Motion involving both translation and rotation:

Angular momentum

$$L_z = I_0 \omega$$

axis of rotation is $\parallel$ z axis

I_0 — moment of inertia about CM $\rightarrow$ Purely rotational motion

Translational motion : $\Rightarrow$ angular momentum $(\vec{R} \times M\vec{V})_z$.

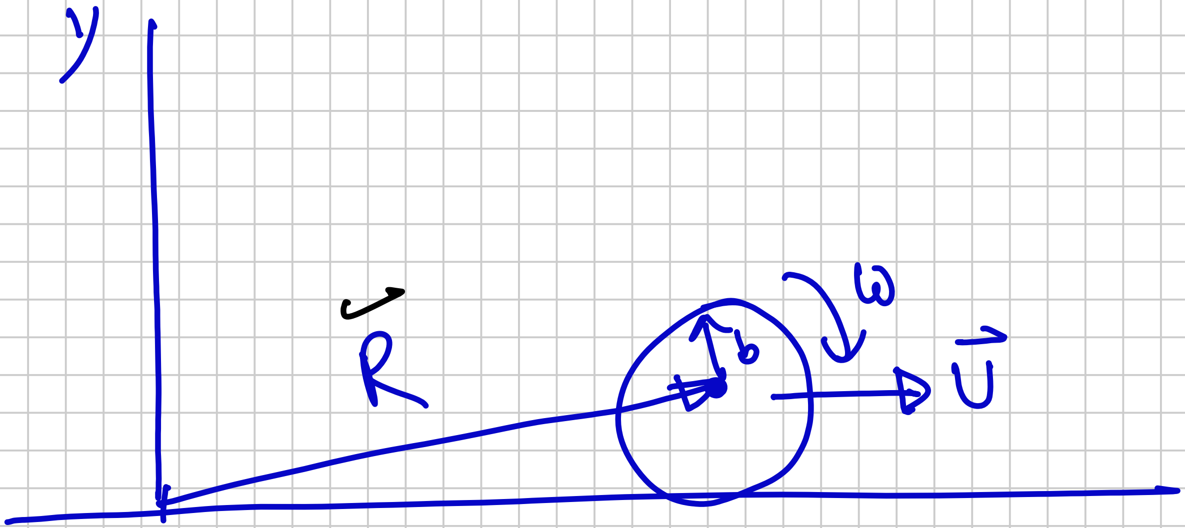
$$\boxed{L_z = I_0 \omega + (\vec{R} \times M\vec{V})_z}$$

See K K for derivation

$\vec{R} \rightarrow$ position vector

$\vec{V} = \dot{\vec{R}}$

Angular momentum of a rolling wheel:



Moment of inertia of the wheel about CM

$$I_0 = \frac{1}{2} M b^2$$

Angular momentum about CM

$$L_0 = -I_0 \omega = -\frac{1}{2} M b^2 \omega$$

Parallel to z-axis

Angular momentum of CM

$$(\vec{R} \times M\vec{v})_z = -Mbv$$

Total angular momentum

$$L_z = -\frac{1}{2} M b^2 \omega - M b v$$

$$= -\frac{1}{2} M b^2 \omega - M b^2 \omega$$

$$= -\frac{3}{2} M b^2 \omega$$

$v = b\omega$

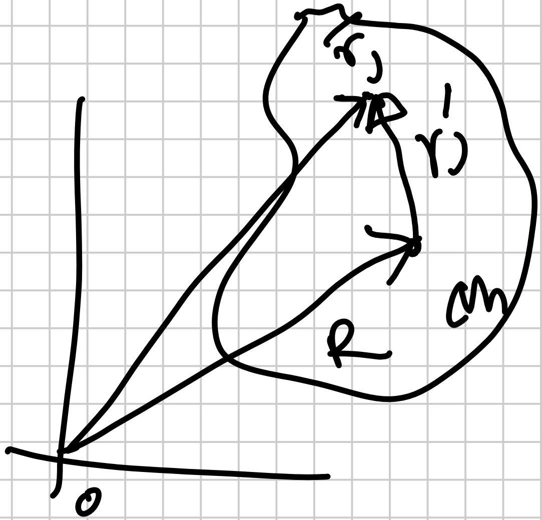
$\times$

Torque $\vec{\tau} = \sum \vec{r}_j \times \vec{f}_j$

$$\vec{r}_j = \vec{R} + \vec{r}_j{}'$$

$$\vec{\tau} = \sum (\vec{r}_j{}' + \vec{R}) \times \vec{f}_j$$

$$= \sum (\vec{r}_j{}' \times \vec{f}_j) + \vec{R} \times \vec{F}$$



$\vec{F} = \sum f_j$

$$\vec{\tau} = \underbrace{\sum \left(\vec{r}_j' \times \vec{f}_j \right)}_{\text{Torque about CM due to various external forces!}} + \underbrace{\vec{R} \times \vec{F}}_{\text{Torque due to total external force acting at CM}}$$

$$\underset{\substack{\uparrow \\ \text{z component of torque about CM}}}{\tau_z = \tau_0} + \left(\vec{R} \times \vec{F} \right)_z \longrightarrow \text{as } \omega = \omega \hat{k}$$

$$\frac{dL_z}{dt} = I_0 \frac{d\omega}{dt} + \frac{d}{dt}\left[\vec{R} \times M\vec{v}\right]_z$$

$$= I_0\alpha + (\vec{R} \times M\vec{a})_z$$

Now $\quad \tau_z = \frac{dL_z}{dt}$

$$\tau_0 + (\vec{R} \times \vec{F})_z = I_0\alpha + (\vec{R} \times M\vec{a})_z$$

$$= I_0\alpha + (\vec{R} \times \vec{F})_z$$

So $\quad \tau_0 = I_0\alpha$

Rotational motion about CM depends only on the torque about CM. & independent of translational motion!

$$\boxed{K.E. = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M v^2}$$

spin

linear motion

a Pure rotation about an axis—no translation

$$L = I\omega$$

$$\tau = I\alpha$$

$$K = \tfrac{1}{2} I\omega^2$$

b Rotation and translation (subscript 0 refers to center of mass)

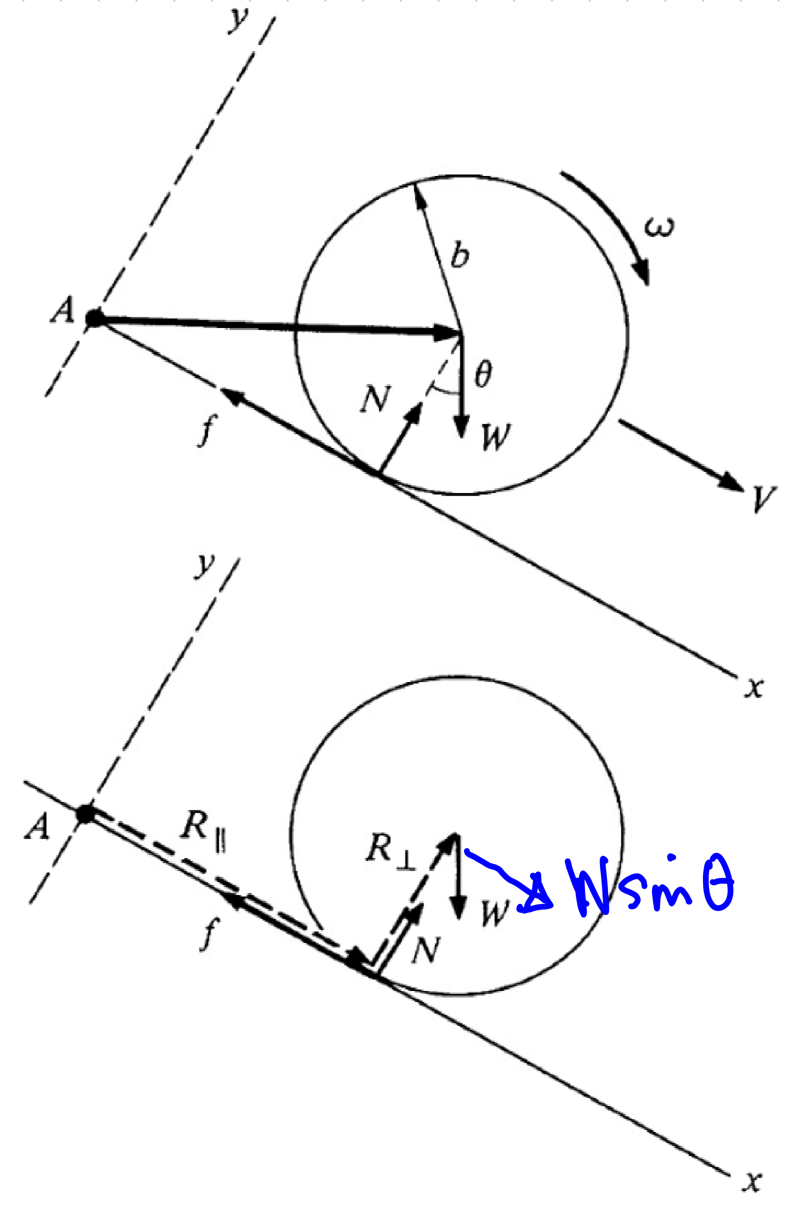
$$L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z$$

