PH101 LECTURE 11 DKG

Note Title 09-Sep-12

Motion under centra force!

Central force $\equiv$! Conservative force
Spherically symmetric

Example — gravitational
— Electrical etc.

Mass "m" in spherially symmetric
Central field of force!

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

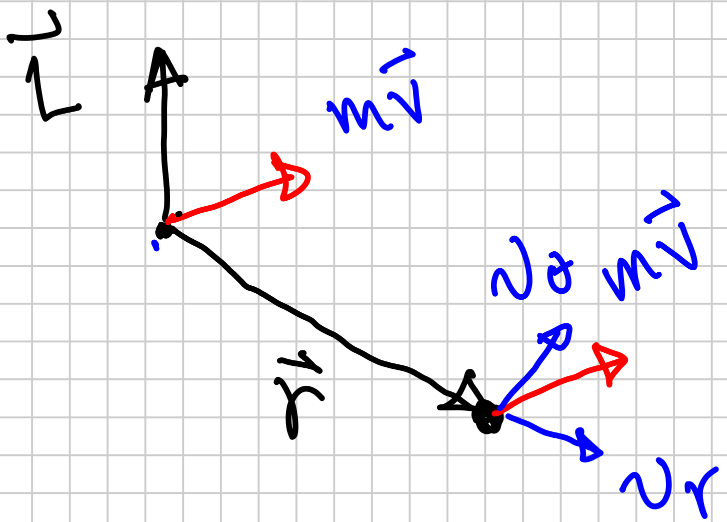
$$\vec{a} = \underbrace{[\ddot{r} - r\dot{\theta}^2]}_{a_r}\hat{r} + \underbrace{[r\ddot{\theta} + 2\dot{r}\dot{\theta}]}_{a_\theta}\hat{\theta}$$

Conservation of angular momentum:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

Now $\quad \tau = \vec{r} \times \vec{F} = r \times m\frac{d\vec{v}}{dt} = \frac{d\vec{L}}{dt}$

orbital momentum $\vec{L} = \vec{r} \times \vec{p}$



$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta}$$

$$L = rmv_\theta = mr^2\frac{d\theta}{dt}$$

$$= mr^2\dot{\theta}$$

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}$$

$$= 0$$

Central force

$F_r = F(r)$

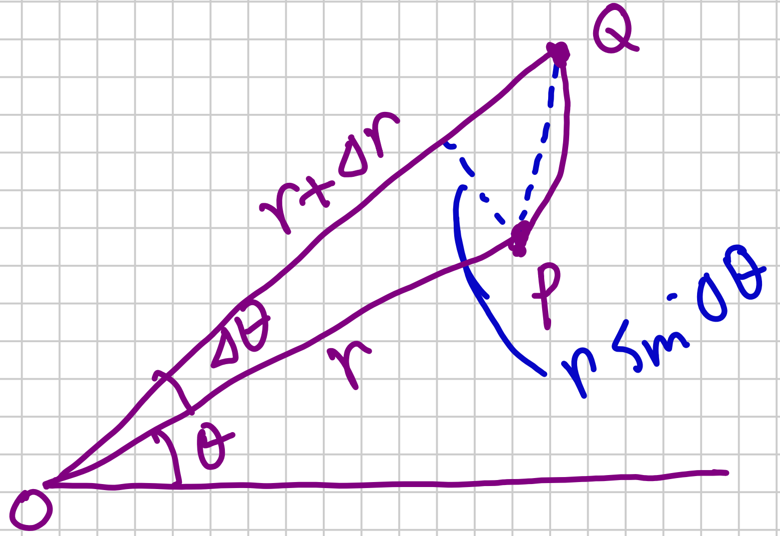
$F_\theta = 0$

$$\Rightarrow r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\cdot\frac{d\theta}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$\boxed{r^2\frac{d\theta}{dt} = \text{constant}}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{L}{2m}$$



$$\Delta A = \frac{1}{2}r(r+\Delta r)\sin\Delta\theta$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{const}$$

[Kepler]

Conservation of energy!

$$\frac{m}{2}\left[v_r^2 + v_\theta^2\right] + U(r) = E$$ mechanical energy

We know $m r v_\theta = L$

E and L Constant of motion!

