On the structure of C-algebras through atomicity and `if-then-else`

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Abstract. This paper investigates the notions of atoms and atomicity in C-algebras and obtains a characterisation of atoms in the C-algebra of transformations. In this connection, various characterisations for the existence of suprema of subsets of C-algebras are obtained. Further, this work presents some necessary conditions and some sufficient conditions for the atomicity of C-algebras and shows that the class of finite atomic C-algebras is precisely the class of finite adas. This paper also uses the intrinsic `if-then-else` action to study the structure of C-algebras and classify the elements of the C-algebra of transformations.

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1. Introduction

There are several studies in the literature (e.g., [1, 2, 3, 8, 10, 11, 12] and many more) on extending two-valued Boolean logic to one that is three-valued, depending on the interpretation of the third truth value, `undefined`. The three-valued logic proposed by McCarthy in [15] models the short-circuit evaluation exhibited by programming languages that evaluates expressions in sequential order, from left to right. In [7], Guzmán and Squier gave a complete axiomatisation of McCarthy's three-valued logic and called the corresponding algebra a C-algebra, or the algebra of conditional logic. While studying `if-then-else` algebras, Manes in [14] defined an ada (algebra of disjoint alternatives) which is a C-algebra equipped with an oracle for the halting problem.

In [9] Jackson and Stokes studied the algebraic theory of computable functions, which can be viewed as possibly non-halting programs, equipped with composition, `if-then-else` and `while-do`. In that work they assumed that the tests form a Boolean algebra. Further, they proposed the problem

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of characterising the algebras of computable functions associated with a C-algebra of non-halting tests. In order to address the problem, recently, Panicker et al. introduced the notion of C-sets and studied an axiomatisation of `if-then-else` over C-algebras in [16, 17]. These works lead to a study of the structure of C-algebras with a twofold aim. First, the structural properties of C-algebras would help us in further understanding the problem in question. Second, we propose to study the structure of C-algebras using C-sets. In the literature, Swamy and his associates studied various properties of the algebraic structure of C-algebras (e.g., see [18, 19, 20, 21]). In contrast, motivated by the above-mentioned context, this work focuses on the structure of C-algebras through atomicity and the intrinsic `if-then-else` action.

The concept of atoms plays a key role in achieving a structural representation of Boolean algebras. In this work, using the partial order defined by Chang in [5] for MV-algebras, we adapt the notion of atoms in Boolean algebras to C-algebras and study their structure in Section 3. We also extend some well-known results related to atomicity for Boolean algebras to C-algebras. Further, using the built-in `if-then-else` action on C-algebras, we introduce a notion of annihilators and investigate various structural properties of C-algebras (see Section 4).

The organisation of this paper is as follows. In Section 2, we recall the definitions of C-algebras and adas along with various results that are useful in this work. In Section 3.1, we introduce the notion of atoms and study some fundamental properties with the goal of studying the structural properties of C-algebras. In Section 3.2, we provide various characterisations for existence of suprema of subsets of C-algebras and define the notion of atomic C-algebras. Focusing on the C-algebra $\mathbb{3}^X$, in Section 3.3, we characterise the atoms in $\mathbb{3}^X$ (cf. Theorem 3.22), and, as a consequence, we establish that the C-algebra $\mathbb{3}^X$ is atomic (cf. Theorem 3.26). Further, in Section 3.4, we obtain some necessary and some sufficient conditions for the atomicity of C-algebras. Finally, in Section 3.4.3, we give a characterisation for finite atomic C-algebras and establish that they are precisely finite adas (cf. Theorem 3.43). A precise characterisation of arbitrary atomic C-algebras is left as an open problem.

To further study the structure of C-algebras, in Section 4.1, we introduce a notion of annihilators in a C-algebra through its inherent `if-then-else` action. The annihilator operator provides us a Galois connection, which in turn, yields closed sets. These closed sets become an internal tool to understand the structure of C-algebras. In Section 4.2, we give a characterisation for the closed sets in the C-algebra $\mathbb{3}^X$ (cf. Theorem 4.8) and show that this collection forms a complete Boolean algebra (cf. Theorem 4.12). We also obtain a partition of $\mathbb{3}^X$ such that the elements of the Boolean algebra $\mathbb{2}^X$ is a single equivalence class (cf. Theorem 4.13). In Section 5, we conclude the paper with a brief discussion on the problems emerging out of this work.

Throughout the paper several examples are provided for illustration of results and remarks.

2. C-algebras and adas

In this section, we provide the necessary definitions and results from the literature and fix the notation. One may refer to a book on universal algebra (e.g., [4]) for other notions used in this paper.

McCarthy in [15] studied a three-valued logic in the context of programming languages. This is a non-commutative regular extension of Boolean logic to three truth values. Here the third truth value U denotes the `undefined` state, while T and F represent `true` and `false` respectively. In this context, the evaluation of expressions is carried out sequentially from left to right, mimicking that of a majority of programming languages. A complete axiomatisation for the class of algebras associated with this logic was given by Guzmán and Squier in [7] and they called the algebra associated with this logic a *C-algebra*.

Definition 2.1. A *C-algebra* is an algebra $\langle M, \vee, \wedge, \neg\rangle$ of type $(2, 2, 1)$, which satisfies the following axioms for all $\alpha, \beta, \gamma \in M$:

$$\neg\neg\alpha = \alpha \tag{2.1}$$

$$\neg(\alpha \wedge \beta) = \neg\alpha \vee \neg\beta \tag{2.2}$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \tag{2.3}$$

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \tag{2.4}$$

$$(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta \wedge \gamma) \tag{2.5}$$

$$\alpha \vee (\alpha \wedge \beta) = \alpha \tag{2.6}$$

$$(\alpha \wedge \beta) \vee (\beta \wedge \alpha) = (\beta \wedge \alpha) \vee (\alpha \wedge \beta) \tag{2.7}$$

Remark 2.2. In view of the axioms (2.1) and (2.2), a C-algebra satisfies the dual of each of its axioms from (2.2) to (2.7). Also, using (2.6) and its dual, one may notice that $\vee$ and $\wedge$ are idempotent operations.

Example 2.3. Every Boolean algebra is a C-algebra. In particular, the two-element Boolean algebra $\mathbb{2}$ is a C-algebra.

Example 2.4. Let $\mathbb{3}$ denote the C-algebra with the universe $\{T, F, U\}$ and the following operations. This is, in fact, McCarthy's three-valued logic. Note that neither $\wedge$ nor $\vee$ is commutative.

$\neg$	
T	F
F	T
U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

Remark 2.5. Since the class of C-algebras is a variety, for any set X, $\mathbb{3}^X$ (the set of all mappings from X to $\mathbb{3}$) is a C-algebra with the operations defined pointwise. In fact, in [7] Guzmán and Squier showed that the elements of the C-algebra $\mathbb{3}^X$ can be viewed as *pairs of sets*. This is a pair (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$. Akin to the well-known correlation between $\mathbb{2}^X$

and the power set $\wp(X)$ of X, for any element $\alpha \in \mathfrak{3}^X$, we can associate the pair of sets $(\alpha^{-1}(T), \alpha^{-1}(F))$. Conversely, for any pair of sets (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$, we can associate the function α where $\alpha(x) = T$ if $x \in A$, $\alpha(x) = F$ if $x \in B$ and $\alpha(x) = U$ otherwise. With this correlation, the operations can be expressed as follows:

$$\neg(A_1, A_2) = (A_2, A_1)$$
$$(A_1, A_2) \wedge (B_1, B_2) = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$$
$$(A_1, A_2) \vee (B_1, B_2) = ((A_1 \cup (A_2 \cap B_1), A_2 \cap B_2)$$

Further, Guzmán and Squier showed that every C-algebra is a subalgebra of $\mathfrak{3}^X$ for some X as stated below. In this paper, $\preceq$ is used for the subalgebra relation.

Theorem 2.6 ([7]). *$\mathfrak{3}$ and $\mathfrak{2}$ are the only subdirectly irreducible C-algebras. Hence, every C-algebra is a subalgebra of a product of copies of $\mathfrak{3}$.*

Definition 2.7. A *C-algebra with T, F, U* is a C-algebra with nullary operations T, F, U, where T is the (unique) left-identity (and right-identity) for $\wedge$, F is the (unique) left-identity (and right-identity) for $\vee$ and U is the (unique) fixed point for $\neg$. Note that U is also a left-zero for both $\wedge$ and $\vee$ while F is a left-zero for $\wedge$.

Notation 2.8. The constants T, F, U of the C-algebra $\mathfrak{3}^X$ will be denoted by $\mathbf{T}, \mathbf{F}, \mathbf{U}$, respectively, and they can be identified by the pairs of sets $(X, \emptyset), (\emptyset, X)$, $(\emptyset, \emptyset)$, respectively.

Unless specified otherwise, in this paper, M always denotes a C-algebra with T, F, U. When M is considered as a subalgebra of $\mathfrak{3}^X$ (i.e., $M \preceq \mathfrak{3}^X$, for some X), the constants T, F, U of M will also be denoted by $\mathbf{T}, \mathbf{F}, \mathbf{U}$, respectively. While we denote elements of M by $a, b, c, \alpha, \beta, \gamma$ and δ, the elements of the C-algebra $\mathfrak{3}^X$ will only be denoted by α, β, γ and δ.

Remark 2.9. The subset $\{\,\alpha \in M : \alpha \vee \neg\alpha = T\,\}$ of M forms a Boolean algebra under the induced operations $\vee, \wedge, \neg, T$ and F.

Definition 2.10. The algebra $M_{\#}$ is the Boolean algebra in Remark 2.9.

We now recall an important expansion of C-algebras, viz., *ada*s (algebras of disjoint alternatives) introduced by Manes in [14].

Definition 2.11. An *ada* is a C-algebra M with T, F, U equipped with an additional unary operation $(\ )^{\downarrow}$, which is an oracle for the halting problem, subject to the following equations for all $\alpha, \beta \in M$:

$$U^{\downarrow} = F = F^{\downarrow} \tag{2.8}$$
$$T^{\downarrow} = T \tag{2.9}$$
$$\alpha \wedge \beta^{\downarrow} = \alpha \wedge (\alpha \wedge \beta)^{\downarrow} \tag{2.10}$$
$$\alpha^{\downarrow} \vee \neg(\alpha^{\downarrow}) = T \tag{2.11}$$
$$\alpha = \alpha^{\downarrow} \vee \alpha \tag{2.12}$$

Example 2.12. The three-element C-algebra $\mathbb{3}$ with the unary operation $(\ )^{\downarrow}$ defined as follows forms an ada.

$$T^{\downarrow} = T$$
$$U^{\downarrow} = F = F^{\downarrow}$$

We use $\tilde{\mathbb{3}}$ to denote this three-element ada.

The C-algebra $\mathbb{3}$ is not functionally-complete. However, the ada $\tilde{\mathbb{3}}$ is functionally-complete. In fact, the variety of adas is generated by the ada $\tilde{\mathbb{3}}$. In [14], Manes showed that $\tilde{\mathbb{3}}$ is the only subdirectly irreducible ada. For any set X, $\tilde{\mathbb{3}}^X$ is an ada with operations defined pointwise. Note that the ada $\tilde{\mathbb{3}}$ is also simple.

Remark 2.13. Since adas are C-algebras with T, F, U, with an additional operation, every C-algebra M freely generates an ada $\widehat{M}$. That is, there exists a C-algebra homomorphism $\phi\colon M \to \widehat{M}$ with the universal property that for each ada A and C-algebra homomorphism $f\colon M \to A$ there exists a unique ada homomorphism $\psi\colon \widehat{M} \to A$ with $\psi(\phi(x)) = f(x)$ for all $x \in M$. In [14], Manes called such an ada the *enveloping ada* of M.

Manes also showed the following result.

