \circ \alpha \in \overline{F}^\theta$ and so $(T, F) \in \theta$ which is a contradiction to Proposition 3.2. Using (2.2) we have $\perp \circ \alpha = U$ and so if $\overline{\perp} \in A_\theta^\alpha \cup B_\theta^\alpha$ we would have $\perp \circ \alpha = U \in \{\overline{T}^\theta, \overline{F}^\theta\}$, a contradiction to Proposition 3.2. Finally we show that the image under ρ_θ is independent of the representative of the equivalence class chosen. Using Proposition 3.3(v) we have $\overline{s} = \overline{t} \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta$. The result follows.

Claim: ρ_θ preserves the constants T, F, U. It is clear that $\rho_\theta(T) = (S_\theta, \emptyset)$, $\rho_\theta(F) = (\emptyset, S_\theta)$ and, using (2.3) and Proposition 3.2, that $\rho_\theta(U) = (A_\theta^U, B_\theta^U) = (\emptyset, \emptyset)$ from which the result follows.

Claim: ρ_θ is a C-algebra homomorphism. We show that $\rho_\theta(\neg\alpha) = \neg(\rho_\theta(\alpha))$. If $\alpha \in \{T, F\}$ the proof is obvious. Suppose that $\alpha \notin \{T, F\}$. Then we have the following.

$$\begin{aligned}
\rho_\theta(\neg\alpha) &= (A_\theta^{\neg\alpha}, B_\theta^{\neg\alpha}) \\
&= (\{\bar{t} : t \circ (\neg\alpha) \in \overline{T}^\theta\}, \{\bar{t} : t \circ (\neg\alpha) \in \overline{F}^\theta\}) \\
&= (\{\bar{t} : \neg(t \circ \alpha) \in \overline{T}^\theta\}, \{\bar{t} : \neg(t \circ \alpha) \in \overline{F}^\theta\}) \qquad \text{using (2.5)} \\
&= (\{\bar{t} : t \circ \alpha \in \overline{F}^\theta\}, \{\bar{t} : t \circ \alpha \in \overline{T}^\theta\}) \\
&= (B_\theta^\alpha, A_\theta^\alpha) \\
&= \neg(\rho_\theta(\alpha))
\end{aligned}$$

Finally we show that $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$. Note that the proof of $\rho_\theta(\alpha \vee \beta) = \rho_\theta(\alpha) \vee \rho_\theta(\beta)$ follows using the double negation and De Morgan's laws, viz., (1.1) and (1.2) respectively in conjunction with the fact that ρ_θ preserves $\neg$ and $\wedge$. In order to prove that $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$ we proceed by considering the following cases.

Case I: $\alpha, \beta \notin \{T, F\}$. We have the following subcases:
Subcase 1: $\alpha \wedge \beta \notin \{T, F\}$. Then $\rho_\theta(\alpha\wedge\beta) = (A_\theta^{\alpha\wedge\beta}, B_\theta^{\alpha\wedge\beta})$, $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha)$ and $\rho_\theta(\beta) = (A_\theta^\beta, B_\theta^\beta)$. Now $(A_\theta^\alpha, B_\theta^\alpha)\wedge(A_\theta^\beta, B_\theta^\beta) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta))$. Thus we have to show that

$$(A_\theta^{\alpha\wedge\beta}, B_\theta^{\alpha\wedge\beta}) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)).$$

We show that the pairs of sets are equal componentwise.

Let $\overline{q} \in A_\theta^{\alpha\wedge\beta}$. Then $q \circ (\alpha \wedge \beta) \in \overline{T}^\theta$

$$\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), T) \in \theta && \text{(using (2.6))} \\
&\Rightarrow ((q \circ \alpha) \wedge ((q \circ \alpha) \wedge (q \circ \beta)), (q \circ \alpha) \wedge T) \in \theta && \text{(since } \theta \text{ is a congruence)} \\
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), q \circ \alpha) \in \theta && \text{(using the properties of } \wedge\text{)} \\
&\Rightarrow (q \circ \alpha, T) \in \theta && \text{(by transitivity of } \theta\text{)}
\end{aligned}$$

so that $\overline{q} \in A_\theta^\alpha$. Along similar lines one can observe that

$$((((q \circ \alpha) \wedge (q \circ \beta)) \wedge (q \circ \beta)), T \wedge (q \circ \beta)) \in \theta.$$

Consequently $(q \circ \beta, T) \in \theta$ so that $\overline{q} \in A_\theta^\beta$. Hence $A_\theta^{\alpha\wedge\beta} \subseteq A_\theta^\alpha \cap A_\theta^\beta$.

For reverse inclusion let $\overline{q} \in A_\theta^\alpha \cap A_\theta^\beta$. Then $(q\circ\alpha, T), (q\circ\beta, T) \in \theta$. Since θ is a congruence we have $((q \circ \alpha) \wedge (q \circ \beta), T \wedge T) = ((q \circ \alpha) \wedge (q \circ \beta), T) = ((q \circ (\alpha \wedge \beta), T) \in \theta$ and so $\overline{q} \in A_\theta^{\alpha\wedge\beta}$. Hence $A_\theta^{\alpha\wedge\beta} = A_\theta^\alpha \cap A_\theta^\beta$.

In order to show that $B_\theta^{\alpha\wedge\beta} \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$ consider $\overline{q} \in B_\theta^{\alpha\wedge\beta}$ that is $(q \circ (\alpha \wedge \beta), F) \in \theta$. Since θ is a maximal congruence consider the following three possibilities:

$(q \circ \alpha, F) \in \theta$**.** Then clearly $\overline{q} \in B_\theta^\alpha$ and so $\overline{q} \in B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, T) \in \theta$**.** Then we have $\overline{q} \in A_\theta^\alpha$. We show that $(q \circ \beta, F) \in \theta$. If this is not the case then either $(q \circ \beta, T) \in \theta$ or $(q \circ \beta, U) \in \theta$. If $(q \circ \beta, T) \in \theta$ then since $(q \circ \alpha, T) \in \theta$ we have $(q \circ (\alpha \wedge \beta), T \wedge T) = (q \circ (\alpha \wedge \beta), T) \in \theta$

using (2.6) and the fact that θ is a congruence. However since $(q \circ (\alpha \wedge \beta), F) \in \theta$ we obtain a contradiction that $(T, F) \in \theta$ (cf. Proposition 3.2). Along similar lines if $(q \circ \beta, U) \in \theta$ then as $(q \circ \alpha, T) \in \theta$ we have $(q \circ (\alpha \wedge \beta), U) \in \theta$ and so $(F, U) \in \theta$ a contradiction to Proposition 3.2. Hence $(q \circ \beta, F) \in \theta$ so that $\overline{q} \in (A_\theta^\alpha \cap B_\theta^\beta) \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, U) \in \theta$**.** Since $(q \circ \beta, q \circ \beta) \in \theta$ we have $(q \circ (\alpha \wedge \beta), U \wedge q \circ \beta) = (q \circ (\alpha \wedge \beta), U) \in \theta$ using (2.6) and the fact that θ is a congruence. However since $(q \circ (\alpha \wedge \beta), F) \in \theta$ we have $(F, U) \in \theta$ a contradiction to Proposition 3.2. Thus this case cannot occur.

To show the reverse inclusion let $\overline{q} \in B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$ that is $\overline{q} \in B_\theta^\alpha$ or $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta$. If $\overline{q} \in B_\theta^\alpha$ then $(q \circ \alpha, F) \in \theta$

$$
\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), F \wedge (q \circ \beta)) \in \theta && \text{(since } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ (\alpha \wedge \beta), F \wedge (q \circ \beta)) \in \theta && \text{(using (2.6))}\\
&\Rightarrow (q \circ (\alpha \wedge \beta), F) \in \theta && \text{(since } F \text{ is a left-zero for } \wedge)
\end{aligned}
$$

from which it follows that $\overline{q} \in B_\theta^{\alpha\wedge\beta}$. In the case where $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta$ that is $(q \circ \alpha, T), (q \circ \beta, F) \in \theta$, along similar lines it follows that $(q \circ (\alpha \wedge \beta), T \wedge F) = (q \circ (\alpha \wedge \beta), F) \in \theta$ and so $\overline{q} \in B_\theta^{\alpha\wedge\beta}$. Therefore $B_\theta^{\alpha\wedge\beta} = B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

Subcase 2: $\alpha \wedge \beta \in \{T, F\}$. Using the fact that $M \leq \mathfrak{Z}^X$ for some set X it is easy to see that if $\alpha, \beta \notin \{T, F\}$ then $\alpha \wedge \beta \neq T$. It follows that the only possibility in this case is that $\alpha \wedge \beta = F$. Therefore $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(F) = (\emptyset, S_\theta)$ and $\rho_\theta(\alpha) \wedge \rho_\theta(\beta) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta))$ and so we have to show that

$$(\emptyset, S_\theta) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)).$$

We first show that $A_\theta^\alpha \cap A_\theta^\beta = \emptyset$. If $A_\theta^\alpha \cap A_\theta^\beta \neq \emptyset$ then let $\overline{q} \in A_\theta^\alpha \cap A_\theta^\beta$ so that $(q \circ \alpha, T) \in \theta, (q \circ \beta, T) \in \theta$

$$
\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), T \wedge T) = ((q \circ \alpha) \wedge (q \circ \beta), T) \in \theta && \text{(as } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ (\alpha \wedge \beta), T) \in \theta && \text{(using (2.6))}\\
&\Rightarrow (q \circ F, T) \in \theta && \text{(since } \alpha \wedge \beta = F)\\
&\Rightarrow (\neg(q \circ F), \neg T) = (\neg(q \circ F), F) \in \theta && \text{(as } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ \neg F, F) = (q \circ T, F) \in \theta && \text{(using (2.5))}
\end{aligned}
$$

which is a contradiction to Proposition 3.3(ii). Hence $A_\theta^\alpha \cap A_\theta^\beta = \emptyset$.