$U(r)$ — P.E of particle in central field

$$\frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} + U(r) = E$$

$$\Rightarrow \frac{1}{2}mv_r^2 + U'(r) = E$$

with $$\boxed{U'(r) = U(r) + \frac{L^2}{2mr^2}}$$ Equivalent P.E. for 1D radial problem

$$U'(r) = U(r) + \underbrace{\frac{L^2}{2mr^2}}$$

Effective Potential Energy

Any fn of r

$U(r) \to 0$ as $r \to \infty$

Independent of θ, $\dot{\theta}$

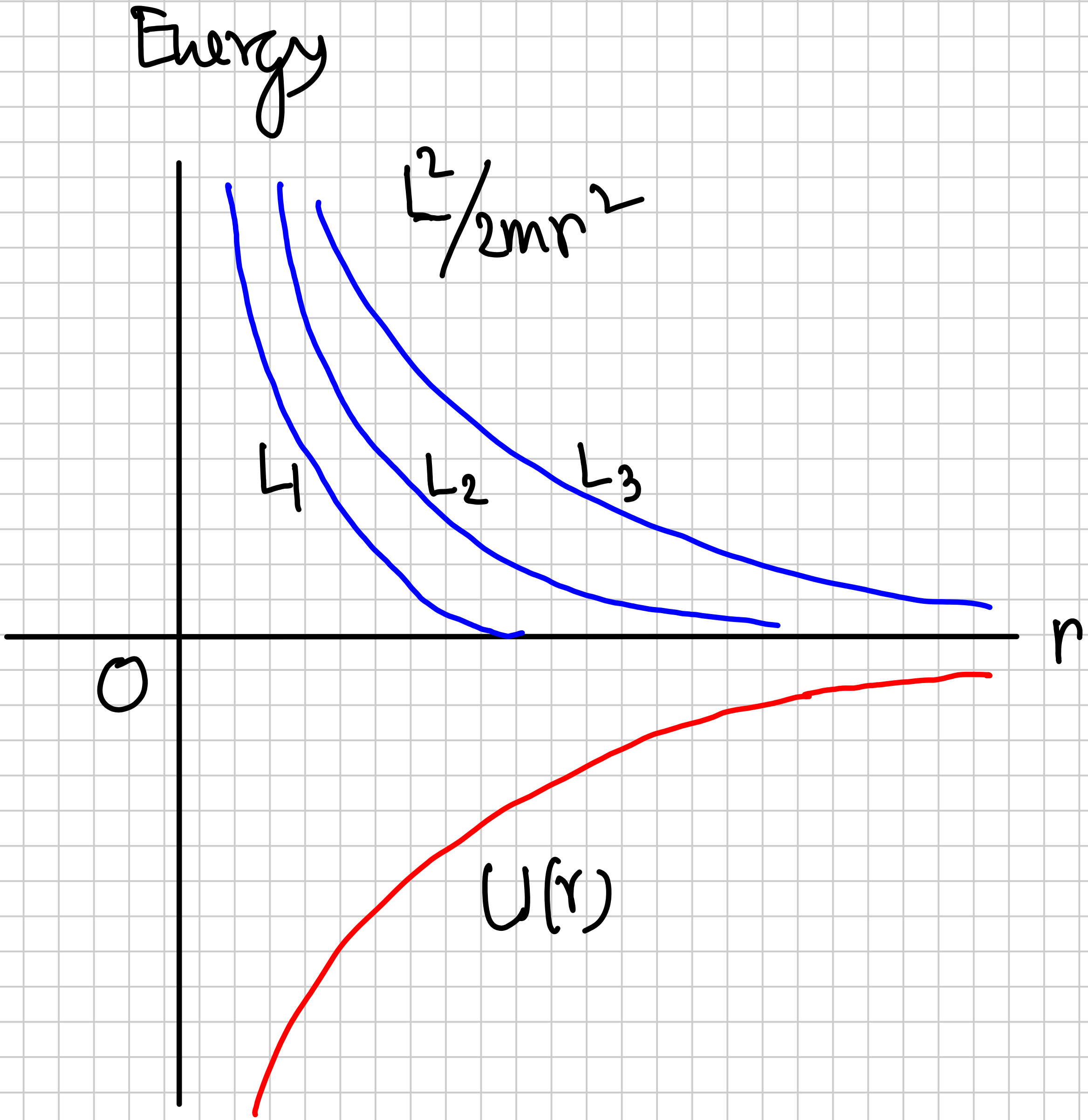
Centrifugal Potential Energy

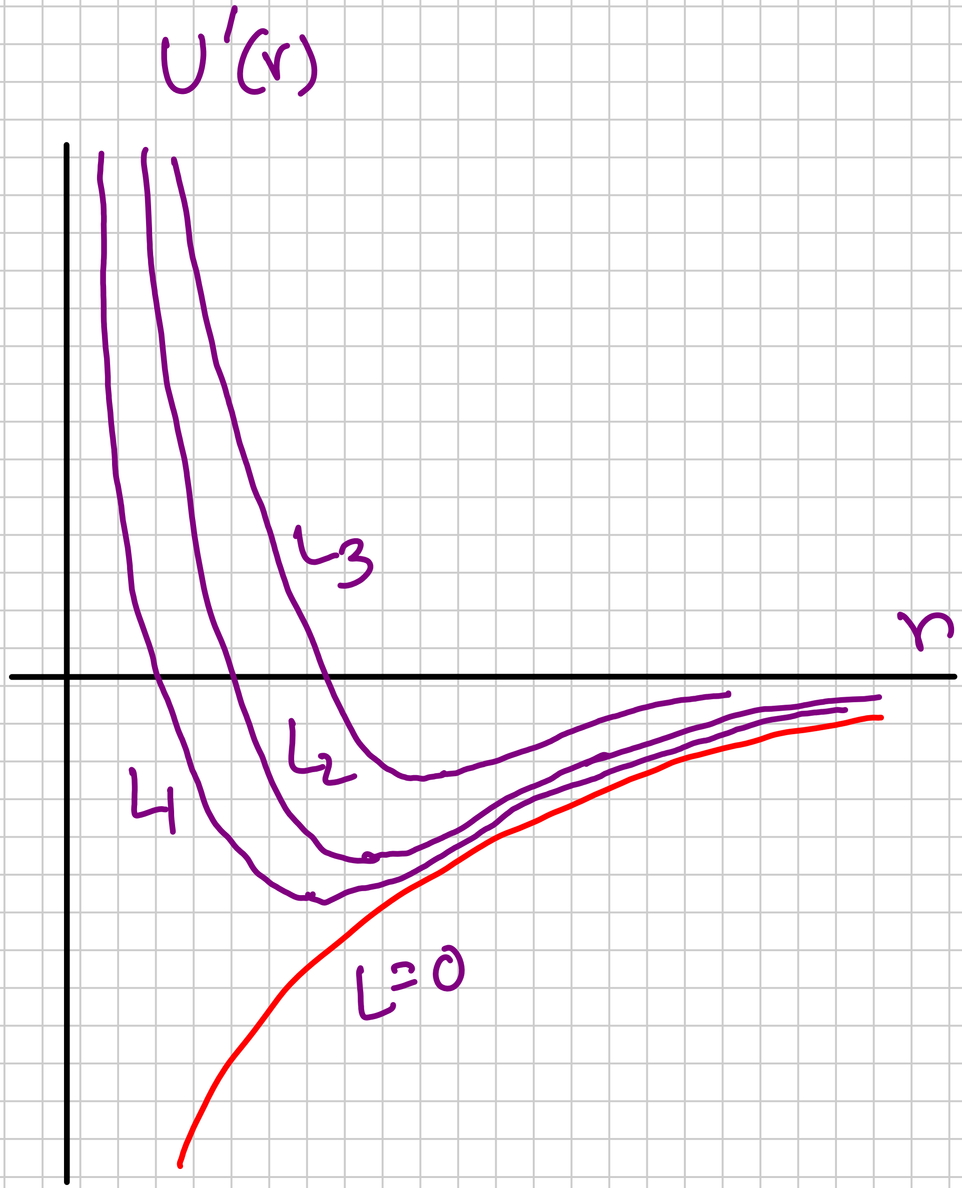
$$F_{Centrifugal} = -\frac{d}{dr}\left(\frac{L^2}{2mr^2}\right) = \frac{L^2}{mr^3}$$