Proposition 2.14 ([14])**.** *Let A be an ada. Then $A^{\downarrow} = \{\, \alpha^{\downarrow} : \alpha \in A \,\}$ forms a Boolean algebra under the induced operations.*

Remark 2.15. In fact, $A^{\downarrow} = A_{\#}$. Also, $A^{\downarrow} = \{\, \alpha \in A : \alpha^{\downarrow} = \alpha \,\}$.

Further, as outlined in the following remark, Manes established that the category of adas and the category of Boolean algebras are equivalent.

Remark 2.16 ([14])**.** Let Q be a Boolean algebra. By Birkhoff's representation of Boolean algebras, suppose Q is a subalgebra of $\mathbb{2}^X$ for some set X. Consider the subalgebra $Q^{\star}$ of the ada $\tilde{\mathbb{3}}^X$ with the universe $Q^{\star} = \{\, (E, F) : E \cap F = \emptyset \,\}$ given in terms of pairs of subsets of X. Note that the map $Q \mapsto (Q^{\star})_{\#}$ is a Boolean algebra isomorphism. Similarly, for an ada A, the map $A \mapsto (A_{\#})^{\star}$ is an ada isomorphism. Hence, the functors based on the assignments above establish that the category of adas and the category of Boolean algebras are equivalent.

Remark 2.17. Since the only finite Boolean algebras are $\mathbb{2}^X$ for finite X, using Remark 2.16, we see that the only finite adas are $\tilde{\mathbb{3}}^X$ for finite X.

3. Atomicity

In order to study the structure of C-algebras through atomicity, in this section, we first introduce the notion of atoms in C-algebras and study their properties. Then we define atomic C-algebras using the concept of suprema of subsets. In this work we provide characterisations for existence of suprema of subsets of C-algebras under various contexts. We prove that $\mathbb{3}^X$ is atomic through a characterisation of its atoms. Subsequently, we present some necessary and some sufficient conditions for the atomicity of C-algebras. Finally, we establish that finite atomic C-algebras are precisely finite adas.

3.1. Atoms and their properties

We begin by defining a partial order on C-algebras. Define the relation $\leq$ on a C-algebra M by $a \leq b$ if $a \vee b = b$. Clearly $\leq$ is reflexive and transitive. Considering $M \preceq \mathfrak{3}^X$, for some set X, one can verify that $\leq$ is also antisymmetric so that $\leq$ is a partial ordering on M.

Remark 3.1. This partial order does not induce a lattice structure on C-algebras. Note that in the C-algebra $\mathfrak{3}$ we have $F \leq T$ and $F \leq U$ while $T \nleq U$ and $U \nleq T$. The Hasse diagram for $\mathfrak{3}$ is given below:

$$\begin{tikzcd}
T & & U \\
& F \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

In fact, $F \leq a$ for all $a \in M$, a C-algebra with T, F, U.

We now define the notion of an atom in C-algebras with T, F, U.

Definition 3.2. Let $A \subseteq M$ with $F \in A$. An element $a \in A \setminus \{F\}$ is said to be an *atom* of A if for all $b \in A$ if $F \leq b \leq a$ and $b \neq a$ then $b = F$. We denote the set of atoms of A by $\mathscr{A}(A)$.

Note that $\mathscr{A}(\mathfrak{3}) = \{T, U\}$ and $\mathscr{A}(\mathfrak{3}^2) = \{(T, F), (F, T), (F, U), (U, F)\}$. If $A = \mathfrak{3}^2 \setminus \{(T, F), (F, T)\}$ then $\mathscr{A}(A) = \{(T, T), (F, U), (U, F)\}$. It is clear that for all finite M, we have $\mathscr{A}(M) \neq \emptyset$.

Proposition 3.3. *If $a, b \in M$ such that $a \leq b$ and $b \in M_\#$ then $a \in M_\#$.*

Proof. Since the identity $a \vee \neg a = a \vee T$ holds in $\mathfrak{3}$, it holds in all C-algebras. We have $a \vee b = b$ since $a \leq b$. Further, $b \in M_\#$ gives $b \vee \neg b = T$. Thus, $a \vee \neg a = a \vee T = a \vee (b \vee \neg b) = (a \vee b) \vee \neg b$ (by Remark 2.2) $= b \vee \neg b = T$ which completes the proof. □

We have the following corollary, which can also be proved independently.

Corollary 3.4. *If $a \in M$ such that $a \leq T$ then $a \in M_\#$.*

Proposition 3.5. *The following hold for all $\alpha, \gamma, \delta \in M$:*

(i) $\alpha \wedge F \leq \alpha$.
(ii) $\alpha \wedge F \leq U$.
(iii) $\alpha \wedge F = U \Leftrightarrow \alpha = U$.
(iv) $\alpha \wedge F = F \Leftrightarrow \alpha \in M_\#$.
(v) $\alpha \wedge F = \alpha \Leftrightarrow \alpha \wedge \beta = \alpha$ *for all* $\beta \in M$.
(vi) $\alpha \leq \gamma \Rightarrow \alpha \wedge \gamma = \alpha$.
(vii) $\alpha \leq \alpha \vee \beta$ *for all* $\beta \in M$.
(viii) $\alpha \leq \delta$ *and* $\gamma \leq \delta \Rightarrow \alpha \vee \gamma \leq \delta$.

Proof. The items (i) and (ii) are straightforward.

(iii) Clearly $U \wedge F = U$. Suppose that $\alpha \wedge F = U$. Since $M \preceq \mathfrak{Z}^X$ for some set X we have $\alpha(x) \wedge F = U$ for all $x \in X$. If $\alpha(x_o) \in \{T, F\}$ for some $x_o \in X$ then $\alpha(x_o) \wedge F = F$, a contradiction. Hence, $\alpha(x) = U$ for all $x \in X$ so that $\alpha = U$ in M.

(iv) Clearly if $\alpha \in M_{\#}$ then $\alpha \wedge F = F$. Note that the identities $\alpha \wedge F = \alpha \wedge \neg\alpha$ and $\neg\alpha \vee \alpha = \alpha \vee \neg\alpha$ hold in all C-algebras since they hold in $\mathfrak{Z}$. Thus, $\alpha \wedge \neg\alpha = F$. Using (2.2) we have $\neg\alpha \vee \alpha = T$ so that $\alpha \vee \neg\alpha = T$. Consequently, $\alpha \in M_{\#}$.

(v) It is clear that $\alpha \wedge \beta = \alpha$ for all $\beta \in M \Rightarrow \alpha \wedge F = \alpha$. Suppose that $\alpha \wedge F = \alpha$. Then for $\beta \in M$ we have $\alpha \wedge \beta = (\alpha \wedge F) \wedge \beta = \alpha \wedge (F \wedge \beta) = \alpha \wedge F = \alpha$.

(vi) Since $\alpha \leq \gamma$ we have $\alpha \vee \gamma = \gamma$. Thus, $\alpha \wedge \gamma = \alpha \wedge (\alpha \vee \gamma) = \alpha$ (by Remark 2.2).

(vii) Using Remark 2.2, $\alpha \vee (\alpha \vee \beta) = (\alpha \vee \alpha) \vee \beta = \alpha \vee \beta$. Thus, $\alpha \leq \alpha \vee \beta$.

(viii) Consider $(\alpha \vee \gamma) \vee \delta = \alpha \vee (\gamma \vee \delta) = \alpha \vee \delta = \delta$. Thus, $\alpha \vee \gamma \leq \delta$. $\square$

Remark 3.6. Note that the converse of Proposition 3.5(vi) is not true in general. For instance $U \wedge F = U$, however $U \nleq F$.

For $A \subseteq M$, we write A^c for the complement of A in M, i.e., $A^c = M \backslash A$. Further, $(M_{\#})^c$ will be simply written as $M_{\#}^c$.

Proposition 3.7. *$\mathscr{A}(M) \cap M_{\#} = \mathscr{A}(M_{\#})$. Moreover,*

$$\mathscr{A}(M) \cap M_{\#}^c \subseteq \{\, a \in M : a \wedge b = a \text{ for all } b \in M \,\}.$$

Proof. Let $a \in \mathscr{A}(M) \cap M_{\#}$. Suppose there exists $b \in M_{\#}$ such that $F \lneq b \lneq a$. But, since $b \in M$, we get a contradiction to $a \in \mathscr{A}(M)$. Conversely, if $a \in \mathscr{A}(M_{\#})$, then clearly $a \in M_{\#}$. If there exists $b \in M$ such that $F \lneq b \lneq a$ then using Proposition 3.3 we have $b \in M_{\#}$. This contradicts $a \in \mathscr{A}(M_{\#})$. Hence, $\mathscr{A}(M) \cap M_{\#} = \mathscr{A}(M_{\#})$.

Let $a \in \mathscr{A}(M) \cap M_{\#}^c$. In order to show that a is a left-zero for $\wedge$, using Proposition 3.5(v) it suffices to show that $a \wedge F = a$. Suppose $a \wedge F \neq a$. Using Proposition 3.5(i) we have $a \wedge F \lneq a$ and so since $a \in \mathscr{A}(M)$ it must follow that $a \wedge F = F$. Then by Proposition 3.5(iv), $a \in M_{\#}$ which contradicts our assumption that $a \in M_{\#}^c$. Hence, $a \wedge F = a$ so that a is a left-zero for $\wedge$. $\square$

Notation 3.8. For $A \subseteq X$, $\varphi_{T,A}$ denotes the element represented by the pair of sets (A, A^c) in $\mathfrak{Z}^X$. Also, the element represented by the pair of sets $(\emptyset, A^c)$ is denoted by $\varphi_{U,A}$. If $A = \{x\}$ then we simply use the notation $\varphi_{T,x}$ and $\varphi_{U,x}$.

The following result gives a necessary condition for atoms of M.

Theorem 3.9. *If $a \in \mathscr{A}(M)$ then $a \wedge b \leq b$ or $a \wedge b = a$ for all $b \in M$.*

Proof. Let $a \in \mathscr{A}(M)$ and $b \in M$. If $a \in \mathscr{A}(M) \cap M_{\#}^c$ then using Proposition 3.7 we have a is a left-zero for $\wedge$ from which the result follows. On the other hand, if $a \in \mathscr{A}(M) \cap M_{\#}$ then consider $M \preceq \mathfrak{Z}^X$ for some set X. Thus, $a = \varphi_{T,A}$ for some $\emptyset \neq A \subseteq X$ so that

$$(a \wedge b)(x) = \begin{cases} b(x), & \text{if } x \in A; \\ F, & \text{otherwise.} \end{cases}$$

Hence, $((a \wedge b) \vee b)(x) = b(x)$ for all $x \in X$ so that $a \wedge b \leq b$. □

Remark 3.10. The condition given in Theorem 3.9 is not sufficient for $\mathscr{A}(M)$. For instance consider (U, U, F, F) in the C-algebra $\mathfrak{Z}^4$. This is a left-zero for $\wedge$ but is not an atom since $(F, F, F, F) \leq (U, F, F, F) \leq (U, U, F, F)$.

Remark 3.11.

(i) For $a \in \mathscr{A}(M)$ and $b \in M$ either $a \leq b$ or $a \wedge b \leq b$ need not hold in general. Let $M = \{(T,T,T,T), (F,F,F,F), (U,U,U,U), (T,T,F,F), (F,F,T,T), (U,U,F,F), (U,U,T,T), (F,F,U,U), (T,T,U,U)\} \preceq \mathfrak{Z}^4$. For $a = (F,F,U,U) \in \mathscr{A}(M)$ and $b = (U,U,T,T) \in M$, we have $a = (F,F,U,U) \nleq (U,U,T,T) = b$ and $a \wedge b = (F,F,U,U) \nleq (U,U,T,T) = b$. Note that in this case $a \wedge b = a$.

(ii) For $a \in \mathscr{A}(M)$ and $b \in M$ it need not be true that $a \wedge b \in \mathscr{A}(M)$. Consider $M = \{(T,T), (F,F), (U,U), (F,U), (T,U)\} \preceq \mathfrak{Z}^2$. Take $a = (T,T) \in \mathscr{A}(M)$ and $b = (T,U) \in M$. Then $a \wedge b = (T,U) \notin \mathscr{A}(M)$.