In order to show that $B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta) = S_\theta$ consider $\overline{q} \in S_\theta$ that is $\overline{q} \neq \overline{\perp}$ which gives $(q, \perp) \notin E_\theta$. We proceed by considering the following three cases:

$(q \circ \alpha, F) \in \theta$**.** Then it is clear that $\overline{q} \in B_\theta^\alpha \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, T) \in \theta$**.** Then we have $\overline{q} \in A_\theta^\alpha$. We show that $(q \circ \beta, F) \in \theta$. Suppose that this is not the case. Since θ is a maximal congruence it implies that either $(q \circ \beta, T) \in \theta$ or $(q \circ \beta, U) \in \theta$. If $(q \circ \beta, T) \in \theta$ then since $(q \circ \alpha, T) \in \theta$ it follows that $(q \circ (\alpha \wedge \beta), T \wedge T) = (q \circ F, T) \in \theta$ so that

$(q \circ T, F) \in \theta$. This is a contradiction to Proposition 3.3(ii). In the case that $(q \circ \beta, U) \in \theta$ proceeding as earlier we have $(q \circ (\alpha \wedge \beta), T \wedge U) = (q \circ F, U) \in \theta$ so that $(q \circ T, U) \in \theta$. It follows from Proposition 3.3(iii) that $(q, \perp) \in E_\theta$ which is a contradiction to the assumption that $\overline{q} \in S_\theta$. Consequently it must be the case that $(q \circ \beta, F) \in \theta$ so that $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, U) \in \theta$. Since θ is a congruence we have $(q \circ \beta, q \circ \beta) \in \theta$

$$\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), U \wedge (q \circ \beta)) \in \theta && \text{(since } \theta \text{ is a congruence)}\\
&\Rightarrow ((q \circ (\alpha \wedge \beta), U) = (q \circ F, U) \in \theta && \text{(since } U \text{ is a left-zero for } \wedge\\
& && \text{and using (2.6))}\\
&\Rightarrow (\neg(q \circ F), \neg U) \in \theta && \text{(since } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ \neg F, \neg U) = (q \circ T, U) \in \theta && \text{(using (2.5))}
\end{aligned}$$

Thus using Proposition 3.3(iii) we have $(q, \perp) \in E_\theta$ which is a contradiction to the assumption that $\overline{q} \in S_\theta$. Hence this case cannot occur.

Thus $B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta) = S_\theta$ which completes the proof in the case where $\alpha, \beta \notin \{T, F\}$.

Case II: $\alpha \in \{T, F\}$. The verification is straightforward by considering $\alpha = T$ and $\alpha = F$ casewise.
Subcase 1: $\alpha = T$. Then $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(T \wedge \beta) = \rho_\theta(\beta) = (S_\theta, \emptyset) \wedge \rho_\theta(\beta) = \rho_\theta(T) \wedge \rho_\theta(\beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Subcase 2: $\alpha = F$. Then $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(F \wedge \beta) = \rho_\theta(F) = (\emptyset, S_\theta) = (\emptyset, S_\theta) \wedge \rho_\theta(\beta) = \rho_\theta(F) \wedge \rho_\theta(\beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Case III: $\beta \in \{T, F\}$. We have the following subcases:
Subcase 1: $\beta = T$. The proof follows along the same lines as *Case II* above since T is the left and right-identity for $\wedge$. Thus $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha \wedge T) = \rho_\theta(\alpha) = \rho_\theta(\alpha) \wedge (S_\theta, \emptyset) = \rho_\theta(\alpha) \wedge \rho_\theta(T) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Subcase 2: $\beta = F$. If $\alpha \in \{T, F\}$ then this reduces to *Case II* proved above and consequently we have $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$ in this case. Thus it remains to consider the case where $\alpha \notin \{T, F\}$. We then have the following subcases depending on $\alpha \wedge \beta$:

$\alpha \wedge \beta \notin \{T, F\}$. Then $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha \wedge F) = (A_\theta^{\alpha \wedge F}, B_\theta^{\alpha \wedge F})$ while $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha)$ and $\rho_\theta(\beta) = \rho_\theta(F) = (\emptyset, S_\theta)$. Thus $\rho_\theta(\alpha) \wedge \rho_\theta(F) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$. We show that

$$(A_\theta^{\alpha \wedge F}, B_\theta^{\alpha \wedge F}) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$$

as earlier by proving that the pairs of sets are equal componentwise.

We show that $A_\theta^{\alpha\wedge F} = \emptyset$ by contradiction. If $A_\theta^{\alpha\wedge F} \neq \emptyset$ then consider $\overline{q} \in A_\theta^{\alpha\wedge F}$. It follows that $(q \circ (\alpha \wedge F), T) \in \theta$

$\Rightarrow ((q \circ (\alpha \wedge F)) \wedge (q \circ F), T \wedge q \circ F) \in \theta$ (since θ is a congruence)

$\Rightarrow ((q \circ (\alpha \wedge F)) \wedge (q \circ F), q \circ F) \in \theta$ (since T is a left-identity for $\wedge$)

$\Rightarrow (((q \circ \alpha) \wedge (q \circ F)) \wedge (q \circ F), q \circ F) \in \theta$ (using (2.6))

$\Rightarrow ((q \circ \alpha) \wedge (q \circ F), q \circ F) \in \theta$ (using the properties of $\wedge$)

$\Rightarrow (q \circ F, T) \in \theta$ (since θ is a congruence)

$\Rightarrow (q \circ T, F) \in \theta$ (from (2.5) and since θ is a congruence)

which is a contradiction to Proposition 3.3(ii). Hence $A_\theta^{\alpha\wedge F} = \emptyset$.

We show that $B_\theta^{\alpha\wedge F} = A_\theta^\alpha \cup B_\theta^\alpha$ using standard set theoretic arguments. Let $\overline{q} \in B_\theta^{\alpha\wedge F}$ and so $(q \circ (\alpha \wedge F), F) \in \theta$ so that $((q \circ \alpha) \wedge (q \circ F), F) \in \theta$. In view of the maximality of θ it suffices to consider three cases. If either $(q \circ \alpha, T) \in \theta$ or $(q \circ \alpha, F) \in \theta$ then $\overline{q} \in A_\theta^\alpha \cup B_\theta^\alpha$. If $(q\circ\alpha, U) \in \theta$ then $((q\circ\alpha)\wedge((q\circ\alpha)\wedge(q\circ F)), U\wedge F) = ((q\circ\alpha)\wedge(q\circ F), U) \in \theta$.Thus $(F, U) \in \theta$ which is a contradiction to Proposition 3.2. Hence this case cannot occur and so $B_\theta^{\alpha\wedge F} \subseteq A_\theta^\alpha \cup B_\theta^\alpha$.

For the reverse inclusion consider $\overline{q} \in A_\theta^\alpha \cup B_\theta^\alpha$ so that $\overline{q} \in A_\theta^\alpha$ or $\overline{q} \in B_\theta^\alpha$. If $\overline{q} \in A_\theta^\alpha$ then $(q \circ \alpha, T) \in \theta$. Since $\overline{q} \in A_\theta^\alpha \subseteq S_\theta$ using Proposition 3.3(iv) we have $(q, \perp) \notin E_\theta \Rightarrow (q \circ F, F) \in \theta$. Consequently $(q \circ (\alpha \wedge F), (T \wedge F)) = (q \circ (\alpha \wedge F), F) \in \theta$ and so $\overline{q} \in B_\theta^{\alpha\wedge F}$. Along similar lines if $\overline{q} \in B_\theta^\alpha$ we have $(q \circ (\alpha \wedge F), F) \in \theta$ so that $\overline{q} \in B_\theta^{\alpha\wedge F}$. Hence $(A_\theta^{\alpha\wedge F}, B_\theta^{\alpha\wedge F}) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$.

$\alpha \wedge \beta \in \{T, F\}$. Using the fact that $M \leq \mathfrak{Z}^X$ for some set X we have $\alpha \wedge F \neq T$ from which it follows that the only case is $\alpha \wedge \beta = \alpha \wedge F = F$. Thus $\rho_\theta(\alpha \wedge F) = \rho_\theta(F) = (\emptyset, S_\theta)$ while $\rho_\theta(\alpha) \wedge \rho_\theta(F) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$. We show that

$$(\emptyset, S_\theta) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha).$$

In order to show that $A_\theta^\alpha \cup B_\theta^\alpha = S_\theta$ consider $\overline{q} \in S_\theta$. If $(q \circ \alpha, T) \in \theta$ or $(q \circ \alpha, F) \in \theta$ then the proof is complete. If $(q \circ \alpha, U) \in \theta$ then since $\overline{q} \neq \overline{\perp}$ that is $(q, \perp) \notin E_\theta$ by Proposition 3.3(iv) we have $(q \circ F, F) \in \theta$. Thus $(q \circ (\alpha \wedge F), U \wedge F) = (q \circ F, U) \in \theta$. Consequently from the transitivity of θ it follows that $(F, U) \in \theta$ which is a contradiction to Proposition 3.2. Hence $(\emptyset, S_\theta) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$.

Thus ρ_θ is a homomorphism of C-algebras with T, F, U. □

Lemma 3.6. *The pair $(\phi_\theta, \rho_\theta)$ is a C-monoid homomorphism from $(S_\perp, M)$ to the functional C-monoid $\big(\mathcal{T}_o(S_{\theta_\perp}), \mathfrak{Z}^{S_\theta}\big)$.*

Proof. In view of Proposition 3.4 and Proposition 3.5 it suffices to show that $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$ and $\rho_\theta(s \circ \alpha) = \phi_\theta(s) \circ \rho_\theta(\alpha)$ hold. In order to show that $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$ we proceed casewise depending on the value of α as per the following:

Case I: $\alpha \in \{T, F\}$. If $\alpha = T$ then $\phi_\theta(\alpha[s,t]) = \phi_\theta(T[s,t]) = \phi_\theta(s) = (S_\theta, \emptyset)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(T)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$. Along similar lines if $\alpha = F$ then $\phi_\theta(\alpha[s,t]) = \phi_\theta(F[s,t]) = \phi_\theta(t) = (\emptyset, S_\theta)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(F)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$.