$$\tau_z = \tau_0 + (\mathbf{R} \times \mathbf{F})_z$$

$$\tau_0 = I_0\alpha$$

$$K = \tfrac{1}{2} I_0\omega^2 + \tfrac{1}{2} M V^2$$

Example   Drum rolling down a plane



The torque about A

$$\begin{aligned}\tau_s &= \tau_0 + (\mathbf{R} \times \mathbf{F})_z \\ &= -R_\perp f + R_\perp(f - W\sin\theta) + R_\parallel(N - W\cos\theta) \\ &= -bW\sin\theta,\end{aligned}$$

As, $R_\perp = b$ and $W\cos\theta = N$.

$$\begin{aligned}L_z &= -I_0\omega + (\mathbf{R} \times M\mathbf{V})_z \\ &= -\tfrac{1}{2}Mb^2\omega - Mb^2\omega \\ &= -\tfrac{3}{2}Mb^2\omega,\end{aligned}$$

Now $\tau_z = \dfrac{dL_z}{dt}$

$$bW\sin\theta = \frac{3}{2}Mb^2\alpha$$

$$\Rightarrow \alpha = \frac{2}{3}\, g\sin\theta / b$$

$$v = b\omega \Rightarrow a = b\alpha$$

$$a = \frac{2}{3} g \sin\theta$$

Work-Energy Theorem:

$$K_b - K_a = W_{ba}$$

we know:

$$= \int_{r_a}^{r_b} \vec{F} \cdot d\vec{r},$$

$$\vec{F} = M \frac{d^2 \vec{R}}{dt^2} = M \frac{d\vec{v}}{dt}$$

$$\vec{F} \cdot d\vec{R} = M \frac{d\vec{v}}{dt} \cdot d\vec{R} = M \frac{d\vec{v}}{dt} \cdot \vec{v} \, dt$$

as $d\vec{R} = \vec{v} \, dt$

$$= d\left(\frac{1}{2} m v^2\right)$$

$$\int_{R_a}^{R_b} \vec{F} \cdot d\vec{r} = \frac{1}{2} M v_b^2 + \frac{1}{2} m A_a^2$$