(iii) For $a, b \in M$ it need not be true that $b \leq a \vee b$. For instance in $\mathfrak{Z}$ we have $T \nleq U \vee T = U$.

(iv) For $a, b \in M$ we need not have $a \wedge b \leq a$ nor $a \wedge b \leq b$ in general. Consider $M = \mathfrak{Z}^3$, $a = (T,U,F)$ and $b = (U,T,F)$. Then $a \wedge b = (U,U,F) \nleq (T,U,F) = a$ and $a \wedge b = (U,U,F) \nleq (U,T,F) = b$.

(v) For $a \in \mathscr{A}(M)$ it need not be true that $a \wedge U \in \mathscr{A}(M)$. Consider $M = \{(T,T), (F,F), (U,U), (F,U), (T,U)\} \preceq \mathfrak{Z}^2$. Note that $\mathscr{A}(M) = \{(T,T), (F,U)\}$. For $a = (T,T)$, $a \wedge \mathbf{U} = (U,U) \notin \mathscr{A}(M)$.

We now provide a characterisation for left-zeros of M in terms of certain elements of the enveloping ada $\widehat{M}$ (cf. Remark 2.13).

Proposition 3.12. *The following are equivalent for all $\beta \in M$:*

(i) *β is a left-zero for $\wedge$.*

(ii) *$\beta \wedge F = \beta$.*

(iii) *$\beta^{\downarrow} = F$ in $\widehat{M}$.*

Proof. Let $\widehat{M} \preceq \tilde{\mathfrak{Z}}^X$ for some set X.

((i) ⇔ (ii)) This is shown in Proposition 3.5(v).

((ii) ⇒ (iii)) Let $\beta \wedge F = \beta$. Then $(\beta \wedge \mathbf{F})(x) = \beta(x)$ gives $\beta(x) \in \{F, U\}$ for all $x \in X$. Thus, $(\beta^{\downarrow})(x) = (\beta(x))^{\downarrow} = F$ for all $x \in X$. Hence, $\beta^{\downarrow} = F$ in $\widehat{M}$.

((iii)) ⇒ (ii)) Let $\beta^{\downarrow} = F$ in $\widehat{M}$. Then $(\beta^{\downarrow})(x) = (\beta(x))^{\downarrow} = F$ for all $x \in X$. It follows that $\beta(x) \in \{F, U\}$ for all $x \in X$ and so $(\beta \wedge \mathbf{F})(x) = \beta(x)$ for all $x \in X$. Hence, $\beta \wedge F = \beta$ in M. □

We now establish a relation between atoms of $M_{\#}$ and those of $M^c_{\#}$ for an ada M.

Theorem 3.13. *Let M be an ada. There exists a bijection between the sets $\mathscr{A}(M) \cap M^c_{\#}$ and $\mathscr{A}(M) \cap M_{\#}$.*

Proof. Let $M \preceq \tilde{\mathfrak{Z}}^X$ for some set X. Consider $G\colon \mathscr{A}(M) \cap M^c_{\#} \to \mathscr{A}(M) \cap M_{\#}$ given by

$$G(\alpha) = \neg((\neg\alpha)^{\downarrow}).$$

Let $\alpha \in \mathscr{A}(M) \cap M^c_{\#}$. It is straightforward to deduce that $G(\alpha) \in M_{\#}$. Since α is a left-zero for $\wedge$ (cf. Proposition 3.7) we have $\alpha = \varphi_{U,A}$ for some $\emptyset \neq A \subseteq X$. It follows that $G(\alpha) = \neg((\neg\alpha)^{\downarrow}) = \varphi_{T,A}$. If $G(\alpha)$ is not an atom of $M_{\#}$ then there exists $\gamma = \varphi_{T,B}$ where $\emptyset \neq B \subsetneq A$ and $\mathbf{F} \lneq \gamma \lneq \delta$. Thus, $\beta = \gamma \wedge \mathbf{U} = \varphi_{U,B}$ and $\mathbf{F} \lneq \beta \lneq \alpha$ which contradicts the fact that $\alpha \in \mathscr{A}(M) \cap M^c_{\#}$. It follows that G is well-defined.

Suppose that $\neg((\neg\alpha)^{\downarrow}) = \neg((\neg\beta)^{\downarrow})$ for some $\alpha, \beta \in \mathscr{A}(M) \cap M^c_{\#}$. Then $(\neg\alpha)^{\downarrow} = (\neg\beta)^{\downarrow} \in M_{\#}$. Then $(\neg\alpha)^{\downarrow} = (\neg\beta)^{\downarrow} = \varphi_{T,A}$ for some $A \subseteq X$. It follows that $\neg\alpha$ and $\neg\beta$ can be represented by the pairs of sets (A, B_α) and (A, B_β) where $B_\alpha, B_\beta \subseteq A^c$. Thus, α and β can be represented by the pairs of sets (B_α, A) and (B_β, A) where $B_\alpha, B_\beta \subseteq A^c$. Since $\alpha, \beta \in \mathscr{A}(M) \cap M^c_{\#}$ we have $\alpha = \varphi_{U,C}$ and $\beta = \varphi_{U,D}$ for some $C, D \subseteq X$. Hence, in the representation for α and β that is (B_α, A) and (B_β, A) respectively we must have $B_\alpha = \emptyset = B_\beta$. It follows that $\alpha = \beta$ and so G is injective.

Let $\beta \in \mathscr{A}(M) \cap M_{\#}$. It follows that $\beta = \varphi_{T,A}$ for some $\emptyset \neq A \subseteq X$. Consider $\alpha = \beta \wedge \mathbf{U} = \varphi_{U,A} \in M^c_{\#}$. Along similar lines as in the proof for the well-definedness of G, we show that $\alpha \in \mathscr{A}(M) \cap M^c_{\#}$. Further, $G(\alpha) = \beta$ so that G is surjective. $\square$

Corollary 3.14. *Let M be a finite ada. Then $|\mathscr{A}(M)|$ is even.*

3.2. Supremum and atomicity

In this section, we study properties of suprema of subsets, using which we introduce the notion of atomicity in C-algebras. We first make an important observation.

Remark 3.15. A representation of an element as a join of atoms need not be unique in a C-algebra. For example in the C-algebra $A = \mathfrak{Z}^2 \setminus \{(T,F),(F,T)\}$, we have $(T,T) = (T,T) \vee (F,U)$ and also $(T,T) = (T,T) \vee (U,F)$.

For $A = \{\, a_i : i \in I \,\} \subseteq M$, if the supremum of A (in short $\sup A$) exists in M, then, in view of Remark 3.15, we denote the $\sup A$ by

$$\bigoplus_{i \in I} a_i$$

rather than writing it as a join of elements of A. We characterise the supremum under various contexts in this work.

Note that if $\phi\colon M \to \mathfrak{Z}^X$ is a C-algebra embedding, then ϕ is order-preserving. Hence, if $M \preceq \mathfrak{Z}^X$ for some X, then the order of M will be preserved in $\mathfrak{Z}^X$.

Proposition 3.16. *Let $M \preceq \mathfrak{Z}^X$ for some set X and let $\{ a_i : i \in I \} \subseteq M$ such that $\bigoplus_{i\in I} a_i$ exists. Given $x \in X$, we have the following:*

(L1) *If $a_i(x) = T$ for some $i \in I$ then $a_j(x) \in \{T, F\}$ for all $j \in I$.*
(L2) *If $a_i(x) = U$ for some $i \in I$ then $a_j(x) \in \{U, F\}$ for all $j \in I$.*

Proof. Suppose that $a_i(x) = T$ and $a_j(x) = U$ for some $i, j \in I$. Let $a = \bigoplus_{i\in I} a_i$. It follows that $a_i \leq a$ and $a_j \leq a$ and therefore $T = a_i(x) \vee a(x) = a(x) = a_j(x) \vee a(x) = U$; a contradiction. Hence, the result follows. □

Proposition 3.17. *Let $\{ \alpha_i : i \in I \} \subseteq \mathfrak{Z}^X$. Then $\bigoplus_{i\in I} \alpha_i$ exists in $\mathfrak{Z}^X$ if and only if* (L1) *and* (L2) *hold. In this case, for $x \in X$,*

$$\bigoplus_{i\in I} \alpha_i(x) = \begin{cases} T, & \textit{if there exists } i \in I \textit{ such that } \alpha_i(x) = T; \\ U, & \textit{if there exists } i \in I \textit{ such that } \alpha_i(x) = U; \\ F, & \textit{otherwise.} \end{cases}$$

Proof. Let $A = \{ \alpha_i : i \in I \}$. If $\bigoplus_{i\in I} \alpha_i$ exists then (L1) and (L2) hold due to Proposition 3.16. Conversely, if (L1) and (L2) hold, define the following element in $\mathfrak{Z}^X$:

$$\alpha(x) = \begin{cases} T, & \text{if there exists } i \in I \text{ such that } \alpha_i(x) = T; \\ U, & \text{if there exists } i \in I \text{ such that } \alpha_i(x) = U; \\ F, & \text{otherwise.} \end{cases}$$

Note that the element α is well-defined due to (L1) and (L2). Let $i \in I$ be given. We ascertain that $\alpha_i \leq \alpha$, that is $\alpha_i \vee \alpha = \alpha$, or in other words $\alpha_i(x) \vee \alpha(x) = \alpha(x)$ for all $x \in X$. Let $x \in X$ be given. If $\alpha_i(x) = T$ then by the definition of α, we have $\alpha(x) = T$ so that $\alpha_i(x) \vee \alpha(x) = T = \alpha(x)$. The case when $\alpha_i(x) = U$ follows along similar lines. If $\alpha_i(x) = F$ it is clear that $\alpha_i(x) \vee \alpha(x) = \alpha(x)$. Thus, α is an upper bound of A. Further, let β be an upper bound of A. Therefore, $\alpha_i \leq \beta$ for all $i \in I$. Let $x \in X$ be given. Consider the following cases:

(i) $\alpha(x) = T$: From the definition of α it follows that $\alpha_i(x) = T$ for some $i \in I$. Since $\alpha_i(x) \leq \beta(x)$ we have $\alpha_i(x) \vee \beta(x) = T = \beta(x)$. Hence, $\alpha(x) \vee \beta(x) = T = \beta(x)$ so that $\alpha(x) \leq \beta(x)$.
(ii) $\alpha(x) = U$: The case follows along similar lines as above.
(iii) $\alpha(x) = F$: Clearly $\alpha(x) \vee \beta(x) = \beta(x)$ so that $\alpha(x) \leq \beta(x)$.

Thus, $\alpha(x) \leq \beta(x)$ for all $x \in X$ from which the result follows. □

Remark 3.18. It can be observed that (L1) holds if and only if (L2) holds. Suppose that (L1) holds but (L2) does not. Then there exist $i, j \in I$ such that $a_i(x) = U$ but $a_j(x) = T$. This clearly contradicts condition (L1). The converse follows similarly.