Case II: $\alpha \notin \{T, F\}$. If $\alpha \notin \{T, F\}$ then using (2.9) we have $\phi_\theta(\alpha[s,t]) = \psi_\theta^{\alpha[s,t]}$ where $\psi_\theta^{\alpha[s,t]}(\overline{v}) = \overline{v \cdot (\alpha[s,t])} = \overline{(v \circ \alpha)[v \cdot s, v \cdot t]}$.

Consider $\rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)] = (A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t]$, where

$$(A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t](\overline{v}) = \begin{cases} \overline{v \cdot s}, & \text{if } \overline{v} \in A_\theta^\alpha, \text{ that is } (v \circ \alpha) \in \overline{T}^\theta; \\ \overline{v \cdot t}, & \text{if } \overline{v} \in B_\theta^\alpha \text{ that is } (v \circ \alpha) \in \overline{F}^\theta; \\ \overline{\perp}, & \text{otherwise.} \end{cases}$$

It suffices to consider the following three cases:
Subcase 1: $(v \circ \alpha) \in \overline{T}^\theta$. using Proposition 1.6(i) we have $((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot s) \in E_\theta$. Consequently $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{v \cdot s}$.

Subcase 2: $(v \circ \alpha) \in \overline{F}^\theta$. Along similar lines if $(v \circ \alpha) \in \overline{F}^\theta$ then $((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot t) \in E_\theta$, by Proposition 1.6(i) and so $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{v \cdot t}$.

Subcase 3: $(v \circ \alpha) \in \overline{U}^\theta$. Then $((v \circ \alpha)[v \cdot s, v \cdot t], \perp) \in E_\theta$, by Proposition 1.6(i) which gives $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{\perp}$.

Thus we have $\psi_\theta^{\alpha[s,t]}(\overline{v}) = (A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t](\overline{v})$ for every $\overline{v} \in S_{\theta_\perp}$ and so $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$.

We show that $\rho_\theta(s \circ \alpha) = \phi_\theta(s) \circ \rho_\theta(\alpha)$ by proceeding casewise depending on the value of α and $s \circ \alpha$.

Case I: $\alpha \notin \{T, F\}, s \circ \alpha \notin \{T, F\}$. Then $\rho_\theta(s \circ \alpha) = (A_\theta^{s \circ \alpha}, B_\theta^{s \circ \alpha})$ and $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha)$. Then $\phi_\theta(s) \circ (A_\theta^\alpha, B_\theta^\alpha) = \psi_\theta^s \circ (A_\theta^\alpha, B_\theta^\alpha) = (C, D)$, where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in A_\theta^\alpha\}$ and $D = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in B_\theta^\alpha\}$. We have to show that

$$(A_\theta^{s \circ \alpha}, B_\theta^{s \circ \alpha}) = (C, D).$$

It is clear that $\overline{q} \in C$

$$\begin{aligned} &\Leftrightarrow \psi_\theta^s(\overline{q}) \in A_\theta^\alpha \\ &\Leftrightarrow \overline{q \cdot s} \in A_\theta^\alpha \\ &\Leftrightarrow ((q \cdot s) \circ \alpha, T) \in \theta \\ &\Leftrightarrow (q \circ (s \circ \alpha), T) \in \theta && \text{(using (2.7))} \\ &\Leftrightarrow \overline{q} \in A_\theta^{s \circ \alpha} \end{aligned}$$

Along similar lines we have $\overline{q} \in D \Leftrightarrow \overline{q} \in B_\theta^{s\circ\alpha}$.

Case II: $\alpha \in \{T, F\}, s \circ \alpha \notin \{T, F\}$. If $\alpha = T$ then $\rho_\theta(s \circ \alpha) = \rho_\theta(s \circ T) = (A_\theta^{s\circ T}, B_\theta^{s\circ T})$. On the other hand $\phi_\theta(s) \circ \rho_\theta(\alpha) = \phi_\theta(s) \circ \rho_\theta(T) = \psi_\theta^s \circ (S_\theta, \emptyset) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in S_\theta\}$ and $D = \emptyset$. We have to show that

$$(A_\theta^{s\circ T}, B_\theta^{s\circ T}) = (C, \emptyset).$$

We show that $B_\theta^{s\circ T} = \emptyset$ by contradiction. If $B_\theta^{s\circ T} \neq \emptyset$ then let $\overline{q} \in B_\theta^{s\circ T}$

$$\begin{aligned} &\Rightarrow (q \circ (s \circ T), F) \in \theta \\ &\Rightarrow ((q \cdot s) \circ T, F) \in \theta && \text{(using (2.7))} \end{aligned}$$

which is a contradiction to Proposition 3.3(ii). Thus $B_\theta^{s\circ T} = \emptyset$.
We now show that $A_\theta^{s\circ T} = C$. It is clear that $\overline{q} \in C$

$$\begin{aligned} &\Leftrightarrow \psi_\theta^s(\overline{q}) \in S_\theta \\ &\Leftrightarrow \overline{q \cdot s} \in S_\theta \\ &\Leftrightarrow (q \cdot s, \perp) \notin E_\theta \\ &\Leftrightarrow ((q \cdot s) \circ T, T) \in \theta && \text{(using Proposition 3.3(iv))} \\ &\Leftrightarrow (q \circ (s \circ T), T) \in \theta && \text{(using (2.7))} \\ &\Leftrightarrow \overline{q} \in A_\theta^{s\circ T} \end{aligned}$$

In the case where $\alpha = F$ the proof follows along similar lines.

Case III: $\alpha \notin \{T, F\}, s \circ \alpha \in \{T, F\}$. We have the following subcases:
Subcase 1: $s \circ \alpha = T$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(T) = (S_\theta, \emptyset)$. On the other hand $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi_\theta^s \circ (A_\theta^\alpha, B_\theta^\alpha) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in A_\theta^\alpha\}$ and $D = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in B_\theta^\alpha\}$. We have to show that

$$(C, D) = (S_\theta, \emptyset).$$

We first show by contradiction that $D = \emptyset$. If $D \neq \emptyset$ consider $\overline{q} \in D$

$$\begin{aligned} &\Rightarrow \psi_\theta^s(\overline{q}) \in B_\theta^\alpha \\ &\Rightarrow \overline{q \cdot s} \in B_\theta^\alpha \\ &\Rightarrow ((q \cdot s) \circ \alpha, F) \in \theta \\ &\Rightarrow (q \circ (s \circ \alpha), F) \in \theta && \text{(using (2.7))} \\ &\Rightarrow (q \circ T, F) \in \theta \end{aligned}$$

which is a contradiction to Proposition 3.3(ii).
In order to show that $C = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$\begin{aligned}
&\Rightarrow (q \circ T, T) \in \theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow (q \circ (s \circ \alpha), T) \in \theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, T) \in \theta && \text{(using (2.7))}\\
&\Rightarrow \overline{q \cdot s} \in A_\theta^\alpha \\
&\Rightarrow \psi_\theta^s(\overline{q}) \in A_\theta^\alpha \\
&\Rightarrow \overline{q} \in C.
\end{aligned}$$

Subcase 2: $s \circ \alpha = F$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(F) = (\emptyset, S_\theta)$ while $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi_\theta^s \circ (A_\theta^\alpha, B_\theta^\alpha) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in A_\theta^\alpha\}$ and $D = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in B_\theta^\alpha\}$. We have to show that

$$(C, D) = (\emptyset, S_\theta).$$

We first show $C = \emptyset$ by contradiction. If $C \neq \emptyset$ consider $\overline{q} \in C$

$$\begin{aligned}
&\Rightarrow \psi_\theta^s(\overline{q}) \in A_\theta^\alpha \\
&\Rightarrow \overline{q \cdot s} \in A_\theta^\alpha \\
&\Rightarrow ((q \cdot s) \circ \alpha, T) \in \theta \\
&\Rightarrow (q \circ (s \circ \alpha), T) \in \theta && \text{(using (2.7))}\\
&\Rightarrow (q \circ F, T) \in \theta \\
&\Rightarrow (q \circ T, F) \in \theta
\end{aligned}$$

which is a contradiction to Proposition 3.3(ii).
In order to show that $D = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$.

$$\begin{aligned}
&\Rightarrow (q \circ F, F) \in \theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow (q \circ (s \circ \alpha), F) \in \theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, F) \in \theta && \text{(using (2.7))}\\
&\Rightarrow \overline{q \cdot s} \in B_\theta^\alpha \\
&\Rightarrow \psi_\theta^s(\overline{q}) \in B_\theta^\alpha \\
&\Rightarrow \overline{q} \in D
\end{aligned}$$

which completes the proof for the case where $\alpha \notin \{T, F\}$ and $s \circ \alpha \in \{T, F\}$.

Case IV: $\alpha \in \{T, F\}, s \circ \alpha \in \{T, F\}$. Note that $s \circ T \neq F$ as a consequence of Proposition 3.3(ii). If $s \circ T = F$ then as θ is a congruence, $(F, F) \in \theta \Rightarrow (s \circ T, F) \in \theta$, a contradiction to Proposition 3.3(ii). Similarly we have $s \circ F \neq T$. In view of the above it suffices to consider the following cases:

Subcase 1: $\alpha = T, s \circ \alpha = T$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(T) = (S_\theta, \emptyset)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi_\theta^s \circ (S_\theta, \emptyset) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in S_\theta\}$ and $D = \emptyset$.

Thus it suffices to show that $C = S_\theta$. Let $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$
\begin{aligned}
&\Rightarrow (q \circ T, T) \in \theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow (q \circ (s \circ T), T) \in \theta &&\\
&\Rightarrow ((q \cdot s) \circ T, T) \in \theta && \text{(using (2.7))}\\
&\Rightarrow (q \cdot s, \perp) \notin E_\theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow \overline{q \cdot s} \in S_\theta &&\\
&\Rightarrow \psi^s_\theta(\overline{q}) \in S_\theta &&\\
&\Rightarrow \overline{q} \in C. &&
\end{aligned}
$$

Thus $C = S_\theta$.

