Real-time Logics

Expressiveness and Decidability

Paritosh K. Pandya

Tata Institute of Fundamental Research
Mumbai

email: pandya@tifr.res.in
Timed Behaviours

- Observable propositions $X_1, X_2, \ldots, X_n$.
- Time frame $(T, <)$ a linear order.
- Behaviour $M$ such that $M(X_i) : T \rightarrow \{\top, \bot\}$.

Some commonly used notions of time and behaviour

- $(\mathbb{R}, <)$ the standard set of real numbers. Continuous time, Canonical behaviors.
- $(\mathbb{Q}, <)$ the set of rational numbers.
- $(\mathbb{N}, <)$ the set of natural numbers. Discrete time.

Behaviours over $(\mathbb{R}, <)$ are called canonical continuous and over $(\mathbb{N}, <)$ canonical discrete.
Finite Variability

- $M$ such that in any finite interval $M(X)$ changes finitely often for any $X$ are called Finitely variable behaviours. Interval based model $(s_0, I_0), (s_1, I_1), \ldots$.

- $(S, <)$ where $S$ is countably infinite set of sampling points from $R$ which is divergent Point based models $(s_0, t_0), (s_1, t_1), \ldots$.

In this talk: Canonical Continuous Models and subclass Finitely Variable models.
Monadic First Order Logic \( MFO \) First order logic with equality over linear order \((T, <)\) and monadic predicates (observable propositions) \(X_i\).

Examples

- \( \forall x \exists y. (x < y \land X_1(y)) \)
  says that \(X_1\) never stops occurring!

- \( \forall x \forall y. (x < y \Rightarrow (\exists z. x < z < y)) \)

Let \( \phi(X_1, \ldots, X_n, y_1, \ldots, y_k) \) be a formula with given free variables. Its model has the form \( M = (T, <, P_1, \ldots, P_n, t_1, \ldots, t_k) \). We can define \( M \models \phi \) as usual.

Monadic Secondorder Logic (MSO)
Temporal Logics

Example $\square(P \rightarrow QUR)$.

A temporal logic $TL(O_1, \ldots, O_k)$ with modalities $O_1, \ldots, O_k$.

**Truth Table** For each modality $O_i$ of arity $k$ we have a MFO formula $[O_i(X_1, \ldots, X_k)] \overset{\text{def}}{=} \phi(t_0, X_1, \ldots, X_k)$ giving its behaviour.

Examples

- $[\lozenge X] \overset{\text{def}}{=} \exists t. t_0 < t \land X(t)$
- $[XUY] \overset{\text{def}}{=} \exists t. t_0 < t \land Y(t) \land \forall z.((t_0 < z < y) \Rightarrow X(z))$.

A popular temporal logic $TL(U, S)$ called $TL$. 
Proposition If every modality of a temporal logic $TL'$ has a truth table then for every $\phi(X_1, \ldots, X_n) \in TL'$ we can construct a $MFO$ formula $\hat{\phi}(t_0, X_1, \ldots, X_n)$ such that $M \models \phi$ iff $M \models \hat{\phi}$.

Every such Temporal logic is a fragment of MFO!
Expressive Completeness

**Definition** A subset $L_1$ of logic $L_2$ is expressively complete for $L_2$ over class of models $C$ provided for all $\phi \in L_2$ there exists $\hat{\phi}_2 \in L_1$ such that $C \models \phi \iff \hat{\phi}$.

**Theorems** ([Kamp68,GPSS80,GHR94]) Logic $TL(U,S)$ is expressively complete for $MFO$ over canonical models (discrete or continuous). There exists $TL(U_S,S_S)$ which is expressively complete for $MFO$ over the class of all linear order.
Decidability

A logic $L$ is **decidable** if there exists an algorithm for finding whether $\phi \in L$ is satisfiable (valid).

**Results**

- $MSO$ over $(\mathbb{N}, <)$ is decidable. [Buchi60]
- $MFO$ over $(\mathbb{R}, <)$ is decidable. [BG85]
- $MSO$ over $(\mathbb{R}, <)$ is undecidable. $MSO$ with quantification over monadic predicates restricted to countable subsets of reals is decidable. [Shelah75]
- $MSO$ over $(\mathbb{Q}, <)$ is decidable. [Rabin69]
- $MSO$ over $(\mathbb{R}, <)$ with finitely variable behaviours is decidable. [Rabin69][Rabinovich98].
Quantitative Realtime Logics

Logic $MFO^+$

Constructs of $MFO$, together with $+1$ function.

Example $(\forall t.((t_0 < t < t_0 + 1 \Rightarrow P(t))) \Rightarrow Q(t_0 + 1))$

Theorem Logic $MFO^+$ is undecidable.