Remark 3.19. In connection to the converse of Proposition 3.16, the conditions (L1) and (L2) do not ensure the existence of suprema in M for its subsets, while suprema may exist in $\mathfrak{Z}^X$. To illustrate this, consider $M \subseteq \mathfrak{Z}^X$ defined using the pair of sets representation by

$$(A, B) \in M \text{ if and only if } A^c \text{ is finite or } B^c \text{ is finite or } (A \cup B) \text{ is finite.}$$

It is routine to verify that M is a subalgebra (with T, F, U) of $\mathfrak{Z}^X$. Let $X = \mathbb{N}$, the set of natural numbers. Consider the set $\{ \alpha_n : n \in \mathbb{N} \}$ of elements of M given by the following: If n is odd, $\alpha_n = (\{n\}, \{n\}^c)$; otherwise, when n is even, $\alpha_n = (\emptyset, \{n\}^c)$. In fact, if n is odd, for any $x \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} T, & \text{if } x = n; \\ F, & \text{otherwise.} \end{cases}$$

On the other hand, if n is even, for any $x \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} U, & \text{if } x = n; \\ F, & \text{otherwise.} \end{cases}$$

Hence, the set $\{ \alpha_n : n \in \mathbb{N} \}$ satisfies (L1) and (L2). So, by Proposition 3.17, $\bigoplus_{n \in \mathbb{N}} \alpha_n$ exists in $\mathfrak{Z}^X$ and, for $x \in \mathbb{N}$,

$$\bigoplus_{n \in \mathbb{N}} \alpha_n(x) = \begin{cases} T, & \text{if } x \text{ is odd;} \\ U, & \text{otherwise.} \end{cases}$$

Clearly, $\bigoplus_{n \in \mathbb{N}} \alpha_n \notin M$ and hence $\{ \alpha_n : n \in \mathbb{N} \}$ has no supremum in M. However, every finite subset of $\{ \alpha_n : n \in \mathbb{N} \}$ has supremum in M. In fact, for any finite subset I of $\mathbb{N}$, $\bigoplus_{n \in I} \alpha_n = (O, \mathbb{N} \setminus I) \in M$, where O is the set of odd numbers in I.

Proposition 3.20. *Let $\{ a_i : 1 \leq i \leq n \}$ be a finite set of elements of M. Then $\bigoplus_{1 \leq i \leq n} a_i$ exists if and only if for every rearrangement of $(a_i)_{i=1}^n$ the join of these elements remain unchanged, i.e., for every bijection*

$$\sigma \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

we have

$$a_{\sigma(1)} \vee a_{\sigma(2)} \vee \cdots \vee a_{\sigma(n)} = a_1 \vee a_2 \vee \cdots \vee a_n.$$

In this case,

$$\bigoplus_{i=1}^{n} a_i = a_1 \vee a_2 \vee \cdots \vee a_n.$$

Proof. Let $A = \{ a_i : 1 \leq i \leq n \}$ and suppose that $\sup A$ exists, say $a = \sup A$. It suffices to show that given $i, j \leq n$ we have $a_i \vee a_j = a_j \vee a_i$. Consider $M \preceq \mathfrak{Z}^X$ for some set X. Let $x \in X$ be given. If $a_i(x) = T$ then from Proposition 3.16 we have $a_j(x) \in \{T, F\}$ so that $a_i(x) \vee a_j(x) = T = a_j(x) \vee$

$a_i(x)$. The proof for the case when $a_i(x) = U$ follows along similar lines. If $a_i(x) = F$ then clearly $a_i(x) \vee a_j(x) = a_j(x) = a_j(x) \vee a_i(x)$. Therefore, since $a_i(x) \vee a_j(x) = a_j(x) \vee a_i(x)$ for all $x \in X$, we have $a_i \vee a_j = a_j \vee a_i$.

Conversely, suppose $a_i \vee a_j = a_j \vee a_i$ for all $i, j \leq n$. We claim that $a_1 \leq a_1 \vee \cdots \vee a_n$. By Remark 2.2, we have $a_1 \vee (a_1 \vee \cdots \vee a_n) = (a_1 \vee a_1) \vee (a_2 \vee \cdots \vee a_n) = a_1 \vee a_2 \vee \cdots \vee a_n$. Therefore, $a_1 \leq a_1 \vee a_2 \vee \cdots \vee a_n$. Along similar lines, using the fact that elements of A commute, we can show that $a_i \leq a_1 \vee a_2 \vee \cdots \vee a_n$ for all $i \leq n$. Thus, $a_1 \vee a_2 \vee \cdots \vee a_n$ is an upper bound of A. Let b be an upper bound of A. Then $a_i \leq b$ for all $1 \leq i \leq n$. Thus, by Remark 2.2, we have $(a_1 \vee a_2 \vee \cdots \vee a_n) \vee b = (a_1 \vee a_2 \vee \cdots \vee a_{n-1}) \vee (a_n \vee b) = (a_1 \vee a_2 \vee \cdots \vee a_{n-1}) \vee b$ and so on. Thus, $(a_1 \vee a_2 \vee \cdots \vee a_n) \vee b = b$ so that $a_1 \vee a_2 \vee \cdots \vee a_n$ is the least upper bound of A and $\sup A = \bigoplus_{i=1}^{n} a_i = a_1 \vee a_2 \vee \cdots \vee a_n$. $\square$

Definition 3.21. Let M be a C-algebra with T, F, U. We say that M is *atomic* if for every $a \in M \setminus \{F\}$ there exists a set of atoms $\{\, a_i : i \in I \,\} \subseteq \mathscr{A}(M)$ such that $a = \bigoplus\limits_{i \in I} a_i$.

Note that the C-algebra $\mathfrak{Z}^2$ is atomic. Whereas, the C-algebra $A = \mathfrak{Z}^2 \setminus \{(T, F), (F, T)\}$ is not atomic. For instance (T, U) is not supremum of any set of atoms in A.

3.3. Atomicity of $\mathfrak{Z}^X$

We consider the C-algebra $\mathfrak{Z}^X$ and first establish a characterisation for its atoms. Then we prove that $\mathfrak{Z}^X$ is atomic for any X.

Theorem 3.22. *Let X be any set. Then $\mathscr{A}(\mathfrak{Z}^X) = \{\, \alpha \in \mathfrak{Z}^X$: there exists a unique $x_o \in X$ such that $\alpha(x_o) \in \{T, U\} \,\}$.*

Proof. Let $\alpha \in A = \{\, \alpha \in \mathfrak{Z}^X$: there exists a unique $x_o \in X$ such that $\alpha(x_o) \in \{T, U\} \,\}$. Let $\beta \in \mathfrak{Z}^X$ such that $\mathbf{F} \leq \beta \lneq \alpha$. Since $\alpha \in A$ we have $\alpha(x) = F$ for all $x \neq x_o$. Thus, since $F \leq \beta(x) \leq \alpha(x)$, we must have $\beta(x) = F$ for all $x \neq x_o$. Further, since $\beta \lneq \alpha$ we must have $\beta(x_o) \lneq \alpha(x_o)$ and so $\beta(x_o) = F$ (cf. Remark 3.1). Consequently, $\beta = \mathbf{F}$. Hence, $\alpha \in \mathscr{A}(M)$.

Conversely, suppose that $\alpha \in \mathscr{A}(M)$ but $\alpha \notin A$. Then either $\alpha(x) = F$ for all $x \in X$ or there exist $x_o, y_o \in X$, where $x_o \neq y_o$, and $\alpha(x_o), \alpha(y_o) \in \{T, U\}$. In the former case, clearly $\alpha = \mathbf{F} \notin \mathscr{A}(M)$, which is a contradiction. In the latter case, consider $\beta \in M$ given by the following:

$$\beta(x) = \begin{cases} \alpha(x), & \text{if } x \neq x_o; \\ F, & \text{if } x = x_o. \end{cases}$$

It is easy to see that $F \leq \beta(x) \leq \alpha(x)$ for all $x \in X$ and so $\mathbf{F} \leq \beta \leq \alpha$. Since $\beta(x_o) = F \lneq \alpha(x_o)$ and $\beta(y_o) = \alpha(y_o) \in \{T, U\}$ we have $\mathbf{F} \lneq \beta \lneq \alpha$. Again, this contradicts $\alpha \in \mathscr{A}(M)$. Hence, the result follows. $\square$

This gives us the following result on the cardinality of $\mathscr{A}(\mathfrak{Z}^X)$.

Corollary 3.23. *For $X \neq \emptyset$ we have $|\mathscr{A}(\mathfrak{Z}^X)| = |X \sqcup X|$, where $\sqcup$ denotes the disjoint union.*

Since every finite ada is isomorphic to $\tilde{\mathfrak{Z}}^X$, for some set X (cf. Remark 2.17) we note that Corollary 3.23 is in fact a stronger version of Corollary 3.14.

Notation 3.24. Let $\alpha \in \mathscr{A}(\mathfrak{Z}^X)$. Using Theorem 3.22, denote by x_α the unique element of X satisfying $\alpha(x_\alpha) \in \{T, U\}$.

Let $\{\, \alpha_i : i \in I \,\}$ be a set of atoms in $\mathfrak{Z}^X$. If $x_{\alpha_i} \neq x_{\alpha_j}$ for all $i \neq j$, then $\{\, \alpha_i : i \in I \,\}$ satisfies conditions (L1) and (L2), and hence, using Proposition 3.17 we have $\sup\{\, \alpha_i : i \in I \,\} = \bigoplus_{i \in I} \alpha_i$ exists. Conversely, if $\bigoplus_{i \in I} \alpha_i$ exists, then using Proposition 3.17 and Theorem 3.22, we have $x_{\alpha_i} \neq x_{\alpha_j}$ for all $i \neq j$. Thus, we have the following characterisation for the supremum of a set of atoms in $\mathfrak{Z}^X$.

Lemma 3.25. *Let $\{\, \alpha_i : i \in I \,\}$ be a set of atoms in $\mathfrak{Z}^X$. Then $\bigoplus_{i \in I} \alpha_i$ exists if and only if $x_{\alpha_i} \neq x_{\alpha_j}$ for all $i \neq j$. Further, using Theorem 3.22 and the expression for the supremum given in Proposition 3.17, we have*

$$\bigoplus_{i \in I} \alpha_i(x) = \begin{cases} \alpha_p(x_{\alpha_p}), & \text{if } x = x_{\alpha_p}; \\ F, & \text{otherwise.} \end{cases}$$

Theorem 3.26. *The C-algebra $\mathfrak{Z}^X$ is atomic for any set X.*

Proof. Let $\beta \in \mathfrak{Z}^X$ such that $\beta \neq \mathbf{F}$. Using the pairs of sets representation of $\mathfrak{Z}^X$ identify β with the pair of sets (A, B). Since $\beta \neq \mathbf{F}$ it follows that $B^c \neq \emptyset$. Consider the family of elements defined by the following for $y \in B^c$:

$$\alpha_y(x) = \begin{cases} \beta(y), & \text{if } x = y; \\ F, & \text{otherwise.} \end{cases}$$

Using Theorem 3.22 since $\alpha_y(y) = \beta(y) \in \{T, U\}$ we have $\alpha_y \in \mathscr{A}(\mathfrak{Z}^X)$ for each $y \in B^c$. Using Lemma 3.25, it is clear that $\bigoplus_{y \in B^c} \alpha_y = \beta$. □

3.4. Necessary and/or sufficient conditions

First we provide some sufficient and some necessary conditions for atomicity of an arbitrary C-algebra M using the atoms of $\mathfrak{Z}^X$ for which M is a subalgebra (cf. Section 3.4.1) and using $M_\#$ (cf. Section 3.4.2). Finally we characterise finite atomic C-algebras in Section 3.4.3.

3.4.1. Using the atoms of $\mathfrak{Z}^X$. We consider $M \preceq \mathfrak{Z}^X$ and study the atomicity of M from information about the atoms of $\mathfrak{Z}^X$. In this connection, we provide a sufficient condition for atomicity of M when X is finite (cf. Corollary 3.31).

Remark 3.27. Let $M \preceq \mathfrak{Z}^X$ for some X. Note that $M \cap \mathscr{A}(\mathfrak{Z}^X) \subseteq \mathscr{A}(M)$. In general the inclusion could be proper. For example, consider

$$M = \{(T,T), (F,F), (U,U), (F,U), (U,F), (T,U), (U,T)\}$$

where $M \preceq \mathfrak{Z}^2$. Then $\mathscr{A}(M) = \{(F,U), (U,F), (T,T)\} \supsetneq M \cap \mathscr{A}(\mathfrak{Z}^2)$ since $(T,T) \notin \mathscr{A}(\mathfrak{Z}^2)$. Thus, not all atoms of M are atoms of $\mathfrak{Z}^X$.

We focus our study on C-algebras $M \preceq \mathfrak{Z}^X$ in which every atom of M is an atom of $\mathfrak{Z}^X$, i.e., $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$. Equivalently, $\mathscr{A}(M) = M \cap \mathscr{A}(\mathfrak{Z}^X)$.