Subcase 2: $\alpha = F, s \circ \alpha = F$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(F) = (\emptyset, S_\theta)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^s_\theta \circ (\emptyset, S_\theta) = (C, D)$ where $C = \emptyset$ and $D = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in S_\theta\}$. The proof follows along similar lines as above. In order to show that $D = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$
\begin{aligned}
&\Rightarrow (q \circ F, F) \in \theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow (q \circ (s \circ F), F) \in \theta &&\\
&\Rightarrow ((q \cdot s) \circ F, F) \in \theta && \text{(using (2.7))}\\
&\Rightarrow (q \cdot s, \perp) \notin E_\theta && \text{(using Proposition 3.3(iv))}\\
&\Rightarrow \overline{q \cdot s} \in S_\theta &&\\
&\Rightarrow \psi^s_\theta(\overline{q}) \in S_\theta &&\\
&\Rightarrow \overline{q} \in D. &&
\end{aligned}
$$

Hence $D = S_\theta$.

Thus $(\phi_\theta, \rho_\theta)$ is a homomorphism of C-monoids. □

Proposition 3.7. *For all $\alpha \in M$ the following statements hold:*

(i) $\rho_\theta(\alpha) = (S_\theta, \emptyset) \Rightarrow (\alpha, T) \in \theta.$
(ii) $\rho_\theta(\alpha) = (\emptyset, S_\theta) \Rightarrow (\alpha, F) \in \theta.$

Proof.

(i) If $\alpha = T$ then the result is obvious. Suppose that $\alpha \neq T$ and $\rho_\theta(\alpha) = (A^\alpha_\theta, B^\alpha_\theta) = (S_\theta, \emptyset)$. It follows that $(t \circ \alpha, T) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 3.3(vi) and (2.4) we have $\overline{1} \in S_\theta$ and so $(1 \circ \alpha, T) = (\alpha, T) \in \theta$.
(ii) Along similar lines if $\alpha \neq F$ then $\rho_\theta(\alpha) = (A^\alpha_\theta, B^\alpha_\theta) = (\emptyset, S_\theta)$ gives $(t \circ \alpha, F) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 3.3(vi) and (2.4) we have $\overline{1} \in S_\theta$ and so $(1 \circ \alpha, F) = (\alpha, F) \in \theta$.

□

Lemma 3.8. *For every $s, t \in S_\perp$ where $s \neq t$ there exists a maximal congruence θ on M such that $\phi_\theta(s) \neq \phi_\theta(t)$.*

Proof. Using Proposition 1.6(iv) we have $\bigcap E_\theta = \Delta_{S_\perp}$ and so since $s \neq t$ there exists a maximal congruence θ on M such that $(s,t) \notin E_\theta$, i.e., $\overline{s} \neq \overline{t}$. For this θ, consider $\phi_\theta \colon S_\perp \to \mathcal{T}_o(S_{\theta_\perp})$. Then $\phi_\theta(s) = \psi_\theta^s$, $\phi_\theta(t) = \psi_\theta^t$. For $\overline{1} \in S_{\theta_\perp}$ we have $\psi_\theta^s(\overline{1}) = \overline{1 \cdot s} = \overline{s}$ while $\psi_\theta^t(\overline{1}) = \overline{1 \cdot t} = \overline{t}$. Since $\overline{s} \neq \overline{t}$ it follows that $\phi_\theta(s) \neq \phi_\theta(t)$. □

Lemma 3.9. *For every $\alpha, \beta \in M$ where $\alpha \neq \beta$ there exists a maximal congruence θ on M such that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$.*

Proof. Using Proposition 1.6(v) since $\alpha \neq \beta$ there exists a maximal congruence θ on M such that $(\alpha, \beta) \notin \theta$. We show that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$. If α or β is in $\{T, F\}$ but $\rho_\theta(\alpha) = \rho_\theta(\beta)$ then using Proposition 3.7 we have $(\alpha, \beta) \in \theta$, a contradiction. In the case where $\alpha, \beta \notin \{T, F\}$ we show that

$$(A_\theta^\alpha, B_\theta^\alpha) \neq (A_\theta^\beta, B_\theta^\beta)$$

by showing that either $A_\theta^\alpha \neq A_\theta^\beta$ or that $B_\theta^\alpha \neq B_\theta^\beta$. Owing to Proposition 3.2 it suffices to consider the following three cases:
Case I: $(\alpha, T) \in \theta$. Note that Proposition 3.3(vi) gives $\overline{1} \in S_\theta$. Thus we have $\overline{1} \in S_\theta$ for which $1 \circ \alpha = \alpha \in \overline{T}^\theta$ and so $\overline{1} \in A_\theta^\alpha$. However $\overline{1} \notin A_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

Case II: $(\alpha, F) \in \theta$. Along similar lines for $\overline{1} \in S_\theta$ we have $1 \circ \alpha = \alpha \in \overline{F}^\theta$ and so $\overline{1} \in B_\theta^\alpha$. It is clear that $\overline{1} \notin B_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

Case III: $(\alpha, U) \in \theta$. In view of Proposition 3.2 it suffices to consider the following cases:
Subcase 1: $(\beta, T) \in \theta$. As earlier we have $\overline{1} \in A_\theta^\beta \setminus A_\theta^\alpha$.

Subcase 2: $(\beta, F) \in \theta$. It is clear that $\overline{1} \in B_\theta^\beta \setminus B_\theta^\alpha$.
Thus $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$ which completes the proof. □

3.3. Embedding into a functional C-monoid

Let $\{\theta\}$ be the collection of all maximal congruences of M. Define the set X to be the disjoint union of S_θ taken over all maximal congruences of M, written

$$X = \bigsqcup_\theta S_\theta \tag{3.2}$$

Set $X_\perp = X \cup \{\perp\}$ with base point $\perp \notin X$. For notational convenience we use the same symbol $\perp$ in $X_\perp$ as well as in $S_\perp$. Which $\perp$ we are referring to will be clear from the context of the statement.

In this subsection we obtain monomorphisms $\phi \colon S_\perp \to \mathcal{T}_o(X_\perp)$ and $\rho \colon M \to \mathfrak{Z}^X$, using which we establish that $(S_\perp, M)$ can be embedded into the functional C-monoid $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$.

Remark 3.10.

(i) Let $q \in S$ be fixed. For different θ's the representation of classes $\overline{q}^{E_\theta}$'s are different in the disjoint union X of S_θ's.

(ii) Let $\{A_\lambda\}, \{B_\lambda\}$ be two families of sets indexed over Λ. Then $\bigsqcup_\lambda (A_\lambda \cap B_\lambda) = \left(\bigsqcup_\lambda A_\lambda\right) \cap \left(\bigsqcup_\lambda B_\lambda\right)$ and $\bigsqcup_\lambda (A_\lambda \cup B_\lambda) = \left(\bigsqcup_\lambda A_\lambda\right) \cup \left(\bigsqcup_\lambda B_\lambda\right)$.

Notation 3.11.

(i) For the pair of sets (A, B), we denote by $\pi_1(A, B)$ the first component A, and by $\pi_2(A, B)$ the second component B.

(ii) For a family of pairs of sets (A_λ, B_λ) where $\lambda \in \Lambda$ we denote by $\bigsqcup_\lambda (A_\lambda, B_\lambda)$ the pair of sets $\left(\bigsqcup_\lambda A_\lambda, \bigsqcup_\lambda B_\lambda\right)$.

Lemma 3.12. *Consider $\phi\colon S_\perp \to \mathcal{T}_o(X_\perp)$ given by*

$$(\phi(s))(x) = \begin{cases} (\phi_\theta(s))(\overline{q}^{E_\theta}), & \text{if } x = \overline{q}^{E_\theta} \in S_\theta \text{ and } (\phi_\theta(s))(\overline{q}^{E_\theta}) \neq \overline{\perp}^{E_\theta}; \\ \perp, & \text{otherwise.} \end{cases}$$

Then ϕ is a monoid monomorphism that maps the zero (and base point) of $S_\perp$ to that of $\mathcal{T}_o(X_\perp)$, that is $\perp \mapsto \zeta_\perp$.

Proof. It is clear that ϕ is well-defined and that $\phi(s) \in \mathcal{T}_o(X_\perp)$ since $(\phi(s))(\perp) = \perp$.

Claim: ϕ is injective. Let $s \neq t \in S_\perp$. Using Lemma 3.8 there exists a maximal congruence θ on M such that $\phi_\theta(s) \neq \phi_\theta(t)$. Hence there exists a $\overline{q}^{E_\theta} (\neq \overline{\perp}^{E_\theta})$ such that $(\phi_\theta(s))(\overline{q}) \neq (\phi_\theta(t))(\overline{q})$. By extrapolation it follows that $(\phi(s))(\overline{q}) \neq (\phi(t))(\overline{q})$ and so $\phi(s) \neq \phi(t)$.

Claim: $\phi(\perp) = \zeta_\perp$. Using Proposition 3.4 we have $\phi_\theta(\perp) = \zeta_{\overline{\perp}^{E_\theta}}$ for all θ and so by definition $(\phi(\perp))(x) = \perp$ for all $x \in X_\perp$.

Claim: $\phi(1) = id_{X_\perp}$. It is clear that $(\phi(1))(\perp) = \perp$. Consider $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. Then by Proposition 3.4 we have $(\phi(1))(\overline{q}^{E_\theta}) = (\phi_\theta(1))(\overline{q}^{E_\theta}) = \overline{q}^{E_\theta}$ and hence $\phi(1) = id_{X_\perp}$.