Proof Method We can encode accepting runs of a 2-counter machine by a formula.

- Encode each configuration within an interval of time length 1.
- Number of alternations of $X_1, X_2$ represent value of counters $C_1, C_2$.
- The configuration can be copied from one unit interval to next.
Guarded Measurements

Logic \(QMFO\) [Hirschfeld-Rabinovich]

\(MFO\) constructs together with two bounded quantifiers below.

- Let \(\phi(t)\) be a QMFO formula with only variable \(t\) free.
  
- Define \((\exists t)^{t_0+1}_{t_0} \phi(t) \overset{\text{def}}{=} \exists t(t_0 < t < t_0 + 1 \land \phi(t))\).

Dually, \((\forall t)^{t_0+1}_{t_0} \phi(t) \overset{\text{def}}{=} \forall t(t_0 < t < t_0 + 1 \Rightarrow \phi(t))\).

- Define \((\exists t)^{t_0}_{t_0-1} \phi(t) \overset{\text{def}}{=} \exists t(t < t_0 < t + 1 \land \phi(t))\).

Dually \((\forall t)^{t_0}_{t_0-1} \phi(t) \overset{\text{def}}{=} \forall t(t < t_0 < t + 1 \Rightarrow \phi(t))\).

Example \(Timer(X, Y) \overset{\text{def}}{=} \forall t(Y(t) \Leftrightarrow (\forall t_1)^{t_0}_{t_0-1} X(t_1))\).

\(Y\) is true at a time iff \(X\) has been invariantly true for the previous (open) unit interval.
Temporal Logic QTL

Logic QTL $TL(U, S, \Diamond_1, \Diamond_1)$ with two constrained modalities below.

- $\Diamond_1 X$ has truth table $(\exists t)^{<t_0+1}_{t_0} X(t)$
- $\Diamond_1 X$ has truth table $(\exists t)^{<t_0}_{t_0-1} X(t)$
- Define duals $\Box_1 \phi \overset{\text{def}}{=} \neg \Diamond_1 \neg \phi$ and $\Box_1 \phi \overset{\text{def}}{=} \neg \Diamond_1 \neg \phi$.

Example: $\Box(Y \Rightarrow \Box_1 X)$. 
Expressive Completeness

Theorem For any temporal logic $TL(\tau)$ which is expressively complete for $MFO$, the logic $TL(\tau, \rightarrow \diamond_1, \leftarrow \diamond_1)$ is expressively complete for $QMFO$. Specifically, $QTL$ is expressively complete for $QMFO$.

Consequence: Any temporal modality which has a truth-table in $QMFO$ can be expressed within $QTL$.

Notation Denote $\rightarrow \diamond_1 X$ by $\rightarrow \diamond_{(0,1)} X$ and $\leftarrow \diamond_1 X$ by $\diamond_{(-1,0)} X$. 
Interval Bounded Modalities

\[ (\exists t)_{\leq t_0 + 1}^\geq X(t) \overset{\text{def}}{=} X(t_0) \lor (\exists t)_{\leq t_0 + 1}^\geq X(t). \]

\[ \Diamond_{[0,1]} X \overset{\text{def}}{=} X \lor \Diamond_{(0,1)} X \]

\[ (\exists t)_{> t_0}^{\leq t_0 + 1} X(t) \overset{\text{def}}{=} (\exists t)_{> t_0}^{t_0 + 1} X(t) \lor \quad \text{First}(t_0, X) \land (\forall t_1)_{> t_0}^{\leq t_0 + 1} (\exists t)_{> t_1}^{t_1 + 1} X(t) \]

\[ \Diamond_{(0,1]} X \overset{\text{def}}{=} \Diamond_{(0,1)} X \lor \left[(\neg X \cup X) \land \Box_{(0,1)} \Diamond_{(0,1)} X\right] \]

\[ \Diamond_{(1,\infty)} X \overset{\text{def}}{=} \Box_{(0,1)} \Diamond X. \]

\[ \Box_{(n,n+1)} X \overset{\text{def}}{=} \Box_{(n-1,n)} \Diamond_{(0,1)} \Box_{(0,1)} X. \]
A Variety of Modalities

Thus, we can define constrained modality $\diamond_I$ where $I$ is non-singular

- interval $I$ has integer end-point or infinity as end-points.
- Non-singular, i.e. not a singleton set or empty.
- Can be closed, open, partially closed.

Logic MITL $TL(U_I, S_I)$ where $I$ is non-singular interval.

Let $X U(n, n + m) Y \overset{\text{def}}{=} (\Box_{(0,n]}(X \land X U Y)) \land \diamond_{(n,n+m)} Y$.