Remark 3.28. The proper subalgebras of $\mathfrak{Z}^2$ are as follows:

$$\begin{aligned}
M_0 &= \{(T,T), (F,F), (U,U)\},\\
M_1 &= \{(T,T), (F,F), (U,U), (F,U), (T,U)\},\\
M_2 &= \{(T,T), (F,F), (U,U), (U,F), (U,T)\},\\
M_3 &= \{(T,T), (F,F), (U,U), (F,U), (T,U), (U,F), (U,T)\}.
\end{aligned}$$

The set of atoms of each subalgebra is as follows:

$$\begin{aligned}
\mathscr{A}(M_0) &= \{(T,T), (U,U)\},\\
\mathscr{A}(M_1) &= \{(T,T), (F,U)\},\\
\mathscr{A}(M_2) &= \{(T,T), (U,F)\},\\
\mathscr{A}(M_3) &= \{(T,T), (F,U), (U,F)\}.
\end{aligned}$$

Since $(T,T) \in \mathscr{A}(M_i)$ for $0 \le i \le 3$ and $(T,T) \notin \mathscr{A}(\mathfrak{Z}^2)$, no proper subalgebra of $\mathfrak{Z}^2$ satisfies $\mathscr{A}(M_i) \subseteq \mathscr{A}(\mathfrak{Z}^2)$.

In fact, for finite X, if $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$ then we prove that $M = \mathfrak{Z}^X$ in Theorem 3.30. To that aim we first have the following result.

Lemma 3.29. *For finite X, let $M \preceq \mathfrak{Z}^X$. If $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$ then $M = \mathfrak{Z}^X$.*

Proof. Let $\beta \in \mathfrak{Z}^X$. If $\beta = \mathbf{F}$, then clearly $\beta \in M$. Suppose that $\beta \ne \mathbf{F}$. Then for the pair of sets representation (A,B) of β, we have $B^c \ne \emptyset$. As in the proof of Theorem 3.26, we have $\beta = \bigoplus_{y \in B^c} \alpha_y$, where, for $y \in B^c$, $\alpha_y \in \mathscr{A}(\mathfrak{Z}^X)$ is given by

$$\alpha_y(x) = \begin{cases} \beta(y), & \text{if } x = y;\\ F, & \text{otherwise.} \end{cases}$$

Since $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$, we have $\alpha_y \in \mathscr{A}(M) \subseteq M$. Further, since X is finite, so is B^c. Consequently, there are only finitely many such α_y in $\bigoplus_{y \in B^c} \alpha_y$. Hence, $\bigoplus_{y \in B^c} \alpha_y \in M$ so that $\beta \in M$. Thus, $M = \mathfrak{Z}^X$. □

Theorem 3.30. *For finite X, let $M \preceq \mathfrak{Z}^X$ such that $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$. Then $M = \mathfrak{Z}^X$.*

Proof. We show that $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$ by generating all atoms from one another in an algorithmic method. Hence, by Lemma 3.29 we have $M = \mathfrak{Z}^X$. It suffices to show that $\varphi_{T,x} \in M$ for each $x \in X$, because if $\varphi_{T,x} \in M$ then $\varphi_{T,x} \wedge \mathbf{U} = \varphi_{U,x} \in M$.

Since X is finite we have M is finite. So, for $\mathbf{T} \in M$ there exists $\alpha \in \mathscr{A}(M)$ such that $\alpha \leq \mathbf{T}$. Then, by Proposition 3.3, $\alpha \in M_{\#} \preceq \mathfrak{2}^X$. Since $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$ we have $\alpha \in \mathscr{A}(\mathfrak{Z}^X)$ so that $\alpha = \varphi_{T,x_1}$ for some $x_1 \in X$.

Define $\beta_1 = \varphi_{T,x_1}$ and so $\neg\beta_1 = \neg\varphi_{T,x_1} = \varphi_{T,X\setminus\{x_1\}}$. If $\neg\beta_1$ is an atom then $X \setminus \{x_1\}$ is a singleton and so $X = \{x_1, x_2\}$ and so the algebra is $\mathfrak{Z}^2$. The only subalgebra in which every atom is an atom of $\mathfrak{Z}^2$ is $\mathfrak{Z}^2$ itself (cf. Remark 3.28) and we are done.

If $\neg\beta_1$ is not an atom then there exists $\varphi_{T,x_2} \in \mathscr{A}(M)$ such that $\varphi_{T,x_2} \leq \neg\beta_1 \leq \mathbf{T}$. Define $\beta_2 = \varphi_{T,x_2}$ and so $\neg\beta_2 = \neg\varphi_{T,x_2} = \varphi_{T,X\setminus\{x_2\}}$. If $\neg\beta_2$ is an atom then we are through. Else there exists $\varphi_{T,x_3} \in \mathscr{A}(M)$ such that $\varphi_{T,x_3} \leq \neg\beta_2 \leq \mathbf{T}$. Define $\beta_3 = \varphi_{T,x_3}$ and so $\neg\beta_3 = \neg\varphi_{T,x_3} = \varphi_{T,X\setminus\{x_3\}}$ and so on.

This process can take at most $|X|$ steps. Further, as mentioned above if $\varphi_{T,x} \in M$ then $\varphi_{T,x} \wedge \mathbf{U} = \varphi_{U,x} \in M$ so that $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$. Hence, $M = \mathfrak{Z}^X$. $\square$

Now, in view of Theorem 3.26, we have the following corollary of Theorem 3.30.

Corollary 3.31. *For finite X, let $M \preceq \mathfrak{Z}^X$ such that $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$. Then M is atomic.*

Remark 3.32. The condition in Corollary 3.31 is not necessary for M to be atomic.

(i) For instance, $\mathscr{A}(M_0) \not\subseteq \mathscr{A}(\mathfrak{Z}^2)$ for the C-algebra M_0 as given in Remark 3.28. However, M_0 is clearly atomic.

(ii) For a nontrivial example, consider

$$M = \left\{ \begin{array}{c} (T,T,T,T),(F,F,F,F),(U,U,U,U), \\ (T,T,F,F),(F,F,T,T),(U,U,F,F), \\ (U,U,T,T),(F,F,U,U),(T,T,U,U) \end{array} \right\} \preceq \mathfrak{Z}^4.$$

Note that $\mathscr{A}(M) = \{(T,T,F,F),(F,F,T,T),(U,U,F,F),(F,F,U,U)\}$ and so $\mathscr{A}(M) \not\subseteq \mathscr{A}(\mathfrak{Z}^4)$. However, M is atomic.

Remark 3.33. Lemma 3.29 and Theorem 3.30 do not hold in general for arbitrary X. To illustrate this, consider the C-algebra $M \preceq \mathfrak{Z}^X$ given in Remark 3.19, for $X = \mathbb{N}$. While M is a proper subalgebra of $\mathfrak{Z}^X$, we show that $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$. First note that, by Theorem 3.22, $\alpha = (A,B) \in \mathscr{A}(\mathfrak{Z}^X)$ if and only if $|B^c| = 1$. Accordingly, we have $\mathscr{A}(\mathfrak{Z}^X) \subseteq \mathscr{A}(M)$. For reverse inclusion, let $\alpha = (A,B) \in \mathscr{A}(M)$. By considering the following three possibilities, we prove that $|B^c| = 1$.

(i) If $A \cup B$ is finite, then choose $x_o \in (A \cup B)^c$ and set $\beta = (A, B \cup \{x_o\})$. Note that $\beta \in M$ and $\mathbf{F} \lneq \beta \lneq \alpha$. But since α is an atom of M, this is not possible. Hence, $A \cup B$ cannot be finite.

(ii) If A^c is finite, then choose $x_o \in A$ and set $\beta = (A \setminus \{x_o\}, B \cup \{x_o\})$. Clearly, $\beta \in M$ and $\mathbf{F} \lneq \beta \lneq \alpha$. Hence, as earlier, A^c cannot be finite.

(iii) Now the possibility is that B^c is finite. Since α is an atom, $B^c \neq \emptyset$. Suppose that $|B^c| \geq 2$. Choose $x_o \in B^c$ and set $D = B \cup \{x_o\}$. Then D^c is finite and $X \neq D \neq B$. Note that $\alpha(x_o) \in \{T, U\}$. If $\alpha(x_o) = U$, then let $C = A$; otherwise, let $C = A \setminus \{x_o\}$. Now $\beta = (C, D) \in M$ and $\mathbf{F} \lneq \beta \lneq \alpha$. This contradicts the fact that α is an atom of M.

Hence, $|B^c| = 1$ so that $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$.

3.4.2. Using $M_\#$. We now investigate the relation between the atomicity of $M_\#$ and that of M. If M is a finite algebra, it is straightforward that $M_\#$ is always atomic even when M is not atomic. However, the question remains in the case where M is infinite. In this section, we give results to produce various atomless and non-atomic C-algebras depending on $M_\#$. Further, we give a necessary condition for atomicity in Theorem 3.37.

Theorem 3.34. *If M is atomic then $M_\#$ is atomic.*

Proof. Suppose M is atomic. Let $a \in M_\# \subseteq M$ then there exist $\{a_i\}_{i \in I} \subseteq \mathscr{A}(M)$ such that $a = \bigoplus_{i \in I} a_i$. Since $a = \bigoplus_{i \in I} a_i = \sup\{\, a_i : i \in I \,\}$ we have $a_i \leq a$. Consequently, by Proposition 3.3, we have $a_i \in M_\#$. Further, using Proposition 3.7, we have $\mathscr{A}(M) \cap M_\# = \mathscr{A}(M_\#)$ so that $a_i \in \mathscr{A}(M_\#)$. Thus, $M_\#$ is atomic. Hence, the result follows. □

Theorem 3.35. *Let M be an ada. If $M_\#$ is atomless then M is atomless.*

Proof. Suppose $M_\#$ is atomless but $\mathscr{A}(M) \neq \emptyset$. Let $\alpha \in \mathscr{A}(M)$. It is clear that $\alpha \notin M_\#$; otherwise, using Proposition 3.7, we have $\alpha \in \mathscr{A}(M) \cap M_\# = \mathscr{A}(M_\#)$, which contradicts $M_\#$ is atomless. Thus, $\alpha \in M_\#^c$ and so $\alpha^\downarrow \neq \alpha$ (cf. Remark 2.15). We consider the following two cases and derive contradictions. Accordingly, the result follows.

Case I: $\alpha^\downarrow \neq F$: From the ada identity $\alpha^\downarrow \vee \alpha = \alpha$, we have $F \lneq \alpha^\downarrow \lneq \alpha$, which contradicts $\alpha \in \mathscr{A}(M)$.

Case II: $\alpha^\downarrow = F$: Since $\alpha \in \mathscr{A}(M) \cap M_\#^c$, α is a left-zero for $\wedge$ (cf. Proposition 3.7). Consider $M \preceq \tilde{\mathfrak{Z}}^X$ for some set X. It follows that $\alpha = \varphi_{_{U,A}}$ for some $A \subseteq X$. This is because if $\alpha(x) = T$ for some $x \in X$ then $\alpha^\downarrow(x) = T$ and so $\alpha^\downarrow \neq \mathbf{F}$, a contradiction. Also $A \neq \emptyset$ since $\alpha \neq \mathbf{F}$. Then

$$\neg\alpha(x) = \neg\varphi_{_{U,A}}(x) = \begin{cases} U, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

Then $(\neg\alpha)^\downarrow \in M$ since M is an ada so that

$$(\neg\alpha)^\downarrow(x) = \begin{cases} F, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

In fact $(\neg\alpha)^{\downarrow} \in M_{\#}$. Consider $\neg((\neg\alpha)^{\downarrow}) \in M_{\#}$. Note that $\neg((\neg\alpha)^{\downarrow})$ is of the form $\varphi_{T,A}$. Since $M_{\#}$ is atomless it follows that there exists $\varphi_{T,B} \in M_{\#}$, where $\emptyset \neq B \subsetneq A$, and $\mathbf{F} \lneq \varphi_{T,B} \lneq \varphi_{T,A}$. Consider $\varphi_{U,B} = \varphi_{T,B} \wedge \mathbf{U} \in M$. Since $\emptyset \neq B \subsetneq A$, we have $\mathbf{F} \lneq \varphi_{U,B} \lneq \varphi_{U,A} = \alpha$, which contradicts $\alpha \in \mathscr{A}(M)$. $\square$

Remark 3.36. Theorem 3.35 allows us to construct an atomless ada from an atomless Boolean algebra. For an atomless Boolean algebra B, the ada $B^{\star}$ will also be atomless. For further reading on atomless Boolean algebras refer to [6].