Claim: $\phi(s \cdot t) = \phi(s) \cdot \phi(t)$. Clearly $(\phi(s \cdot t))(\perp) = \perp = (\phi(s) \cdot \phi(t))(\perp)$. Let $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. Suppose that $(\phi(s \cdot t))(\overline{q}) = \perp$ so that $(\phi_\theta(s \cdot t))(\overline{q}) = \overline{\perp}$

$$\begin{aligned} &\Rightarrow ((\phi_\theta(s) \cdot \phi_\theta(t))(\overline{q}) = \overline{\perp} && \text{(using Proposition 3.4)} \\ &\Rightarrow \phi_\theta(t)(\phi_\theta(s)(\overline{q})) = \overline{\perp} \\ &\Rightarrow \phi(t)(\phi_\theta(s)(\overline{q})) = \perp \end{aligned}$$

Noting that there are only two possibilities for $\phi(s)(\overline{q})$ we see that if $\phi(s)(\overline{q}) = \phi_\theta(s)(\overline{q})$ then we are through. On the other hand if $\phi(s)(\overline{q}) = \perp$ that is $\phi_\theta(s)(\overline{q}) = \overline{\perp}$ then we have $(\phi(s \cdot t))(\overline{q}) = \perp = (\phi(s) \cdot \phi(t))(\overline{q})$ which completes the proof in this case.

Consider the case where $(\phi(s \cdot t))(\overline{q}) \neq \perp$. Using Proposition 3.4 it follows that $(\phi(s \cdot t))(\overline{q}) = (\phi_\theta(s \cdot t))(\overline{q}) = (\phi_\theta(s) \cdot \phi_\theta(t))(\overline{q}) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))$ and so $(\phi_\theta(s))(\overline{q}) \neq \perp$. Consequently $\phi(t)(\phi(s)(\overline{q})) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))$ since $(\phi_\theta(s))(\overline{q}) \neq \perp$. It follows that $(\phi(s \cdot t))(\overline{q}) = (\phi(s) \cdot \phi(t))(\overline{q})$ which completes the proof. $\square$

Lemma 3.13. *The function $\rho\colon M \to \mathfrak{Z}^X$ defined by*

$$\rho(\alpha) = \sqcup_\theta \rho_\theta(\alpha)$$

is a monomorphism of C-algebras with T, F, U.

Proof. *Claim: ρ is well defined.* Let $\alpha \in M$. Using Remark 3.10(i) we have $\pi_1(\rho(\alpha)) \cap \pi_2(\rho(\alpha)) = \emptyset$ due to the distinct representation of equivalence classes. Also by Proposition 3.5 we have $\pi_1(\rho_\theta(\alpha)), \pi_2(\rho_\theta(\alpha)) \subseteq S_\theta$ and so $\perp \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha))$ that is $\rho(\alpha)$ is can be identified with a pair of sets over X.

Claim: ρ is injective. Let $\alpha \neq \beta \in M$. By Lemma 3.9 there exists a θ such that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$. Without loss of generality we infer that there exists a $\overline{q}^{E_\theta} \in \pi_1(\rho_\theta(\alpha)) \setminus \pi_1(\rho_\theta(\beta))$. Since $\rho(\alpha)$ is formed by taking the disjoint union of the individual images under $\rho_\theta(\alpha)$, using Remark 3.10(i) we can say that $\overline{q} \in \pi_1(\rho(\alpha)) \setminus \pi_1(\rho(\beta))$ that is $\rho(\alpha) \neq \rho(\beta)$.

Claim: ρ preserves the constants T, F, U. It follows easily from Proposition 3.5 that $\rho(T) = (X, \emptyset)$, $\rho(F) = (\emptyset, X)$ and $\rho(U) = (\emptyset, \emptyset)$.

Claim: $\rho(\neg\alpha) = \neg(\rho(\alpha))$. If $\alpha \in \{T, F\}$ then the result is obvious. If $\alpha \notin \{T, F\}$ then $\neg\alpha \notin \{T, F\}$. Using Proposition 3.5 we have $\rho(\neg\alpha) = (\sqcup A_\theta^{\neg\alpha}, \sqcup B_\theta^{\neg\alpha}) = (\sqcup B_\theta^{\alpha}, \sqcup A_\theta^{\alpha})$. Thus $\rho(\neg\alpha) = (\sqcup B_\theta^{\alpha}, \sqcup A_\theta^{\alpha}) = \neg(\rho(\alpha))$.

Claim: $\rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \rho(\beta)$. In view of Remark 3.10(ii) we have $\sqcup((A_\lambda, B_\lambda) \wedge (C_\lambda, D_\lambda)) = (\sqcup A_\gamma, \sqcup B_\gamma) \wedge (\sqcup C_\gamma, \sqcup D_\gamma)$ for the family of pairs of sets $(A_\lambda, B_\lambda), (C_\lambda, D_\lambda)$ where $\lambda \in \Lambda$ over X. In view of the above and Proposition 3.5 we have $\sqcup\rho_\theta(\alpha \wedge \beta) = \sqcup(\rho_\theta(\alpha) \wedge \rho_\theta(\beta)) = (\sqcup\rho_\theta(\alpha)) \wedge (\sqcup\rho_\theta(\beta)) = \rho(\alpha) \wedge \rho(\beta)$ which completes the proof. $\square$

Lemma 3.14. *The pair (ϕ, ρ) is a C-monoid monomorphism from $(S_\perp, M)$ to the functional C-monoid $\left(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\right)$.*

Proof. In view of Lemma 3.12 and Lemma 3.13 it suffices to show $\phi(\alpha[s,t]) = (\rho(\alpha))[\phi(s), \phi(t)]$ and $\rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha)$.

In order to show that $\phi(\alpha[s,t]) = (\rho(\alpha))[\phi(s), \phi(t)]$ we show that $\phi(\alpha[s,t])(x) = (\rho(\alpha))[\phi(s), \phi(t)](x)$ for all $x \in X_\perp$. Thus we have the following cases:

Case I: $x = \perp$. It is clear that $\phi(\alpha[s,t])(\perp) = \perp = (\rho(\alpha))[\phi(s),\phi(t)](\perp)$ since $\pi_1(\rho(\alpha)), \pi_2(\rho(\alpha)) \subseteq X$ and $\perp \notin X$.

Case II: $x \in X$. Consider $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. We have the following subcases:

Subcase 1: $\phi(\alpha[s,t])(\overline{q}) = \perp$. then $\phi_\theta(\alpha[s,t])(\overline{q}) = \overline{\perp}$ and so using Lemma 3.6 we have $\phi_\theta(\alpha[s,t])(\overline{q}) = \overline{\perp} = (\rho_\theta(\alpha))[\phi_\theta(s),\phi_\theta(t)](\overline{q})$. It follows that either $\overline{q} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha))$ or that $\overline{q} \in \pi_1(\rho_\theta(\alpha))$ and $\phi_\theta(s)(\overline{q}) = \overline{\perp}$ or, similarly, that $\overline{q} \in \pi_2(\rho_\theta(\alpha))$ and $\phi_\theta(t)(\overline{q}) = \overline{\perp}$. Thus we have the following:

- $\overline{q} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha))$**.** In view of Remark 3.10(i) it follows that $\overline{q} \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha))$ and so $(\rho(\alpha))[\phi(s),\phi(t)](\overline{q}) = \perp$.
- $\overline{q} \in \pi_1(\rho_\theta(\alpha))$ **and** $\phi_\theta(s)(\overline{q}) = \overline{\perp}$**.** Then $\overline{q} \in \pi_1(\rho(\alpha))$ and $\phi(s)(\overline{q}) = \perp$ and so $(\rho(\alpha))[\phi(s),\phi(t)](\overline{q}) = \perp$.
- $\overline{q} \in \pi_2(\rho_\theta(\alpha))$ **and** $\phi_\theta(t)(\overline{q}) = \overline{\perp}$**.** Along similar lines we have $(\rho(\alpha))[\phi(s),\phi(t)](\overline{q}) = \perp$.

Subcase 2: $\phi(\alpha[s,t])(\overline{q}) \neq \perp$. Then $\phi(\alpha[s,t])(\overline{q}) = \phi_\theta(\alpha[s,t])(\overline{q})$ and so using Lemma 3.6 we have $\phi(\alpha[s,t])(\overline{q}) = (\rho_\theta(\alpha))[\phi_\theta(s),\phi_\theta(t)](\overline{q})$. It follows that

$$\phi(\alpha[s,t])(\overline{q}) = (\rho_\theta(\alpha))[\phi_\theta(s),\phi_\theta(t)](\overline{q}) = \begin{cases} \phi_\theta(s)(\overline{q}), & \text{if } \overline{q} \in \pi_1(\rho_\theta(\alpha)); \\ \phi_\theta(t)(\overline{q}), & \text{if } \overline{q} \in \pi_2(\rho_\theta(\alpha)); \\ \perp, & \text{otherwise.} \end{cases}$$

- $\overline{q} \in \pi_1(\rho_\theta(\alpha))$**.** It follows that $\overline{q} \in \pi_1(\rho(\alpha))$ and so $(\rho(\alpha))[\phi(s),\phi(t)](\overline{q}) = \phi(s)(\overline{q})$. Note that $\phi_\theta(s)(\overline{q}) \neq \overline{\perp}$ else $\phi(\alpha[s,t])(\overline{q}) = \perp$, a contradiction. Thus $\phi(s)(\overline{q}) = \phi_\theta(s)(\overline{q})$ so that $\phi(\alpha[s,t])(\overline{q}) = (\rho(\alpha))[\phi(s),\phi(t)](\overline{q})$.
- $\overline{q} \in \pi_2(\rho_\theta(\alpha))$**.** The proof follows along similar lines as above.
- $\overline{q} \notin (\pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha)))$**.** This case cannot occur since we assumed that $\phi(\alpha[s,t])(\overline{q}) \neq \perp$.

Thus $\phi(\alpha[s,t]) = (\rho(\alpha))[\phi(s),\phi(t)]$.

