Model checking pushdown systems

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- Real time: discrete, dense domains
Extended automata

- A generic way of modelling such systems is by finite state automata with **guarded transitions**.
- An extended automaton is equipped with a finite set of **variables** $X = \{x_1, \ldots, x_n\}$ with variable $x_i$ taking values in set $V_i$.
- We have a finite set of **guards** $G$: each guard is a predicate over $X$.
- With each transition is associated an **action**, which is possibly a nondeterministic assignment to $X$. 

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Extended automata: semantics

A configuration is a tuple \((q, v_1, \ldots, v_n)\) where \(q\) is a state and \(v_i\) is a valuation for \(x_i\).

The transition system of the extended automaton is over configurations:

\((q, v_1, \ldots, v_n) \Rightarrow (q', v'_1, \ldots, v'_n)\) if the automaton has a transition \(q^{g,a} \rightarrow q'\), the values \(v_i\) satisfy guard \(g\) and the tuple \((v'_1, \ldots, v'_n)\) is a possible result of applying \(a\) to \((v_1, \ldots, v_n)\).
Some classes of extended automata

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- Pushdown systems: Variables – stack; guards – emptiness check; actions: push / pop.
Reachability problem

- Given: An extended automaton $E$, a set $I$ of initial configurations and a set $D$ of dangerous configurations.
- Decide if some $d \in D$ is reachable from some $c_0 \in I$.
- The sets $I$ and $D$ may be infinite.
Symbolic search

- Let $post(C)$ denote the set of immediate successors of a possibly infinite set of configurations $C$.
- Forward search: Initialize $C$ to $I$.
- Iterate $C := C \cup post(C)$ until $C \cap D \neq \emptyset$ or a fixed point is reached.
- Question: When is symbolic search effective?
Sufficient conditions

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• Any chain $C_1 \subseteq C_2 \subseteq \ldots$ reaches a fixpoint finitely.
Timed automata

- Variables are clocks: non-negative real valued.
- Transitions guarded by boolean combinations of comparisons with integer bounds, actions reset a subset of clocks.
- Equivalent configurations: when states are the same and values are equivalent with respect to constraints.
- Regions: equivalence classes of configurations.
- Choose $C$ to be the powerset of regions.
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- Checking emptiness of $C \cap D$: check if $C$ contains some configuration with some state of $Q_D$ as its first element.
- Checking equality of regions is decidable.
- Fixedpoint condition follows from the fact that the set of regions is finite.
Lossy channel systems

- Automata extended with unbounded queues.
- Send transitions: no guard, action: add message to channel.
- Receive transitions: guard: non-emptiness of channel; action removes first message.
- Loss transitions: no guard, self loop, removes an arbitrary message.
Symbolic reachability

- Order configurations by the *subword ordering*.
- Choose $\mathcal{C}$ to be all upward closed sets of configurations.
- Forward search does not work, satisfies conditions 1 to 5 but not 6.
- When $\mathcal{D}$ is a set of upward closed configurations, backward search works.
Backward symbolic search

- Key idea: Use Higman’s lemma to show that any upward closed set can be finitely represented by its set of minimal elements w.r.t. the pointwise order $\geq$.
- Checking that if $C$ is upward closed, so is $\operatorname{pre}(C)$ is easy.
- To show that a fixed point is reached in finitely many steps, again appeal to Higman’s lemma.
Forward symbolic search

- Choose $C$ to be the set of simple regular expressions.
- SREs satisfy the first 5 conditions, but the fixpoint cannot be effectively computed.
- One approach: find loops by (a kind of) static analysis (Abdallah et al LICS 99).
- Another: use Angluin’s learning algorithms (Varadhan et al FSTTCS 04).
Pushdown Systems

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• Applications: analysis of boolean programs, data-flow analysis, checkpoint algorithms (suspend computations to inspect stack content, for instance, to enforce security requirements).
Automata with stack

• Automata extended with one stack.
• Guards: Check the topmost symbol on stack.
• Actions: replace topmost symbol by a fixed word.
• Configuration \((q, v)\): \(q\) holds values of global variables, \(v\) holds values of program pointer, values of local variables, return address.
Symbolic reachability

- Choose $C$ to be the family of regular configurations.
- Each is represented by a DFA.
- $I$ is typically finite and hence regular. Equality of regular sets is decidable.
- If $C$ is regular, showing that $\text{pre}(C')$ or $\text{post}(C')$ is regular is straightforward.
- Büchi’s theorem asserts that the fixedpoint of a chain is regular and can be effectively computed.
Model checking-1

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• When valuations are arbitrary – that is, the set of pushdown configurations in which an atomic proposition is true, is an arbitrary subset of the possible ones, model checking is undecidable.
Model checking-2

- When valuations are simple – that is, the truth of an atomic proposition in a pushdown configuration depends only on the control state and topmost stack symbol, we can use a Büchi-like technique to get decidability.
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- Over pushdown systems, model checking $CTL^*$ reduces to model checking $LTL$ over regular valuations.
• Fix $P$, a countable set of atomic propositions. LTL formulae are defined by the following syntax:

$$\alpha ::= p \in P \mid \neg \alpha \mid \alpha \lor \beta \mid \Box \alpha \mid \alpha U \beta$$

• A model is a word $w : \mathcal{N} \rightarrow 2^P$, and the notion $w \models \alpha$ is defined as usual.

• Derived modalities: $\Diamond \alpha = True U \alpha$ and $\Box \alpha = \neg \Diamond \neg \alpha$. 
LTL:2

• Let $\mathcal{L}(\alpha) = \{w | w \models \alpha\}$.