Theorem MITL and QTL have same expressive power.
A Variety of Modalities (Cont)

- **Nearest Next [Wilke, Raskin]**

\[ \triangledown (n, n+m) X \overset{\text{def}}{=} (\neg X \cup X) \land \lozenge (m, m+n) X \land \neg \lozenge (0, n] X. \]

- **Age Constraints [MP93]**

\[ \text{Age}(X) > n \overset{\text{def}}{=} \lozenge (\neg n, 0) X. \]

\[ \text{Age}(X) = n \overset{\text{def}}{=} \Box (\neg n, 0) X \land \Box (\neg n, -(n-1)) \lozenge (0, 0) \neg X. \]
Summary

- Logic $QTL$ is expressively complete for $QMSO$.
- Modalities from most know decidable timed logics can be defined within $QTL$ and $QMSO$.
- Question Is $QMSO$ decidable?
Decidability of QMFO

Overview

- Timer Normal Form $TNF \subset QMFO$
- Transform $QMFO \rightarrow TNF$ (equivalent formula)
- Transform $TNF \rightarrow MFO$ (equisatisfiable formula)
- Decidability of $MFO$
Timer Normal Form (TNF)

In QTL, define \( Timer(X, Y) \overset{\text{def}}{=} \square(Y \iff \square_1 X) \).

In QMFO define

\[
Timer(X, Y) \overset{\text{def}}{=} \forall t(Y(t) \iff (\forall t_1)_{t_0-1} X(t_1))
\]

\[
Timer_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \overset{\text{def}}{=} \bigwedge_i Timer(X_i, Y_i)
\]
Timer Normal Form (TNF)

In QTL, define $\text{TIMER}(X,Y) \overset{\text{def}}{=} \Box(Y \Leftrightarrow \Box_1 X)$.

In QMFO define

$$\text{TIMER}(X,Y) \overset{\text{def}}{=} \forall t (Y(t) \Leftrightarrow (\forall t_1)_{t_0} X(t_1))$$

$\text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \overset{\text{def}}{=} \bigwedge_i \text{Timer}(X_i, Y_i)$

Formula is in first order TNF if it has the following form where $\phi \in MFO$ and $\overline{W}$ is list of monadic predicates.

$$\exists \overline{W}.(\text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \land \phi)$$
**Timer Normal Form (TNF)**

- In QTL, define \( \text{TIMER}(X, Y) \overset{\text{def}}{=} \Box (Y \Leftrightarrow \Box_1 X) \).
- In \( QMFO \) define
  \[
  \text{TIMER}(X, Y) \overset{\text{def}}{=} \forall t (Y(t) \Leftrightarrow (\forall t_1)_{t_0-1}^{t_0} X(t_1))
  \]
- \( \text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \overset{\text{def}}{=} \bigwedge_i \text{Timer}(X_i, Y_i) \)
- Formula is in **first order TNF** if it has the following form where \( \phi \in MFO \) and \( \overline{W} \) is list of monadic predicates.
  \[
  \exists \overline{W}. (\text{Timer}_n(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \land \phi)
  \]
- If in above definition, if \( \phi \in MSO \) we have **second order TNF**. If \( \phi \in TL \) we have **TL TNF**.
Reducing Future to Past

Aim: To represent $\diamondsuit_1$ by $\square_1$ (using $\mathcal{U}$, $\mathcal{S}$).
Consider witness $\square(Y \Leftrightarrow \diamondsuit_1 Y)$. This implies
Reducing Future to Past

Aim: To represent $\Diamond_1$ by $\square_1$ (using $\mathcal{U}, S$).

Consider witness $\square(Y \Leftrightarrow \Diamond_1 Y)$. This implies

$$\psi_1 \overset{\text{def}}{=} \square(Y \Rightarrow YUX).$$
Reducing Future to Past

Aim: To represent $\Diamond_1$ by $\Box_1$ (using $U, S$).

Consider witness $\Box(Y \Leftrightarrow \Diamond_1 Y)$. This implies

\[ \psi_1 \triangleq \Box(Y \Rightarrow YUX). \]

\[ \psi_2 \triangleq \Box(X \Rightarrow \Box_1 Y). \]
Reducing Future to Past

**Aim:** To represent $\Diamond_1$ by $\Box_1$ (using $\mathcal{U}, \mathcal{S}$).

Consider witness $\Box(Y \leftrightarrow \Diamond_1 Y)$. This implies

- $\psi_1 \overset{\text{def}}{=} \Box(Y \Rightarrow Y \mathcal{U} X)$.
- $\psi_2 \overset{\text{def}}{=} \Box(X \Rightarrow \Box_1 Y)$.
- $\psi_3 \overset{\text{def}}{=} \Box((\Box_1 \neg X) \Rightarrow \Diamond_{[-1,0]} \neg Y)$.