We now show that $\rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha)$. In order to prove this we proceed by showing that

$$\pi_i(\rho(s \circ \alpha)) = \pi_i(\phi(s) \circ \rho(\alpha))$$

for $i \in \{1, 2\}$.

Let $\overline{q} \in \pi_1(\rho(s \circ \alpha)) = \sqcup \pi_1(\rho_\theta(s \circ \alpha))$. Then $\overline{q}^{E_\theta} \in S_\theta$ for some θ and $\overline{q}^{E_\theta} \in \pi_1(\rho_\theta(s \circ \alpha))$

$$\begin{aligned}
&\Rightarrow \overline{q}^{E_\theta} \in \pi_1(\phi_\theta(s) \circ \rho_\theta(\alpha)) && \text{(using Lemma 3.6)}\\
&\Rightarrow \phi_\theta(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho_\theta(\alpha)) \subseteq S_\theta\\
&\Rightarrow \phi_\theta(s)(\overline{q}^{E_\theta}) \neq \overline{\perp}\\
&\Rightarrow \phi(s)(\overline{q}^{E_\theta}) = \phi_\theta(s)(\overline{q}^{E_\theta})\\
&\Rightarrow \phi(s)(\overline{q}^{E_\theta}) \in \sqcup \pi_1(\rho_\theta(\alpha))\\
&\Rightarrow \phi(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho(\alpha))\\
&\Rightarrow \overline{q}^{E_\theta} \in \pi_1(\phi(s) \circ \rho(\alpha))
\end{aligned}$$

and so $\pi_1(\rho(s \circ \alpha)) \subseteq \pi_1(\phi(s) \circ \rho(\alpha))$.

For the reverse inclusion assume that $\overline{q} \in \pi_1(\phi(s) \circ \rho(\alpha))$. Consequently we have $\overline{q}^{E_\theta} \in S_\theta$ for some θ and $\phi(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho(\alpha)) \subseteq X$

$$\begin{aligned}
&\Rightarrow \phi(s)(\overline{q}^{E_\theta}) \neq \perp\\
&\Rightarrow \phi(s)(\overline{q}^{E_\theta}) = \phi_\theta(s)(\overline{q}^{E_\theta})(\neq \overline{\perp}^{E_\theta})\\
&\Rightarrow \phi_\theta(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho_\theta(\alpha)) && \text{(using Remark 3.10(i))}\\
&\Rightarrow \overline{q}^{E_\theta} \in \pi_1(\phi_\theta(s) \circ \rho_\theta(\alpha))\\
&\Rightarrow \overline{q}^{E_\theta} \in \pi_1(\rho_\theta(s \circ \alpha)) && \text{(using Lemma 3.14)}\\
&\Rightarrow \overline{q}^{E_\theta} \in \sqcup \pi_1(\rho_\theta(s \circ \alpha)) = \pi_1(\rho(s \circ \alpha))
\end{aligned}$$

from which it follows that $\pi_1(\phi(s) \circ \rho(\alpha)) \subseteq \pi_1(\rho(s \circ \alpha))$. Proceeding along exactly the same lines we can show that $\pi_2(\rho(s \circ \alpha)) = \pi_2(\phi(s) \circ \rho(\alpha))$ which completes the proof. □

3.4. Proof of Theorem 3.1

Let $\{\theta\}$ be the collection of all maximal congruences of M. Consider the set X as in (3.2). The functions $\phi\colon S_\perp \to \mathcal{T}_o(X_\perp)$ and $\rho\colon M \to \mathfrak{Z}^X$ as defined in Lemma 3.12 and Lemma 3.13, respectively, are monomorphisms. Further, by Lemma 3.14, the pair (ϕ, ρ) is a monomorphism from $(S_\perp, M)$ to the functional C-monoid $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$. From the construction of X it is also evident that if M and $S_\perp$ are finite then there are only finitely many maximal congruences θ on M and finitely many equivalence classes E_θ on $S_\perp$ and so X must be finite.

Corollary 3.15. *An identity is satisfied in every C-monoid $(S_\perp, M)$ where M is an ada if and only if it is satisfied in all functional C-monoids.*

In view of Corollary 3.15 and (2.1), we have the following result.

Corollary 3.16. *In every C-monoid $(S_\perp, M)$ where M is an ada we have $(f \circ T)[f, f] = f$.*

4. Conclusion and future work

The notion of C-sets axiomatize the program construct `if-then-else` considered over possibly non-halting programs and non-halting tests. In this work, we extended the axiomatization to C-monoids which include the composition of programs as well as composition of programs with tests. For the class of C-monoids where the C-algebra is an ada we obtained a Cayley-type theorem which exhibits the embedding of such C-monoids into functional C-monoids. Using this, we obtained a mechanism to determine the equivalence of programs through functional C-monoids. As future work, one may study such a representation for the general class of C-monoids with no restriction on the C-algebra. Note that the term $f \circ T$ in the standard functional model of a C-monoid represents the aspect of the domain of the function, as used in [5, 12]. It is interesting to study the relation between these two concepts in the current set up.

Appendix. Verification of examples

A.1. **Verification of Example** 2.2

We use the pairs of sets representation given by Guzmán and Squier in [7] and identify $\alpha \in \mathfrak{Z}^X$ with a *pair of sets* (A, B) of X where $A = \alpha^{-1}(T)$ and $B = \alpha^{-1}(F)$. In this representation $\mathbf{T} = (X, \emptyset), \mathbf{F} = (\emptyset, X)$ and $\mathbf{U} = (\emptyset, \emptyset)$. Thus the operation $\circ$ is given as follows:

$$(f \circ \alpha)(x) = \begin{cases} T, & \text{if } f(x) \in A; \\ F, & \text{if } f(x) \in B; \\ U, & \text{otherwise.} \end{cases}$$

In other words $f \circ \alpha$ can be identified with the pair of sets (C, D) where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$.

Axiom (2.2): Let α be identified with the pair of sets (A, B). Then $\zeta_\perp \circ \alpha = (\emptyset, \emptyset) = \mathbf{U}$ as $\zeta_\perp(x) = \perp \notin (A \cup B)$.

Axiom (2.3): Consider $\mathbf{U} = (\emptyset, \emptyset)$. Then $f \circ \mathbf{U} = (\emptyset, \emptyset) = \mathbf{U}$.

Axiom (2.4):

$$\begin{aligned}(1 \circ \alpha)(x) &= \begin{cases} T, & \text{if } id_{X_\perp}(x) \in A; \\ F, & \text{if } id_{X_\perp}(x) \in B; \\ U, & \text{otherwise} \end{cases} \\ &= \alpha(x).\end{aligned}$$

Thus $1 \circ \alpha = \alpha$.

Axiom (2.5): Let α be identified with the pair of sets (A, B). Then $f \circ \alpha = (C, D)$ where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$. Thus $\neg(f \circ \alpha) = (D, C)$. Also $f \circ (\neg\alpha) = f \circ (B, A) = (E, F)$ where $E = \{x \in X : f(x) \in B\}$ and $F = \{x \in X : f(x) \in A\}$. It follows that $(E, F) = (D, C)$.

Axiom (2.6): Let α, β be represented by the pairs of sets (A_1, A_2) and (B_1, B_2) respectively. Then $\alpha \wedge \beta = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$. Also let $f \circ \alpha = (C_1, C_2)$ where $C_1 = \{x \in X : f(x) \in A_1\}$ and $C_2 = \{x \in X : f(x) \in A_2\}$, and $f \circ \beta = (D_1, D_2)$ where $D_1 = \{x \in X : f(x) \in B_1\}$ and $D_2 = \{x \in X : f(x) \in B_2\}$. Then $(C_1, C_2) \wedge (D_1, D_2) = (C_1 \cap D_1, C_2 \cup (C_1 \cap D_2))$. Thus $C_1 \cap D_1 = \{x \in X : f(x) \in A_1 \cap B_1\}$ and $C_2 \cup (C_1 \cap D_2) = \{x \in X : f(x) \in A_2 \cup (A_1 \cap B_2)\}$. Hence $f \circ (\alpha \wedge \beta) = (f \circ \alpha) \wedge (f \circ \beta)$.

Axiom (2.7): Consider $f, g \in \mathcal{T}_o(X_\perp)$ and $\alpha \in \mathfrak{Z}^X$ represented by the pair of sets (A, B).

$$((f \cdot g) \circ \alpha)(x) = \begin{cases} T, & \text{if } g(f(x)) \in A; \\ F, & \text{if } g(f(x)) \in B; \\ U, & \text{otherwise.} \end{cases}$$

Let $g \circ \alpha = (C, D)$ where $C = \{x \in X : g(x) \in A\}$ and $D = \{x \in X : g(x) \in B\}$.

$$(f \circ (g \circ \alpha))(x) = \begin{cases} T, & \text{if } f(x) \in C; \\ F, & \text{if } f(x) \in D; \\ U, & \text{otherwise.} \end{cases}$$

We may consider the following three cases.
Case I: *$x \in X$ such that $g(f(x)) \in A$:* Then $((f \cdot g) \circ \alpha)(x) = T$. Also $f(x) \in C$ as $g(f(x)) \in A$. Thus $(f \circ (g \circ \alpha))(x) = T$.
Case II: *$x \in X$ such that $g(f(x)) \in B$:* Then $((f \cdot g) \circ \alpha)(x) = F$. Similarly $g(f(x)) \in B$ means that $f(x) \in D$. Thus $(f \circ (g \circ \alpha))(x) = F$.
Case III: *$x \in X$ such that $g(f(x)) \notin (A \cup B)$:* Then $((f \cdot g) \circ \alpha)(x) = U$. Since $f(x)$ is in neither C nor D it follows that $(f \circ (g \circ \alpha))(x) = U$.

Axiom (2.8): Consider $\alpha \in \mathfrak{Z}^X$ represented by the pair of sets (A, B).

$$(\alpha[f, g] \cdot h)(x) = h(\alpha[f, g](x)) = \begin{cases} h(f(x)), & \text{if } x \in A; \\ h(g(x)), & \text{if } x \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

Hence $\alpha[f, g] \cdot h = \alpha[f \cdot h, g \cdot h]$.