Theorem 3.37. *Let M be a finite C-algebra with T, F, U such that $|M| > 3$ and $T \in \mathscr{A}(M)$. Then M is not atomic.*

Proof. Since $T \in \mathscr{A}(M)$ it is clear that $M_{\#} = \{T, F\}$. Since $|M| > 3$ there exists $\gamma \in M \setminus \{T, F, U\}$ and since M is finite, there exists $\alpha \in \mathscr{A}(M)$ such that $\alpha \leq \gamma$. Clearly $\alpha \in \mathscr{A}(M) \cap M_{\#}^{c}$.

Consider $M \preceq \mathfrak{Z}^{X}$ for some set X. Then $\alpha = \varphi_{U,A}$ for some $\emptyset \neq A \subseteq X$. Suppose that $A = X$. Then $\alpha = \mathbf{U} \in \mathscr{A}(M)$ and it follows that $M = \{\mathbf{T}, \mathbf{F}, \mathbf{U}\}$. This is because if there is some $\beta \in M_{\#}^{c} \setminus \{\mathbf{U}\}$ then using Proposition 3.5(ii) we have $\mathbf{F} \lneq \beta \wedge \mathbf{F} \lneq \mathbf{U}$, which is not possible as $\mathbf{U} \in \mathscr{A}(M)$. However, since M is non-trivial, we arrived at a contradiction. Thus, $A \neq X$.

Suppose that M is atomic. Consider $\neg\alpha \in M_{\#}^{c}$. There exist finitely many $a_i \in \mathscr{A}(M)$ such that $\neg\alpha = \bigoplus a_i$. Clearly $\mathbf{T} \notin \{a_i\}$ since $\mathbf{T} \vee a = \mathbf{T} \neq \neg\alpha$. Since $M_{\#} = \{\mathbf{T}, \mathbf{F}\}$ we have $\mathscr{A}(M) \setminus \{\mathbf{T}\} \subseteq M_{\#}^{c}$. Thus, $a_i = \varphi_{U,A_i}$ for $\emptyset \neq A_i \subseteq X$. However, $\neg\alpha = \neg\varphi_{U,A}$ where $\emptyset \neq A \subsetneq X$ and so we have

$$\neg\alpha(x) = \begin{cases} U, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

On the other hand, $\neg\alpha = \bigoplus \varphi_{U,A_i}$ gives

$$\neg\alpha(x) = \begin{cases} U, & \text{if } x \in A_i \text{ for some } i; \\ F, & \text{otherwise.} \end{cases}$$

This is a contradiction, since $A \subsetneq X$ there exists $x_o \in X$ such that $\neg\alpha(x_o) = T$. Hence, M is not atomic. $\square$

Remark 3.38. Although $M_{\#}^{c}$ is a C-algebra under the induced operations $\neg$, $\wedge$ and $\vee$ of M, it does not contain the constants T and F, and is therefore not a subalgebra of M (with T, F, U). It is therefore natural to consider $\overline{M_{\#}^{c}} = M_{\#}^{c} \cup \{T, F\}$, which is clearly a C-algebra with T, F, U.

Corollary 3.39. *Let M be a finite C-algebra with T, F, U such that $|M| > 3$. Then $\overline{M_{\#}^{c}} = M_{\#}^{c} \cup \{T, F\}$ is never atomic.*

Proof. Since $(\overline{M_{\#}^{c}})_{\#} = \{T, F\}$ we have $T \in \mathscr{A}(\overline{M_{\#}^{c}})$. The result follows from Theorem 3.37. $\square$

Remark 3.40. The converse of Theorem 3.37 need not be true. That is, if M be a C-algebra with T, F, U such that M is not atomic then T need not be in $\mathscr{A}(M)$. Consider

$$M = \left\{ \begin{array}{c} (T,T,T),(F,F,F),(U,U,U),(U,F,F),(U,T,T), \\ (F,F,T),(T,T,F),(F,F,U),(T,T,U),(U,U,F), \\ (U,T,F),(U,F,T),(U,F,U),(U,T,U),(U,U,T) \end{array} \right\} \preceq \mathfrak{Z}^3.$$

Then, since $\mathscr{A}(M) = \{(U,F,F),(F,F,T),(T,T,F),(F,F,U)\}$, $(T,T,T) \notin \mathscr{A}(M)$. However, M is not atomic since (U,T,F) can only be written as join of atoms (U,F,F) and (T,T,F) but the supremum of these atoms does not exists.

3.4.3. Characterisation of finite atomic C-algebras. Here we prove that the class of finite atomic C-algebras (with T, F, U) is precisely that of finite adas. For this, we give a set-theoretic requirement for the existence of suprema of subsets.

Lemma 3.41. *Let $M \preceq \mathfrak{Z}^X$ for some set X and let $\{\alpha_i\}_{i\in I} \subseteq M$ where $I = \{1, 2, \ldots, n\}$. Suppose α_i is represented by the pairs of sets (A_i, B_i) for all $i \in I$. Then $\bigoplus_{i\in I} \alpha_i$ exists if and only if $A_i \cap (A_j \cup B_j)^c = \emptyset$ for all $i, j \in I$.*

Proof. If $A_i \cap (A_j \cup B_j)^c = \emptyset$ for all $i, j \in I$ then

$$\begin{aligned} \alpha_1(x) \vee \alpha_2(x) \vee \cdots \vee \alpha_n(x) &= \begin{cases} T, & \text{if } x \in A_1; \\ U, & \text{if } x \in (A_1 \cup B_1)^c; \\ \alpha_2(x) \vee \cdots \vee \alpha_n(x), & \text{otherwise.} \end{cases} \\ &= \begin{cases} T, & \text{if } x \in A_1; \\ U, & \text{if } x \in (A_1 \cup B_1)^c; \\ T, & \text{if } x \in A_2; \\ U, & \text{if } x \in (A_2 \cup B_2)^c; \\ \alpha_3(x) \vee \cdots \vee \alpha_n(x), & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the well-definedness of this expression follows from the fact that $A_i \cap (A_j \cup B_j)^c = \emptyset$ so that we do not have $x \in A_1 \cap (A_2 \cup B_2)^c$ or $x \in A_2 \cap (A_1 \cup B_1)^c$. This process yields the following:

$$\alpha_1(x) \vee \alpha_2(x) \vee \cdots \vee \alpha_n(x) = \begin{cases} T, & \text{if } x \in \bigcup A_i; \\ U, & \text{if } x \in \bigcup (A_i \cup B_i)^c; \\ F, & \text{otherwise} \end{cases}$$

which is well-defined and establishes that the join is independent of the order of the elements. Consequently, using Proposition 3.20, $\bigoplus_{i\in I} \alpha_i$ exists and can be expressed as follows:

$$\bigoplus_{i\in I}\alpha_i(x) = \begin{cases} T, & \text{if } x \in A_i \text{ for some } i \in I; \\ U, & \text{if } x \in (A_i \cup B_i)^c \text{ for some } i \in I; \\ F, & \text{otherwise.} \end{cases}$$

Conversely, suppose $\bigoplus\limits_{i\in I}\alpha_i$ exists. If $x \in A_p \cap (A_q \cup B_q)^c$ for some $x \in X$ and some $p, q \in I$ where $p \neq q$, then $\alpha_p(x) = T$ and $\alpha_q(x) = U$; which contradicts Proposition 3.16. Hence, the result follows. □

Using the criterion of Lemma 3.41, the following result is immediate.

Corollary 3.42. *Let $(\emptyset \neq)$ I be finite and $\{\alpha_i\}_{i\in I} \subseteq M$ such that $\bigoplus\limits_{i\in I}\alpha_i$ exist. Then for any $(\emptyset \neq)$ $J \subseteq I$, $\bigoplus\limits_{j\in J}\alpha_j$ exists.*

We now characterise finite atomic C-algebras in the following result.

Theorem 3.43. *Let M be a finite C-algebra with T, F, U. M is atomic if and only if M is an ada.*

Proof. $(\Leftarrow)$ In view of Remark 2.17 we have M is isomorphic to $\tilde{\mathfrak{Z}}^X$ for some finite set X. Since its reduct $\mathfrak{Z}^X$ is atomic (cf. Theorem 3.26), M is an atomic C-algebra.

$(\Rightarrow)$ Let M be atomic but not an ada. Then $M \subsetneq \widehat{M}$ where $\widehat{M}$ is the enveloping ada of M. Consider $\widehat{M} \preceq \tilde{\mathfrak{Z}}^X$, for some finite set X. Thus, $M \preceq \widehat{M} \preceq \mathfrak{Z}^X$ as C-algebras.

Since $M \subsetneq \widehat{M}$ there exists $\gamma \in M$ such that $\gamma^\downarrow \notin M$. Therefore, there exists $x_1 \in X$ such that $\gamma(x_1) = T$ since otherwise $\gamma^\downarrow = \mathbf{F} \in M$. Further, there exists $x_2 \in X$ such that $\gamma(x_2) = U$ since otherwise $\gamma^\downarrow = \gamma \in M$, a contradiction. Hence, γ can be identified with the pair of sets (A, B) where $A \neq \emptyset \neq (A \cup B)^c$.

Since M is atomic there exist $\{\alpha_i\}_{i\in I} \subseteq \mathscr{A}(M) \cap M_\#$ and $\{\beta_j\}_{j\in J} \subseteq \mathscr{A}(M) \cap M_\#^c$, where I and J are finite, such that

$$\gamma = (\bigoplus \alpha_i) \oplus (\bigoplus \beta_j).$$

It is clear that each α_i can be identified with the pair of sets (A_i, A_i^c) and that each β_j can be identified with the pair of sets $(\emptyset, B_j^c)$ where $A_i, B_j \subseteq X$. In other words $\alpha_i = \varphi_{T,A_i}$ and $\beta_j = \varphi_{U,B_j}$.

Since we have ascertained that $A \neq \emptyset \neq (A \cup B)^c$ we have $I \neq \emptyset \neq J$. Since $\bigoplus$ is defined, using Lemma 3.41 we have $A_i \cap (\emptyset \cup B_j^c)^c = A_i \cap B_j = \emptyset$ for all $i \in I$ and $j \in J$. Further, $\bigcup A_i = A$.

Since I is finite we have $\bigoplus \alpha_i \in M_\# \subseteq M$. Also $\bigoplus \alpha_i = \gamma^\downarrow$ since $\gamma^\downarrow$ is represented by the pair of sets (A, A^c) and $\bigoplus \alpha_i$ is represented by the pair of sets $(\bigcup A_i, (\bigcup A_i)^c)$. Thus $\gamma^\downarrow \in M$, which is a contradiction. The result follows. □

4. Further study using if-then-else

In [16], Panicker et al. introduced the notion of a C-set to study an axiomatisation of `if-then-else` that included models of possibly non-halting programs and tests that were drawn from a C-algebra. Standard examples of C-sets include $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$, where $\mathcal{T}_o(X_\perp)$ is the set of all functions on a pointed set $X_\perp$ which fix $\perp$, with the action

$$\_\,[\_\,,\_]\colon \mathfrak{Z}^X \times \mathcal{T}_o(X_\perp) \times \mathcal{T}_o(X_\perp) \to \mathcal{T}_o(X_\perp)$$

given by

$$\alpha[f,g](x) = \begin{cases} f(x), & \text{if } \alpha(x) = T; \\ g(x), & \text{if } \alpha(x) = F; \\ \perp, & \text{otherwise.} \end{cases} \tag{4.1}$$

Given a C-algebra M with T, F, U, by treating M as a pointed set with base point U, the pair (M, M) is a C-set under the action

$$\_\,[\![\_\,,\_]\!]\colon M \times M \times M \to M$$

given by

$$\alpha[\![\beta,\gamma]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma).$$

This inherent `if-then-else` operation enables us to study further structural properties of C-algebras. We directly refer to [16] for the results on C-sets which are used in this work.