Let $\psi \overset{\text{def}}{=} \psi_1 \land \psi_2 \land \psi_3$. 
Reducing Future to Past

**Aim:** To represent $\Diamond_1$ by $\Box_1$ (using $U, S$).

Consider witness $\Box(Y \iff \Diamond_1 Y)$. This implies

- $\psi_1 \overset{\text{def}}{=} \Box(Y \Rightarrow YUX)$.
- $\psi_2 \overset{\text{def}}{=} \Box(X \Rightarrow \Box_1 Y)$.
- $\psi_3 \overset{\text{def}}{=} \Box(\Box_1 \neg X \Rightarrow \Diamond_{[-1,0]} \neg Y)$.

Let $\psi \overset{\text{def}}{=} \psi_1 \land \psi_2 \land \psi_3$.

Also, $\psi_1 \land \psi_3 \Rightarrow (Y \Rightarrow (\Diamond_1 X))$ and $\psi_2 \Rightarrow ((\Diamond_1 X) \Rightarrow Y)$.
Reducing Future to Past

Aim: To represent $\Diamond_1$ by $\Box_1$ (using $U$, $S$).

Consider witness $\Box(Y \iff \Diamond_1 Y)$. This implies

- $\psi_1 \overset{\text{def}}{=} \Box(Y \Rightarrow YUX)$.
- $\psi_2 \overset{\text{def}}{=} \Box(X \Rightarrow \Box_1 Y)$.
- $\psi_3 \overset{\text{def}}{=} \Box((\Box_1 \neg X) \Rightarrow \Diamond_{[-1,0]} \neg Y)$

Let $\psi \overset{\text{def}}{=} \psi_1 \land \psi_2 \land \psi_3$.

Also, $\psi_1 \land \psi_3 \Rightarrow (Y \Rightarrow (\Diamond_1 X))$ and $\psi_2 \Rightarrow (((\Diamond_1 X) \Rightarrow Y)$.

Hence, $\Box(Y \iff \Diamond_1 Y) \iff \psi$.

$\Diamond_1 Y \iff \exists Y.(Y \land \psi)$. 
Reducing \( \psi \)

\[
\Box(Y \Rightarrow YUX) \land \Box(X \Rightarrow \Box_1 Y) \land \Box((\Box_1 \neg X) \Rightarrow \Diamond [-1,0] \neg Y).
\]
Reducing $\psi$

\[\Box(Y \Rightarrow YUX) \land \Box(X \Rightarrow \Box_1 Y) \land \Box((\Box_1 \neg X) \Rightarrow \Diamond[(-1,0) \neg Y]).\]

- $\Diamond[(-1,0) \neg Y] \overset{\text{def}}{=} (\Diamond_1 \neg Y \lor (YS \neg Y) \land \Box_1 \Diamond_1 \neg Y)$ and
- $\Diamond_1 Z \iff \neg \Box_1 \neg Z$. 

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Reducing $\psi$

$$\square(Y \implies YU\neg X) \land \square(X \implies \square_1Y) \land \square((\square_1\neg X) \implies \lozenge[-1,0]\neg Y).$$

- $\lozenge[-1,0]\neg Y \overset{\text{def}}{=} (\lozenge_1\neg Y \lor (YS\neg Y) \land \square_1 \lozenge_1\square_1\neg Y)$ and
- $\lozenge_1Z \iff \neg\square_1\neg Z$.

Hence, $\psi \iff$

$$\square(Y \implies YU\neg X) \land \square(X \implies \square_1Y) \land$$

$$\square((\square_1\neg X) \implies (\neg\square_1Y \lor (YS\neg Y) \land \square_1\neg\square_1Y)).$$
Eliminating $\mathcal{F}_{1} X$ using Timers

- Consider subformula $\mathcal{F}_{1} X$.
- Introduce $\text{Timer}(X, W) \overset{\text{def}}{=} \mathcal{F}(W \Leftrightarrow \mathcal{F}_{1} X)$.
- Hence, $\psi \Leftrightarrow \exists W. (\text{Timer}(X, W) \land \psi[\mathcal{F}_{1} X/W]$
Eliminating $\square_1 X$ using Timers

- Consider subformula $\square_1 X$.
- Introduce $\text{Timer}(X, W) \overset{\text{def}}{=} \square (W \Leftrightarrow \square_1 X)$.
- Hence, $\psi \iff \exists W.(\text{Timer}(X, W) \land \psi[\square_1 X/W]\notag$