• We know that for every formula $\alpha$, we can construct a nondeterministic Büchi automaton $B_\alpha$ such that $\mathcal{L}(\alpha) = L(B_\alpha)$, where $B_\alpha$ of size $O(2^{\|\alpha\|})$.

• Typically, we define a transition system $T = (S, \rightarrow, s_0, V)$ where $V : S \rightarrow 2^P$ is a valuation, and interpret formulas on runs of $T$. We thus define the model checking problem: $T \models \alpha$, if every infinite run of $T$ satisfies $\alpha$. 
A pushdown system is a tuple 
\[ S = (C, \Gamma, \Delta, c_0, b) \]  
where:  
- \( C \) is a finite set of control locations,  
- \( \Gamma \) is the stack alphabet,  
- \( \Delta \) is the transition relation,  
- \( c_0 \) is the initial location and  
- \( b \) is the bottom stack symbol.

\[ \Gamma \subseteq (C \times \Gamma) \times (C \times \Gamma^*) \], and a transition is written as: 
\[ (c, a) \rightarrow (d, w) \].

A configuration is an element of \( C \times \Gamma^* \).
Pushdown systems-2

- With a pushdown system $S$, we associate a transition system $T_S$ with configurations as states, $(c_0, b)$ as the initial state and the transition relation $\Rightarrow$ is the least one satisfying:
  if $(c, a) \rightarrow (d, w)$ then for all $u \in \Gamma^*$, $(c, au) \Rightarrow (d, wu)$.

- Without loss of generality, we assume that $b$ is never removed from stack, and that every transition increases the stack by at most one.
LTL on pushdown systems

Let $S = (C, \Gamma, \Delta, c_0, b)$ be a pushdown system, $\alpha$ an LTL formula, and $V : P \rightarrow 2^{C \times \Gamma^*}$. The model checking problem comes in three forms:

- Does $(c_0, b) \models \alpha$?
- Is there any configuration that violates $\alpha$?
- Is there any reachable configuration that violates $\alpha$?

All these problems are undecidable, in general.
Simple valuations

- A set of configurations $C$ is said to be simple if $C \subseteq \{(c, aw) \mid w \in \Gamma^*\}$ for some $c \in C$, $a \in \Gamma$.
- A valuation $V$ is simple, if for every $p \in P$, $V(p)$ is a union of simple sets.
Regular valuations

- A valuation $V$ is said to be regular if for every $p \in P$, $V(p)$ is recognizable and does not contain any configuration with an empty stack.

- Then, for every $p \in P$ and $c \in C$, we have a DFA $A_p^c$ over the alphabet $\Gamma$ such that $V(p) = \{(c, w) \mid c \in C, w^R \in L(A_p^c)\}$.

- That is, $p$ is true at $(c, w)$ iff $A_p^c$ enters a final state after reading the stack bottom up.
\(S\)-automata

- For a PDS \(S = (C, \Gamma, \Delta, c_0, b)\), an \(S\)-automaton is a tuple \(A = (Q, \Gamma, \delta, C, F)\) where \(Q\) is a finite set of states, \(\Gamma\) (the stack alphabet of \(S\)) is its input alphabet, \(\delta : (Q \times \Gamma) \rightarrow 2^Q\) is its transition function, \(C\) is its set of initial states and \(F\) is the set of accepting states.

- \(\delta\) is extended as usual, and we say that a configuration \((c, w)\) is accepted by \(A\) iff \(\delta(c, w) \cap F \neq \emptyset\).

- A set of \(S\)-configurations \(C'\) is regular if it is accepted by some \(S\)-automaton.
The main idea

Consider the model checking problem for the initial configuration.

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- The problem is reduced to that of emptiness for Büchi pushdown systems.
- The emptiness problem for Büchi pushdown systems is reduced to that of computing the set of predecessors of certain regular sets of configurations.
- The set of predecessors is regular, and an algorithm is given for computing it; this is Büchi’s saturation procedure.
Step 1

- Given a PDS $S = (C, \Gamma, \Delta, c_0, b)$, and a formula $\alpha$, first construct $A_\alpha = (Q, \delta, q_0, F')$ on $2^P$.

- Construct the product $B = ((C \times Q), \Gamma, \Delta', (c_0, q_0), b, G)$ by “synchronizing” $S$ and $A_\alpha$.

- $((c, q), a) \rightarrow' ((c', q'), w))$ if $(c, a) \rightarrow (c', w)$ in $S$ and $q' \in \delta(q, \sigma)$, where $\sigma$ is the set of propositions true in $(c, a)$.

- Note that we are using the simplicity of valuations here.
Step 2

- Consider a transition $(c, a) \rightarrow (c', w)$ in $B$.
- It is repeating if there exists $v \in \Gamma^*$ such that $(c, av)$ can be reached from $(c, a)$ visiting $G$.
- Let $Rep$ denoting repeating heads of transitions and let $R$ denote the set $\{(c, aw) \mid (c, a) \in Rep, w \in \Gamma^*\}$.
- We can show that $L(B)$ is nonempty iff $(c_0, b) \in pre^*(R)$.
- $Rep$ is easily computed by an edge marking algorithm.
Regular valuations

- Suppose we have $P_\alpha = \{p_1, \ldots, p_k\}$. Consider all the DFAs $M_i^c$ for each $c \in C$.
- We form a vector of these automata in a canonical fashion with its (product) state from a set $States$.
- The crucial idea is to carry the state vector as part of the stack in a larger pushdown system with $\Gamma' = (\Gamma \times States$).
- Care is needed to ensure consistent configurations.