4.1. Annihilators and closed sets

In this section we define annihilators using the `if-then-else` action on a C-algebra. Then we investigate the internal structure of C-algebras through closed sets which are obtained via a Galois connection of the annihilator operator.

For each $\alpha \in M$, by treating $\alpha[\![\,\_\,,\_\,]\!]$ as a binary operation, we define the notion of annihilators in Definition 4.1. Hereafter we use $\alpha, \beta, \gamma, \delta$ to denote elements of M treated as binary operations while a, b, c are used otherwise. As earlier, the elements of the C-algebra $\mathfrak{Z}^X$ will also be denoted by $\alpha, \beta, \gamma, \delta$.

Definition 4.1. For $a \in M$,

$$Ann(a) = \{\, \alpha \in M : \alpha[\![a,a]\!] = U \,\}.$$

We extend Ann in a natural manner to subsets of M through the operator $Ann\colon \wp(M) \to \wp(M)$ given by

$$Ann(S) = \bigcap_{a \in S} Ann(a).$$

In other words $Ann(S) = \{\, \alpha \in M : \text{ for all } a \in S,\ \alpha[\![a,a]\!] = U \,\}$.

Proposition 4.2. *The following hold in any C-algebra M with T, F, U:*

(i) *$Ann(U) = M$.*
(ii) *For any $a \in M$, $U \in Ann(a)$.*
(iii) *For any $a \in M_\#$, $Ann(a) = \{U\}$.*

(iv) $Ann(M) = \{U\}$.
(v) $b \in Ann(a) \Leftrightarrow a \in Ann(b)$.
(vi) $B \subseteq Ann(A) \Leftrightarrow A \subseteq Ann(B)$.
(vii) $A \subseteq B \Rightarrow Ann(B) \subseteq Ann(A)$.

Proof. (i) By [16, Proposition 2.8], we have $\alpha[\perp, \perp] = \perp$ in an arbitrary C-set $(S_\perp, M)$. Hence, for the C-set (M, M) we get $\alpha[\![U, U]\!] = U$ for each $\alpha \in M$ so that $Ann(U) = M$.

(ii) Note that $U[\![a, a]\!] = (U \wedge a) \vee (\neg U \wedge a) = U$ so that $U \in Ann(a)$ for all $a \in M$.

(iii) Using Proposition 4.2(ii) it is clear that $\{U\} \subseteq Ann(a)$. For the reverse inclusion since $M \preceq \mathfrak{Z}^X$ for some set X, for $a \in M_\#$ we have $a(x) \in \{T, F\}$ for all $x \in X$. Suppose that $\alpha(x_o) \in \{T, F\}$ for some $x_o \in X$ so that $(\alpha(x_o) \wedge a(x_o)) \vee (\neg\alpha(x_o) \wedge a(x_o)) \in \{T, F\}$. However,

$$Ann(a) = \{\, \alpha \in M : (\alpha(x) \wedge a(x)) \vee (\neg\alpha(x) \wedge a(x)) = \mathbf{U} \text{ for all } x \in X \,\},$$

a contradiction. Consequently, $\alpha(x) = U$ for all $x \in X$ hence $\alpha = \mathbf{U}$.

(iv) Since $Ann(M) = \bigcap\limits_{a \in M} Ann(a)$, we have $Ann(M) = \{U\}$ (cf. Proposition 4.2(iii)).

(v) Consider $M \preceq \mathfrak{Z}^X$ for some set X and $b \in Ann(a)$ so that $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ for all $x \in X$. For $x \in X$ we have the following cases for $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x))$:

$b(x) = T$**.** Since $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ we have $(T \wedge a(x)) \vee (F \wedge a(x)) = a(x) \vee F = a(x) = U$. Thus, $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (U \wedge b(x)) \vee (U \wedge b(x)) = U$.

$b(x) = F$**.** Along similar lines since $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ we have $(F \wedge a(x)) \vee (T \wedge a(x)) = F \vee a(x) = a(x) = U$. Hence, $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (U \wedge b(x)) \vee (U \wedge b(x)) = U$.

$b(x) = U$**.** Then $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (a(x) \wedge U) \vee (\neg a(x) \wedge U) = U$ for $a(x) \in \{T, F, U\}$.

Hence, $a[\![b, b]\!] = U$ so that $a \in Ann(b)$. The converse follows along similar lines.

(vi) This follows as a direct consequence of Proposition 4.2(v).

(vii) Let $A \subseteq B$, $\beta \in Ann(B)$ and $a \in A$. Since $\beta \in Ann(b)$ for each $b \in B$ and $a \in A \subseteq B$ we have $\beta \in Ann(a)$. Thus, $Ann(B) \subseteq Ann(A)$. $\square$

We now recall the notions of closure operators, closed sets and Galois connection which are well-known in the literature. For more details one may refer to [13].

Given a set X, a function $C \colon \wp(X) \to \wp(X)$ is termed a *closure operator* on X if for all $A, B \subseteq X$ it satisfies the following:

$$A \subseteq C(A) \qquad \text{(extensive)}$$

$$C^2(A) = C(A) \qquad \text{(idempotent)}$$

$$A \subseteq B \Rightarrow C(A) \subseteq C(B) \qquad \text{(isotone)}$$

A subset $A \subseteq X$ is called a *closed subset* if $C(A) = A$.

Remark 4.3. The set of all closed sets of X ordered by set inclusion $\subseteq$ is a partially ordered set (poset). In fact, it forms a complete lattice.

Let A and B be posets and $F\colon A \to B$ and $G\colon B \to A$ be two antitone functions. The pair (F, G) is said to be an *antitone Galois connection* if for all $a \in A, b \in B$,

$$b \leq F(a) \Leftrightarrow a \leq G(b).$$

Remark 4.4. Given an antitone Galois connection (F, G) of posets A and B, the composite functions $FG\colon B \to B$ and $GF\colon A \to A$ form closure operators. Further, $FGF = F$ and $GFG = G$.

Now we have the following result which follows from Proposition 4.2(vi) and Proposition 4.2(vii).

Proposition 4.5. *The pair (Ann, Ann) is an antitone Galois connection.*

In view of Remark 4.4, we have the following corollary of Proposition 4.5.

Corollary 4.6. *The function $Ann^2\colon \wp(M) \to \wp(M)$ is a closure operator. Further, $Ann^3 = Ann$.*

Let $\mathfrak{I}$ be the collection of closed sets of M under the closure operator Ann^2, i.e., $\mathfrak{I} = \{I \subseteq M : Ann^2(I) = I\}$. Through these closed sets, we now investigate the structure of C-algebras focusing on $\mathfrak{Z}^X$. First observe the following general property of closed sets in $\mathfrak{I}$.

Proposition 4.7. *Let $I \in \mathfrak{I}$ such that $I \neq M$. Then $I \cap M_\# = \emptyset$.*

Proof. Suppose that there exists $a \in I \cap M_\#$ where $I \in \mathfrak{I}$ such that $I \neq M$. For any $\alpha \in Ann(a)$ we have $\alpha = U$ using Proposition 4.2(iii) since $a \in M_\#$. Further, since $Ann(I) = \bigcap\limits_{a \in I} Ann(a)$ we have $Ann(I) = \{U\}$. Using Proposition 4.2(i) we have $Ann^2(I) = Ann(U) = M$. It follows that $I = Ann^2(I) = M$ since $I \in \mathfrak{I}$, which is a contradiction. Thus, $I \cap M_\# = \emptyset$. $\square$

4.2. Closed sets of $\mathfrak{Z}^X$

We consider the C-algebra $\mathfrak{Z}^X$ and give a characterisation of the closed sets in $\mathfrak{I}$ with respect to operator Ann^2. To that aim, in this section we consider the C-algebra in question to be precisely $\mathfrak{Z}^X$ for an arbitrary set X.

Theorem 4.8. *Let $I \subseteq \mathfrak{Z}^X$. $I \in \mathfrak{I}$ if and only if there exists $Y \subseteq X$ such that*

(P1) *for all $\alpha \in I$, for all $y \in Y$, $\alpha(y) = U$,*

(P2) *for all $f\colon Y^c \to \mathfrak{Z}$ there exists $\alpha \in I$ such that $\alpha \restriction Y^c = f$.*

Proof. ($\Leftarrow$) Let $I \subseteq \mathfrak{Z}^X$ such that there exists $Y \subseteq X$ which satisfies both the given conditions (P1) and (P2). We show that $Ann^2(I) = I$. Since Ann^2 is extensive it suffices to show that $Ann^2(I) \subseteq I$.

Let $\beta \in Ann^2(I)$. For $\beta \restriction Y^c$ there exists $\alpha \in I$ such that $\alpha \restriction Y^c = \beta \restriction Y^c$. Moreover, $\alpha(y) = U$ for all $y \in Y$. We show that $\beta(y) = U$ for all $y \in Y$ so that $\beta = \alpha$ from which it follows that $\beta \in I$.

Suppose $\beta(y_o) \in \{T, F\}$ for some $y_o \in Y$. Since $\beta \in Ann^2(I)$ we have $(\beta[\![\gamma, \gamma]\!])(y_o) = U$ for all $\gamma \in Ann(I)$ and so $\gamma(y_o) = U$ for all $\gamma \in Ann(I)$. Consider $\delta \in \mathfrak{Z}^X$ given by

$$\delta(x) = \begin{cases} T, & \text{if } x \in Y, \\ U, & \text{otherwise.} \end{cases}$$

Since $\alpha(y) = U$ for all $y \in Y$, for all $\alpha \in I$, we infer that $\delta[\![\alpha, \alpha]\!] = \mathbf{U}$ so that $\delta \in Ann(I)$. However, $\delta(y_o) = T \neq U$, a contradiction. Hence, $\beta \in I$ and so $I \in \mathfrak{I}$.

($\Rightarrow$) Let $I \in \mathfrak{I}$. Consider the following.

$$A = \{\, x \in X : \alpha(x) = U \text{ for all } \alpha \in I \,\}$$
$$B = \{\, x \in X : \alpha(x) \in \{T, F\} \text{ for some } \alpha \in I \,\} = X \setminus A.$$

We show that $Y = A$ is the required set. It is clear that $\alpha(y) = U$ for all $\alpha \in I$ and for all $y \in A$. Let $f \colon B \to \mathfrak{Z}$. Consider its extension $\hat{f} \colon X \to \mathfrak{Z}$ given by the following:

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in B; \\ U, & \text{if } x \in A. \end{cases}$$

Thus, $\hat{f} \restriction B = \hat{f} \restriction A^c = f$. Let $\beta \in \mathfrak{Z}^X$. It is clear that

$$\beta \in Ann(I) \Leftrightarrow \beta(z) = U \text{ for all } z \in B.$$

Consider $\beta \in Ann(I)$. It follows that $(\hat{f}[\![\beta, \beta]\!])(z) = U$ for all $z \in B$. Also since $\hat{f}(y) = U$ for all $y \in A$ we have $(\hat{f}[\![\beta, \beta]\!])(y) = U$ and so $\hat{f}[\![\beta, \beta]\!] = U$ from which it follows that $\hat{f} \in Ann^2(I) = I$. This completes the proof. $\square$

Theorem 4.8 equips us with a mechanism to identify the collection of closed sets in $\mathfrak{I}$ with respect to Ann^2.