Axiom (2.9): Let $\alpha \in \mathfrak{Z}^X$ be represented by the pair of sets (A, B).

$$(h \cdot \alpha[f,g])(x) = \alpha[f,g](h(x)) = \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\ g(h(x)), & \text{if } h(x) \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

Let $h \circ \alpha$ be represented by the pair of sets (C, D) where
$C = \{x \in X : h(x) \in A\}$ and $D = \{x \in X : h(x) \in B\}$.

$$\begin{aligned} (h \circ \alpha)[h \cdot f, h \cdot g](x) &= \begin{cases} (h \cdot f)(x), & \text{if } x \in C; \\ (h \cdot g)(x), & \text{if } x \in D; \\ \perp, & \text{otherwise} \end{cases} \\ &= \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\ g(h(x)), & \text{if } h(x) \in B; \\ \perp, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $h \cdot \alpha[f,g] = (h \circ \alpha)[h \cdot f, h \cdot g]$.

Axiom (2.10): Let $\alpha, \beta \in \mathfrak{Z}^X$ be represented by the pairs of sets (A_1, A_2) and (B_1, B_2) respectively. For $f, g \in \mathcal{T}_o(X_\perp)$ we have the following:

$$h(x) = \alpha[f,g](x) = \begin{cases} f(x), & \text{if } x \in A_1; \\ g(x), & \text{if } x \in A_2; \\ \perp, & \text{otherwise.} \end{cases}$$

Also $h \circ \beta = (C_1, C_2)$ where $C_1 = \{x \in X : h(x) \in B_1\}$ and
$C_2 = \{x \in X : h(x) \in B_2\}$. Similarly $f \circ \beta = (D_1, D_2)$ where
$D_1 = \{x \in X : f(x) \in B_1\}$ and $D_2 = \{x \in X : f(x) \in B_2\}$. Let
$g \circ \beta = (E_1, E_2)$ where $E_1 = \{x \in X : g(x) \in B_1\}$ and
$E_2 = \{x \in X : g(x) \in B_2\}$. Thus
$\alpha[\![f \circ \beta, g \circ \beta]\!] = \big((A_1, A_2) \wedge (D_1, D_2)\big) \vee \big(\neg(A_1, A_2) \wedge (E_1, E_2)\big)$.
This evaluates to

$$\begin{aligned} \alpha[\![f \circ \beta, g \circ \beta]\!] &= \big(A_1 \cap D_1, A_2 \cup (A_1 \cap D_2)\big) \vee \big(A_2 \cap E_1, A_1 \cup (A_2 \cap E_2)\big) \\ &= \Big((A_1 \cap D_1) \cup \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big), \\ &\quad (A_2 \cup (A_1 \cap D_2)) \cap (A_1 \cup (A_2 \cap E_2))\Big) \\ &= (S_1, S_2) \text{ (say)} \end{aligned}$$

We show that $(C_1, C_2) = (S_1, S_2)$ by standard set theoretic arguments. First we prove that $C_1 \subseteq S_1$. Let $x \in C_1$. Then $h(x) \in B_1$. Consider the following cases:
Case I: $x \in A_1$: Then $h(x) = f(x) \in B_1$ hence $x \in D_1$. Therefore
$x \in A_1 \cap D_1$ and so $x \in S_1$.
Case II: $x \in A_2$: Then $h(x) = g(x) \in B_1$ hence $x \in E_1$. Hence
$x \in A_2 \cap E_1 \subseteq A_2$ we have $x \in S_1$.

Case III: $x \notin (A_1 \cup A_2)$: Then $h(x) = \perp \notin B_1$ a contradiction to our assumption that $h(x) \in B_1$. It follows that this case cannot occur.
We show that $S_1 \subseteq C_1$. Let $x \in S_1$. Thus $x \in A_1 \cap D_1$ or $x \in \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big)$. If $x \in A_1 \cap D_1$ then $h(x) = f(x)$ as $x \in A_1$ and $f(x) \in B_1$ as $x \in D_1$. Thus $h(x) \in B_1$ and so $x \in C_1$. If $x \in \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big)$, then $x \in (A_2 \cap E_1)$. Thus $h(x) = g(x)$ as $x \in A_2$ and $g(x) \in B_1$ as $x \in E_1$. Hence $h(x) \in B_1$, thus $x \in C_1$.
We show that $C_2 \subseteq S_2$. Let $x \in C_2$ hence $h(x) \in B_2$. Consider the following cases:
Case I: $x \in A_1$: Then $h(x) = f(x) \in B_2$, therefore $x \in D_2$. Hence $x \in A_1 \cap D_2 \subseteq A_1$ and so $x \in S_2$.
Case II: $x \in A_2$: Then $h(x) = g(x) \in B_2$ therefore $x \in E_2$. Thus $x \in A_2 \cap E_2 \subseteq A_2$ and so $x \in S_2$.
Case III: $x \notin (A_1 \cup A_2)$: Then $h(x) = \perp \notin B_2$ which is a contradiction. It follows that this case cannot occur.
Finally we show that $S_2 \subseteq C_2$. Since $A_1 \cap A_2 = \emptyset$ it follows that $x \in A_1 \cap D_2$ or $x \in A_2 \cap E_2$. If $x \in A_1 \cap D_2$ then $h(x) = f(x) \in B_2$ and hence $x \in C_2$. If $x \in A_2 \cap E_2$ then $h(x) = g(x) \in B_2$ hence $x \in C_2$.
Thus $\alpha[f, g] \circ \beta = \alpha[\![f \circ \beta, g \circ \beta]\!]$.

A.2. **Verification of Example** 2.3

Let $f, g, h \in S_\perp^X$ and $\alpha, \beta \in \mathfrak{Z}^X$.
Axiom (2.2): It is easy to see that $(\zeta_\perp \circ \alpha)(x) = U$ for all $x \in X$.

Axiom (2.3): It is clear that $(f \circ \mathbf{U})(x) = U$.

Axiom (2.4): Since $S_\perp$ is non-trivial we must have $1 \neq \perp$. If not then for $a \in S_\perp \setminus \{\perp\}$ we have $a = a \cdot 1 = a \cdot \perp = \perp$ a contradiction. It follows that $\zeta_1 \neq \zeta_\perp$. Hence $(\zeta_1 \circ \alpha)(x) = \alpha(x)$ as $\zeta_1(x) = 1 \neq \perp$.

Axiom (2.5): We have

$$\begin{aligned}
(f \circ (\neg\alpha))(x) &= \begin{cases} (\neg\alpha)(x), & \text{if } f(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= \begin{cases} \neg(\alpha(x)), & \text{if } f(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= \neg(f \circ \alpha)(x).
\end{aligned}$$

Thus $f \circ (\neg\alpha) = \neg(f \circ \alpha)$.

Axiom (2.6): We have

$$\begin{aligned}
(f \circ (\alpha \wedge \beta))(x) &= \begin{cases} (\alpha \wedge \beta)(x), & \text{if } f(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= \begin{cases} \alpha(x) \wedge \beta(x), & \text{if } f(x) \neq \perp; \\ U \wedge U, & \text{otherwise} \end{cases} \\
&= (f \circ \alpha)(x) \wedge (f \circ \beta)(x).
\end{aligned}$$

Thus $f \circ (\alpha \wedge \beta) = (f \circ \alpha) \wedge (f \circ \beta)$.

Axiom (2.7): Since $S_\perp$ has no zero-divisors we have
$f(x) \cdot g(x) = \perp \Leftrightarrow f(x) = \perp$ or $g(x) = \perp$. Consequently

$$\begin{aligned}
((f \cdot g) \circ \alpha)(x) &= \begin{cases} \alpha(x), & \text{if } (f \cdot g)(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= \begin{cases} \alpha(x), & \text{if } f(x) \cdot g(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= \begin{cases} \alpha(x), & \text{if } f(x) \neq \perp \text{ and } g(x) \neq \perp; \\ U, & \text{otherwise} \end{cases} \\
&= (f \circ (g \circ \alpha))(x).
\end{aligned}$$

Thus $(f \cdot g) \circ \alpha = f \circ (g \circ \alpha)$.

Axiom (2.8): We have

$$\begin{aligned}
(\alpha[f,g] \cdot h)(x) = \alpha[f,g](x) \cdot h(x) &= \begin{cases} f(x) \cdot h(x), & \text{if } \alpha(x) = T; \\ g(x) \cdot h(x), & \text{if } \alpha(x) = F; \\ \perp, & \text{otherwise} \end{cases} \\
&= \alpha[f \cdot h, g \cdot h](x).
\end{aligned}$$

Thus $\alpha[f,g] \cdot h = \alpha[f \cdot h, g \cdot h]$.

Axiom (2.9): Consider

$$h \cdot \alpha[f,g](x) = h(x) \cdot \alpha[f,g](x) = \begin{cases} h(x) \cdot f(x), & \text{if } \alpha(x) = T; \\ h(x) \cdot g(x), & \text{if } \alpha(x) = F; \\ \perp, & \text{otherwise.} \end{cases}$$

On the other hand

$$(h \circ \alpha)[h \cdot f, h \cdot g](x) = \begin{cases} h(x) \cdot f(x), & \text{if } (h \circ \alpha)(x) = T; \\ h(x) \cdot g(x), & \text{if } (h \circ \alpha)(x) = F; \\ \perp, & \text{otherwise.} \end{cases}$$

Note that if $h(x) = \perp$ then $h \cdot \alpha[f,g](x) = \perp = (h \circ \alpha)[h \cdot f, h \cdot g](x)$.
Suppose that $h(x) \neq \perp$ then $(h \circ \alpha)(x) = \alpha(x)$. It is clear that in this case

as well $h \cdot \alpha[f,g](x) = (h \circ \alpha)[h \cdot f, h \cdot g](x)$ holds. Thus
$h \cdot \alpha[f,g] = (h \circ \alpha)[h \cdot f, h \cdot g]$.