We obtain, $\psi \iff \exists T_1, T_2, T_3. \quad \text{Timer}(\neg X, T_1) \land \text{Timer}(Y, T_2) \land \text{Timer}(\neg T_2, T_3) \land$

$\square(Y \Rightarrow YUX) \land \square(X \Rightarrow T_2) \land$

$\square(T_1 \Rightarrow (\neg T_2 \lor ((YS\neg Y) \land T_3))))$
Eliminating $\square_1 X$ using Timers

- Consider subformula $\square_1 X$.
- Introduce $Timer(X, W) \overset{\text{def}}{=} \square(W \Leftrightarrow \square_1 X)$.
- Hence, $\psi \Leftrightarrow \exists W. (Timer(X, W) \land \psi[\square_1 X/W])$

We obtain, $\psi \Leftrightarrow$

$$\exists T_1, T_2, T_3. \ Timer(\neg X, T_1) \land Timer(Y, T_2) \land Timer(\neg T_2, T_3) \land$$
$$\square(Y \Rightarrow YUX) \land \square(X \Rightarrow T_2) \land$$
$$\square(T_1 \Rightarrow (\neg T_2 \lor ((YS\neg Y) \land T_3))))$$

Recall that $\diamondsuit_1 X \Leftrightarrow \exists Y. (Y \land \psi)$
**Reduction to Timer Normal Form**

**Theorem** For any \( \phi(t, Z) \) in \( QMFO \), we can associate auxiliary monadic predicates \( X, Y \) and formula \( \phi(t, Z, X, Y) \) in \( MFO \) such that

\[
\phi(t, Z) \iff \exists X, Y. (Timer_n(X, Y) \land \phi(t, Z, X, Y))
\]

- The theorem is true even when \( \phi \in QMOSO \) and gives a reduction to \( MSO \).
- The theorem is true even when \( \phi \in QTL \) and gives a reduction to \( TL(U, S) \).
Elimination of Metric

Let $\overline{X} = X_1, \ldots, X_n$ and $\overline{Y} = Y_1, \ldots, Y_n$. We transform $\text{Timer}(\overline{X}, \overline{Y})$ into $\text{Timer}(X, \overline{Y})$ in $MFO$ such that satisfiability is preserved (equisatisfiable).

MFO Properties of Timers

Formula $A_i$ is conjunction of

- $Y_i$ is true at 0
- $Y_i$ is finitely variable.
- Set of point where $Y_i$ is true is closed. I.e. if $Y_i$ holds for $(a, b)$ it also holds for $[a, b]$.

Formula $B_i$ is conjunction of

- For $t > 0$ if $Y_i(t)$ then $X_i$ is true is small left neighbourhood of $t$. 
Metric Elimination (Cont)

- If $X_i$ continuously true from $t$ onwards then $Y_i$ becomes continuously true from some future point $t' > t$.
- If $Y_i(t)$ and $X_i$ holds in $[t, t')$ then $Y_i(t')$.

Formula $C_{i,j}$ is conjunction of

- If $Y_i(t) \land \neg Y_j(t)$ then for some $t' < t$ we have $X_i$ holds invariantly for $(t', t)$ but $X_j$ does not hold invariantly in $(t', t)$.
- If $Y_i$ and $Y_j$ become true at $t$ then for every previous $t'$ we have $X_i$ is true over $(t', t)$ iff $X_j$ is true over $(t', t)$.

Let $\overline{Timer}(X, Y) \overset{\text{def}}{=} \bigwedge_i A_i \land B_i \land \bigwedge_{i,j} C_{i,j}$. 
Timer Elimination Theorem

Theorem [HR03] The predicates

\[ P_1, \ldots, P_n, Q_1, \ldots, Q_n \models \text{Timer}(X, Y) \] iff there is an order preserving bijection \( \rho : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[ \rho(P_1), \ldots, \rho(P_n), \rho(Q_1), \ldots, \rho(Q_n) \models \text{Timer}(\overline{X}, \overline{Y}) \]

For \( \phi \in MFO \), we have \( M \models \phi \) iff \( \rho(M) \models \phi \).
Decidability of QMFO

Theorem For every \( \phi \in QMFO \) (or \( QMSO, QTL \)) we can construct \( \overline{\phi} \in MFO \) (or \( MSO, TL \)) which is equisatisfiable.

Corollary

\( QMFO \) is decidable over continuous canonical models.

\( QMSO \) is decidable over finitely variable models.

(Alternative proof of \( MITL \) decidability.)

\( QTL \) is decidable over continuous canonical models.
References