Work associated with

$$\tau_0 = I_0 \alpha = I_0 \frac{d\omega}{dt}$$

Rotational K.E. $= \frac{1}{2} I_0 \omega^2$

$$\tau_0 \, d\theta = I_0 \frac{d\omega}{dt} \cdot \omega \, dt$$

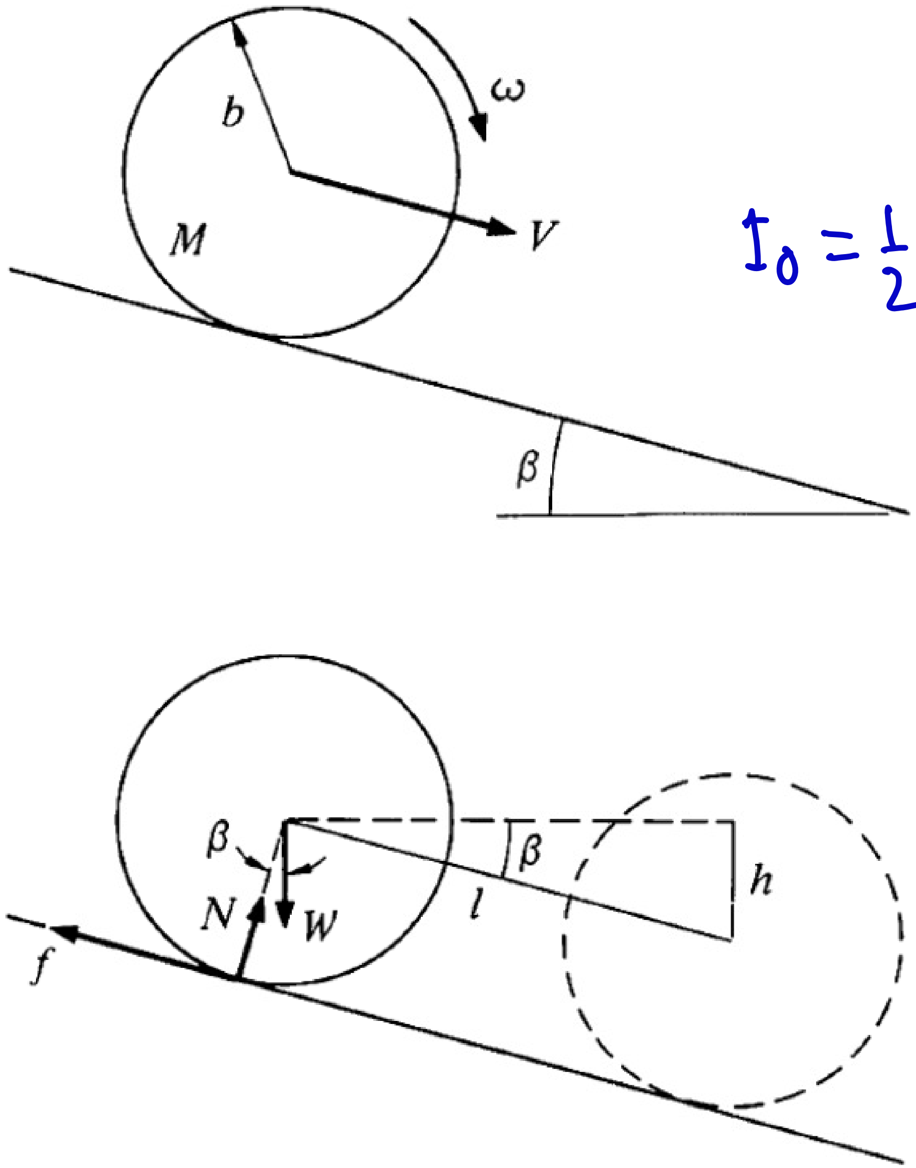
$$= d\left(\frac{1}{2} I_0 \omega^2\right)$$

$$\underbrace{\int_{\theta_a}^{\theta_b} \tau_0 \, d\theta}_{\text{work done by applied torque}} = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2$$

$$K_b - K_a = W_{ba}$$

$$K = \frac{1}{2} M v^2 + \frac{1}{2} I_0 \omega^2$$

Drum rolling down a plane: Energy Method



$I_0 = \frac{1}{2}Mb^2$

Energy eq^n for translational motion

$$\int_a^b \vec{F} \cdot d\vec{r} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2$$

$$(W\sin\beta - f)\,l = \frac{1}{2}MV^2 \quad (1)$$

where $l = \dfrac{h}{\sin\beta}$

Consider rotational motion!

$$\int_{\theta_a}^{\theta_b} \tau\, d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2$$

a, $f b \theta = \frac{1}{2} I_0 \omega^2$ θ rotational angle about CM.

without slipping $b\theta = l$

$\Rightarrow f l = \frac{1}{2} I_0 \omega^2$ $\omega = \frac{V}{b}$

$= \frac{1}{2} \frac{I_0 V^2}{b^2}$

$\Rightarrow W_h = \frac{1}{2} \left(\frac{I_0}{b^2} + M \right) V^2$ — Workdone!

$= \frac{3}{4} M V^2$.

Friction force is not dissipative!