Definition 4.9. For $A \subseteq X$ define $I_A \subseteq \mathfrak{Z}^X$ by

$$I_A = \{\, \alpha \in \mathfrak{Z}^X : \alpha(y) = U \text{ for all } y \in A \,\}. \tag{4.2}$$

Proposition 4.10. $\mathfrak{I} = \{\, I_A : A \subseteq X \,\}$.

Proof. For $A \subseteq X$ consider I_A as defined by (4.2). It follows in a straightforward manner that I_A satisfies (P1) and (P2) of Theorem 4.8 for $Y = A$ so that $I_A \in \mathfrak{I}$.

Conversely for $I \in \mathfrak{I}$ using Theorem 4.8 we have $Y \subseteq X$ such that (P1) and (P2) are satisfied. We show that $I = I_Y$. Clearly $I \subseteq I_Y$ due to (P1). Conversely assume that $\alpha \in I_Y$ that is $\alpha(y) = U$ for all $y \in A$. Using (P2) of Theorem 4.8 we have $\beta \restriction Y^c = \alpha \restriction Y^c$ for some $\beta \in I$. Property (P1) of Theorem 4.8 ensures that $\beta(y) = U$ for all $y \in Y$. It follows that $\alpha = \beta$ so that $\alpha \in I$. $\square$

We prove the following set theoretic relations on closed sets. These are useful in establishing that $\mathfrak{I}$ is a Boolean algebra in Theorem 4.12.

Lemma 4.11. *For $A \subseteq X$ and $I_A \in \mathfrak{I}$ the following hold:*
 (i) $Ann(I_A) = I_{A^c}$.
 (ii) $I_A \cap I_B = I_{A \cup B}$.
 (iii) $Ann(Ann(I_A) \cap Ann(I_B)) = I_{A \cap B}$.

Proof. (i) Let $\alpha \in Ann(I_A)$. In view of Definition 4.9 and Proposition 4.10 it suffices to show that $\alpha(y) = U$ for all $y \in A^c$. For each $y \in A^c$ consider $\beta_y \in \mathfrak{Z}^X$ given by

$$\beta_y(x) = \begin{cases} T, & \text{if } x = y; \\ U, & \text{otherwise.} \end{cases}$$

It is straightforward to see that $\beta_y \in I_A$ for all $y \in A^c$. Thus, $\alpha[\![\beta_y, \beta_y]\!] = U$ for all $y \in A^c$ and so $(\alpha[\![\beta_y, \beta_y]\!])(y) = U$ for all $y \in A^c$. Since $\beta_y(y) = T$ it follows that $\alpha(y) = U$ for all $y \in A^c$.

For the reverse inclusion consider $\alpha \in I_{A^c}$ and $\beta \in I_A$. Using Definition 4.9 and Proposition 4.10 we have $\alpha(y) = U$ for all $y \in A^c$, so that $(\alpha[\![\beta, \beta]\!])(y) = U$ for all $y \in A^c$. Thus, $\beta(y) = U$ for all $y \in A$ so that $(\alpha[\![\beta, \beta]\!])(y) = U$ for all $y \in A$. Thus, $\alpha \in Ann(I_A)$ and consequently $Ann(I_A) = I_{A^c}$.

(ii) Consider $\alpha \in I_A \cap I_B$ and $y \in A \cup B$. It suffices to show that $\alpha(y) = U$. If $y \in A$ then $\alpha(y) = U$ since $\alpha \in I_A$. Along similar lines, $\alpha(y) = U$ if $y \in B$ so that $\alpha \in I_{A \cup B}$.

For the reverse inclusion consider $\alpha \in I_{A \cup B}$. For $y \in A \subseteq A \cup B$ we have $\alpha(y) = U$ so that $\alpha \in I_A$. Proceeding along similar lines we can show that $\alpha \in I_B$ from which the result follows.

(iii) Using Lemma 4.11(i) and Lemma 4.11(ii) we have $Ann(Ann(I_A) \cap Ann(I_B)) = Ann(I_{A^c} \cap I_{B^c}) = Ann(I_{A^c \cup B^c}) = I_{(A^c \cup B^c)^c} = I_{A \cap B}$. ☐

While it is known that $\mathfrak{I}$ is a complete lattice (cf. Remark 4.3), we now explicitly show that $\mathfrak{I}$ is a complete Boolean algebra using its internal structure.

Theorem 4.12. *The set $\mathfrak{I}$ of closed sets of $\mathfrak{Z}^X$ with respect to Ann^2 is a Boolean algebra with respect to the operations*

$$\begin{aligned} \neg I &= Ann(I) \\ I_1 \wedge I_2 &= I_1 \cap I_2 \\ I_1 \vee I_2 &= Ann(Ann(I_1) \cap Ann(I_2)) \end{aligned}$$

and $\{\mathbf{U}\}$ and $\mathfrak{Z}^X$ as the constants 0 and 1 respectively. Moreover, $\mathfrak{I} \cong 2^X$ and is therefore complete.

Proof. We rely on the representation of $\mathfrak{I}$ as given in Proposition 4.10. In view of Lemma 4.11 we show that the operations given as follows define a

Boolean algebra on $\mathfrak{I}$:

$$\neg I_A = I_{A^c}$$
$$I_A \wedge I_B = I_{A\cup B}$$
$$I_A \vee I_B = I_{A\cap B}$$

The verification is straightforward and involves set theoretic arguments.

Let $A, B \subseteq X$. Using Lemma 4.11 we have $I_A \wedge I_B = I_{A\cup B} = I_{B\cup A} = I_B \wedge I_A$ and similarly $I_A \vee I_B = I_{A\cap B} = I_{B\cap A} = I_B \vee I_A$. Along similar lines the axioms of associativity, idempotence, absorption and distributivity can be verified so that $\langle \mathfrak{I}, \vee, \wedge\rangle$ is a distributive lattice.

Note that $I_X = \{\mathbf{U}\}$ while $I_\emptyset = \mathfrak{Z}^X$. Using Lemma 4.11 we have $I_A \wedge I_X = I_{A\cup X} = I_X$ and $I_A \vee I_\emptyset = I_{A\cap\emptyset} = I_\emptyset$ for all $A \subseteq X$. Also $I_A \wedge Ann(I_A) = I_A \wedge I_{A^c} = I_{A\cup A^c} = I_X$ and $I_A \vee Ann(I_A) = I_A \vee I_{A^c} = I_{A\cap A^c} = I_\emptyset$.

Hence, $\langle \mathfrak{I}, \vee, \wedge, \neg, I_X, I_\emptyset\rangle$ is a Boolean algebra. It is straightforward to verify that the assignment given by $I_A \mapsto A^c$ from $\mathfrak{I}$ to $\mathfrak{2}^X$ is a Boolean algebra isomorphism. □

In the following result, using closed sets we give a classification of elements of $M = \mathfrak{Z}^X$ which segregates the elements of $\mathfrak{2}^X (= M_\#)$ into one class.

Theorem 4.13. *For each $A \subseteq X$ define $S_A = \{\, \alpha \in \mathfrak{Z}^X : Ann(\alpha) = I_A \,\}$. The collection $\{\, S_A : A \subseteq X \,\}$ forms a partition of $\mathfrak{Z}^X$ in which all the elements of $\mathfrak{2}^X$ form a single equivalence class.*

Proof. We first show that $S_A \neq \emptyset$ for any $A \subseteq X$. To that aim consider $\alpha \in \mathfrak{Z}^X$ given by

$$\alpha(x) = \begin{cases} U, & \text{if } x \in A^c; \\ T, & \text{otherwise.} \end{cases}$$

Using Definition 4.9 and Proposition 4.10 it is straightforward to verify that $Ann(\alpha) = I_A$. Consequently, $S_A \neq \emptyset$.

It is evident that $\alpha \in S_A \cap S_B$ is a violation of the well-definedness of $Ann(\alpha)$ from which it follows that $S_A \cap S_B = \emptyset$ for $A, B \subseteq X$ where $A \neq B$.

Note that for any $\alpha \in \mathfrak{Z}^X$ we have $Ann(\alpha) \in \mathfrak{I}$ that is $Ann(\alpha) = I_A$ for some $A \subseteq X$, since $Ann^2(Ann(\alpha)) = Ann^3(\alpha) = Ann(\alpha)$ using Corollary 4.6. Thus, $Ann(\alpha) = I_A$ for some $A \subseteq X$ so that $\alpha \in S_A$. Therefore,

$$\bigcup_{A\subseteq X} \{\, S_A : A \subseteq X \,\} = \mathfrak{Z}^X$$

and hence the collection $\{\, S_A : A \subseteq X \,\}$ forms a partition of $\mathfrak{Z}^X$.

Further, for $\alpha \in \mathfrak{2}^X$ we have $Ann(\alpha) = \{\mathbf{U}\} = I_X$ so that $\alpha \in S_X$. Conversely any $\alpha \in S_X$ would satisfy $Ann(\alpha) = I_X = \{\mathbf{U}\}$. If $\alpha(x_o) = U$ for some $x_o \in X$ then it follows that $\beta \in \mathfrak{Z}^X$ given by

$$\beta(x) = \begin{cases} T, & \text{if } x = x_o; \\ U, & \text{otherwise.} \end{cases}$$

satisfies $\beta[\![\alpha, \alpha]\!] = \mathbf{U}$ and so $\mathbf{U} \neq \beta \in Ann(\alpha)$ which is a contradiction. Thus, $\alpha(x) \in \{T, F\}$ for all $x \in X$ from which it follows that $\alpha \in 2^X$. Hence, the equivalence class $S_X = 2^X$. $\square$

We conclude this section with some remarks on annihilators.

Remark 4.14.

(i) The statement $Ann(\alpha) = \{U\} \Leftrightarrow \alpha \in M_\#$ holds in 3^X but need not be true in general.

For example, consider $A = \{(T,T), (F,F), (U,U), (F,U), (T,U)\} \leq 3^2$ and $(T,U) \in A$. Then $Ann(T,U) = \{\, \beta \in A : \beta[\![(T,U),(T,U)]\!] = (U,U)\,\}$. Hence, for $(x,y) \in Ann(T,U)$ we have $((x \wedge T) \vee (\neg x \wedge T), (y \wedge U) \vee (\neg y \wedge U)) = (U,U)$ and so $(x \vee \neg x, U) = (U,U)$. It follows that $x = U$ and so $Ann(T,U) = \{(U,U)\}$. However, $(T,U) \notin A_\#$.

Note that the only closed sets of A are A and $\{(U,U)\}$. Further, in this example, the collection of closed sets is a Boolean algebra.

(ii) For $I \subseteq M$ where $M \preceq 3^X$ we have $Ann_M(I) = Ann_{3^X}(I) \cap M$.

Let $\alpha \in Ann_M(I)$. Clearly $\alpha \in M$ and $\alpha \in Ann_{3^X}(I)$. Conversely suppose $\alpha \in Ann_{3^X}(I) \cap M$. Then it is clear that $\alpha \in Ann_M(I)$.

(iii) Thus, on applying Ann to the previous statement and making appropriate substitutions we have $Ann^2_M(I) = Ann_{3^X}(Ann_{3^X}(I) \cap M) \cap M$.

5. Conclusion

The interdependence between the structure of a Boolean algebra and its atomicity is well known in the literature. In this paper, we studied the atomicity of a non-commutative extension of Boolean algebras, viz., C-algebras. We also studied the structure of a C-algebra via closed sets defined using its inherent `if-then-else` operation. In addition to demonstrating the relationship between the atomicity of the Boolean subalgebra $M_\#$ and that of a C-algebra M, in both the approaches, we proved several structural properties of arbitrary C-algebras. This work especially focuses on the C-algebra of transformations, 3^X. In fact, we proved that 3^X is atomic, for any set X, and characterised the class of finite atomic C-algebras. In this connection, we determined various necessary and sufficient conditions for the existence of suprema of subsets of the C-algebra 3^X. Using an alternative approach, we classified the elements of 3^X by means of closed sets. However, these problems for arbitrary C-algebras are open for further investigation.

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