Axiom (2.10): Consider

$$(\alpha[f,g] \circ \beta)(x) = \begin{cases} \beta(x), & \text{if } \alpha[f,g](x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \beta(x), & \text{if } (f(x) \neq \bot, \alpha(x) = T) \text{ or } (g(x) \neq \bot, \alpha(x) = F); \\ U, & \text{otherwise.} \end{cases}$$

We have $(\alpha[\![f \circ \beta, g \circ \beta]\!])(x) = (\alpha(x) \wedge (f \circ \beta)(x)) \vee (\neg\alpha(x) \wedge (g \circ \beta)(x))$.
If $f(x) \neq \bot$ and $\alpha(x) = T$ we have $(\alpha[\![f \circ \beta, g \circ \beta]\!])(x) =$
$(T \wedge \beta(x)) \vee (F \wedge (g \circ \beta)(x)) = \beta(x) \vee F = \beta(x) = (\alpha[f,g] \circ \beta)(x)$.
If $g(x) \neq \bot$ and $\alpha(x) = F$ we have $(\alpha[\![f \circ \beta, g \circ \beta]\!])(x) =$
$(F \wedge (f \circ \beta)(x)) \vee (T \wedge \beta(x)) = F \vee \beta(x) = \beta(x) = (\alpha[f,g] \circ \beta)(x)$.
In all other cases it can be easily ascertained that
$(\alpha[\![f \circ \beta, g \circ \beta]\!])(x) = U = (\alpha[f,g] \circ \beta)(x)$. Thus $\alpha[f,g] \circ \beta = \alpha[\![f \circ \beta, g \circ \beta]\!]$.

A.3. **Verification of Example** 2.4

Axiom (2.2): It is clear that $\bot \circ \alpha = U$.

Axiom (2.3): It is obvious that $t \circ U = U$.

Axiom (2.4): Since $S_\bot$ is non-trivial it follows that $1 \neq \bot$. Consequently
$1 \circ \alpha = \alpha$.

Axiom (2.5): If $s = \bot$ then $s \circ (\neg\alpha) = U = \neg(s \circ \alpha)$. If $s \neq \bot$ then
$s \circ (\neg\alpha) = \neg\alpha = \neg(s \circ \alpha)$. Thus $s \circ (\neg\alpha) = \neg(s \circ \alpha)$.

Axiom (2.6): If $s = \bot$ then $s \circ (\alpha \wedge \beta) = U$ and
$(s \circ \alpha) \wedge (s \circ \beta) = U \wedge U = U$. If $s \neq \bot$ then
$s \circ (\alpha \wedge \beta) = \alpha \wedge \beta = (s \circ \alpha) \wedge (s \circ \beta)$. Thus $s \circ (\alpha \wedge \beta) = (s \circ \alpha) \wedge (s \circ \beta)$.

Axiom (2.7): Consider $s, t \in S_\bot$ such that $s \cdot t = \bot$. Then
$(s \cdot t) \circ \alpha = \bot \circ \alpha = U$. Since $S_\bot$ has no non-zero zero-divisors we have
$s = \bot$ or $t = \bot$ and so $s \circ (t \circ \alpha) = U$ in either case. If $s \cdot t \neq \bot$ then
$(s \cdot t) \circ \alpha = \alpha$ and $s \circ (t \circ \alpha) = t \circ \alpha = \alpha$ as neither s nor t are $\bot$. Thus
$(s \cdot t) \circ \alpha = s \circ (t \circ \alpha)$.

Axiom (2.8): As $\alpha \in \{T, F, U\}$ we consider the following three cases:
Case I: $\alpha = T$: Then $\alpha[s,t] \cdot u = T[s,t] \cdot u = s \cdot u = T[s \cdot u, t \cdot u]$.
Case II: $\alpha = F$: Then $\alpha[s,t] \cdot u = F[s,t] \cdot u = t \cdot u = F[s \cdot u, t \cdot u]$.
Case III: $\alpha = U$: Then $\alpha[s,t] \cdot u = U[s,t] \cdot u = \bot \cdot u = \bot = U[s \cdot u, t \cdot u]$.
Thus $\alpha[s,t] \cdot u = \alpha[s \cdot u, t \cdot u]$.

Axiom (2.9): Consider the following cases:
Case I: $r = \perp$: Then
$r \cdot \alpha[s,t] = \perp \cdot \alpha[s,t] = \perp = U[r \cdot s, r \cdot t] = (\perp \circ \alpha)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.
Case II: $r \neq \perp$: We again consider the following three cases:
Case i: $\alpha = T$:
$r \cdot \alpha[s,t] = r \cdot T[s,t] = r \cdot s = T[r \cdot s, r \cdot t] = (r \circ T)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.
Case ii: $\alpha = F$:
$r \cdot \alpha[s,t] = r \cdot F[s,t] = r \cdot t = F[r \cdot s, r \cdot t] = (r \circ F)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.
Case iii: $\alpha = U$:
$r \cdot \alpha[s,t] = r \cdot U[s,t] = r \cdot \perp = \perp = U[r \cdot s, r \cdot t] = (r \circ U)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.
Thus $r \cdot \alpha[s,t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Axiom (2.10): Consider the following three cases:
Case I: $\alpha = T$: $\alpha[s,t] \circ \beta = T[s,t] \circ \beta = s \circ \beta = T[\![s \circ \beta, t \circ \beta]\!]$.
Case II: $\alpha = F$: $\alpha[s,t] \circ \beta = F[s,t] \circ \beta = t \circ \beta = F[\![s \circ \beta, t \circ \beta]\!]$.
Case III: $\alpha = U$: $\alpha[s,t] \circ \beta = U[s,t] \circ \beta = \perp \circ \beta = U = U[\![s \circ \beta, t \circ \beta]\!]$.
Thus $\alpha[s,t] \circ \beta = \alpha[\![s \circ \beta, t \circ \beta]\!]$.

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References

[1] Bergman, G.M.: Actions of Boolean rings on sets. Algebra Universalis **28**, 153–187 (1991)

[2] Bloom, S.L., Ésik, Z., Manes, E.G.: A Cayley theorem for Boolean algebras. Amer. Math. Monthly **97**, 831–833 (1990)

[3] Bloom, S.L., Tindell, R.: Varieties of "if-then-else". SIAM J. Comput. **12**, 677–707 (1983)

[4] Bochvar, D.A.: Ob odnom tréhznacnom isčislenii i égo priménénii k analiza paradoksov klassičéskogo răssirénnogo funkcional'nogo isčisléniá (in Russian). *matématičeskij sbornik*, 4: 287–308, 1939. Translated to English by M. Bergmann "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus". History and Philosophy of Logic **2**, 87–112 (1981)

[5] Desharnais, J., Jipsen, P., Struth, G.: Domain and antidomain semigroups. In: Relations and Kleene algebra in computer science, *Lecture Notes in Comput. Sci.*, vol. 5827, pp. 73–87. Springer, Berlin (2009)

[6] Guessarian, I., Meseguer, J.: On the axiomatization of "if-then-else". SIAM J. Comput. **16**, 332–357 (1987)

[7] Guzmán, F., Squier, C.C.: The algebra of conditional logic. Algebra Universalis **27**, 88–110 (1990)

[8] Heyting, A.: Die formalen regeln der intuitionistischen logik, sitzungsberichte der preuszischen akademie der wissenschaften, physikalischmathematische klasse,(1930), 42–56 57–71 158–169 in three parts. Sitzungsber. preuss. Akad. Wiss **42**, 158–169 (1934)

[9] Igarashi, S.: Semantics of ALGOL-like statements. In: Symposium on Semantics of Algorithmic Languages, pp. 117–177. Springer (1971)

[10] Jackson, M., Stokes, T.: Agreeable semigroups. J. Algebra **266**, 393–417 (2003)

[11] Jackson, M., Stokes, T.: Semigroups with if-then-else and halting programs. Int. J. Algebra Comput. **19**, 937–961 (2009)

[12] Jackson, M., Stokes, T.: Monoids with tests and the algebra of possibly non-halting programs. J. Log. Algebr. Methods Program. **84**, 259–275 (2015)

[13] Kennison, J.F.: Triples and compact sheaf representation. J. Pure Appl. Algebra **20**, 13–38 (1981)

[14] Kleene, S.: On notation for ordinal numbers. The Journal of Symbolic Logic **3**, 150–155 (1938)

[15] Lukasiewicz, J.: On three-valued logic. Ruch Filozoficzny **5**,(1920), English translation in Borkowski, L.(ed.) 1970. Jan Lukasiewicz: Selected Works (1920)

[16] Manes, E.G.: Adas and the equational theory of if-then-else. Algebra Universalis **30**, 373–394 (1993)

[17] Manes, E.G.: A transformational characterization of if-then-else. Theoret. Comput. Sci. **71**, 413–417 (1990)

[18] McCarthy, J.: A basis for a mathematical theory of computation. In: Computer programming and formal systems, pp. 33–70. North-Holland, Amsterdam (1963)

[19] Mekler, A.H., Nelson, E.M.: Equational bases for if-then-else. SIAM J. Comput. **16**, 465–485 (1987)

[20] Panicker, G., Krishna, K.V., Bhaduri, P.: Axiomatization of `if-then-else` over possibly non-halting programs and tests. Int. J. Algebra Comput. **27**, 273–297 (2017)

[21] Pigozzi, D.: Equality-test and if-then-else algebras: Axiomatization and specification. SIAM J. Comput. **20**, 766–805 (1991)

[22] Sethi, R.: Conditional expressions with equality tests. J. ACM **25**, 667–674 (1978)

[23] Stokes, T.: Sets with B-action and linear algebra. Algebra Universalis **39**, 31–43 (1998)

[24] Stokes, T.: Comparison semigroups and algebras of transformations. Semigroup Forum **81**, 325–334 (2010)

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