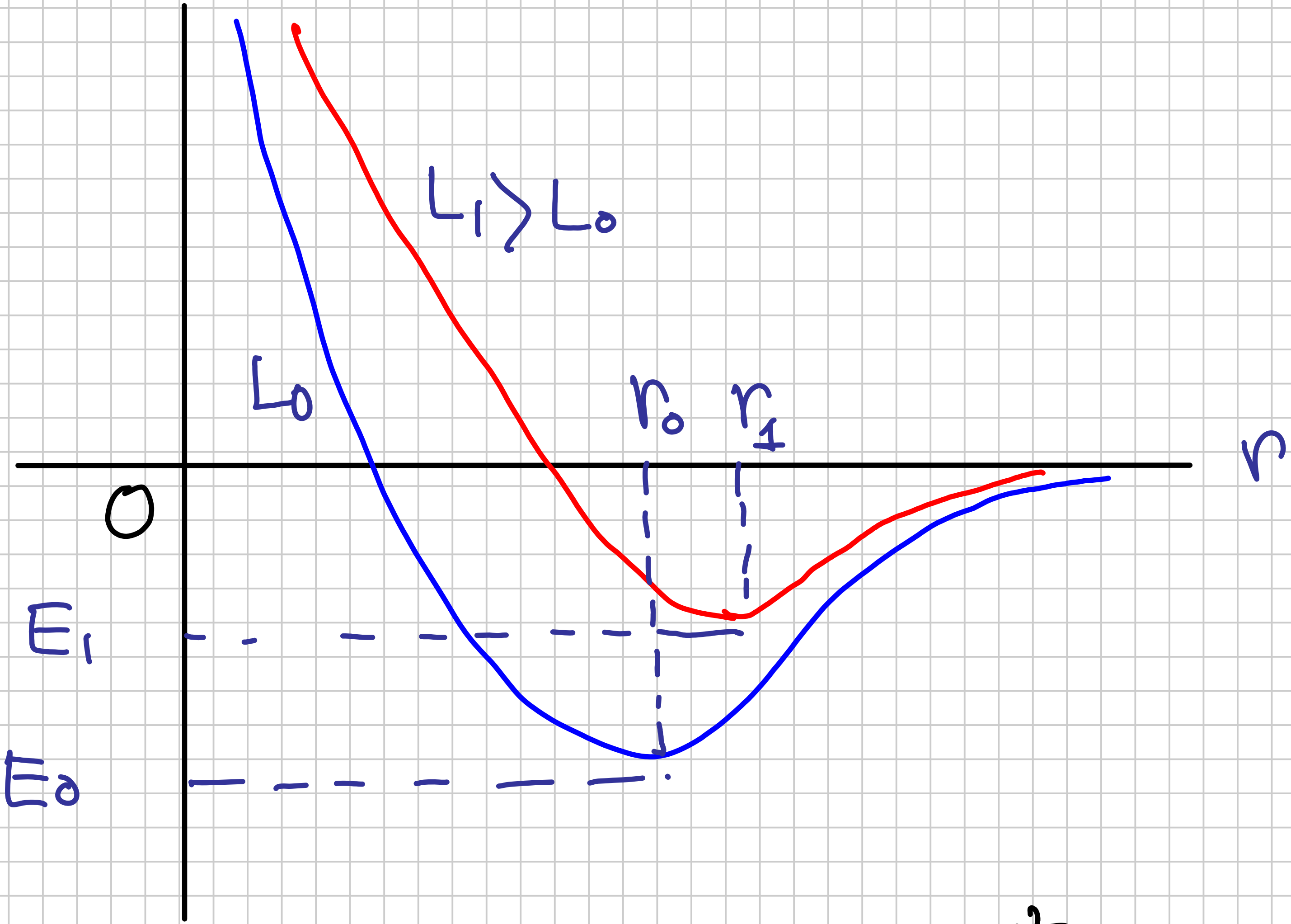
$$L = mr^2 \frac{d\theta}{dt}$$

$$F_{Centrifugal} = mr\left(\frac{d\theta}{dt}\right)^2 = m\omega^2 r$$

Centrifugal P.E = Part of K.E.


Energy
L^2/2mr^2
L1
L2
L3
r
O
U(r)


U'(r)
L3
r
L2
L1
L=0



If $v_r = 0$ Energy $E = \frac{L^2}{2mr^2} - \frac{GMm}{r}$

Put $\frac{GMm}{r} = \frac{L^2}{mr^2} \Rightarrow E = -\frac{L^2}{2mr^2} = -\frac{GMm}{2r}$

$r \propto \frac{1}{E}$

Example

$$U(r) = -\frac{GMm}{r} \qquad \text{Gravitational field!}$$

So, $$\frac{mv_r^2}{2} + \frac{L^2}{2mr^2} - \frac{GMm}{r} = E \qquad \text{and}$$

$$U'(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \rightarrow \text{Effective potential}$$

$$\frac{dU'}{dr} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0$$

$$\Rightarrow \frac{L^2}{mr} = GMm = \text{Constant}$$

Orbital radius $r \propto L^2$

$$U'(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

$$\frac{dU'}{dr} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2}$$

$$\frac{L^2}{mr_0} = GMm$$

$$\frac{d^2U'}{dr^2} = \frac{3L^2}{mr^4} - \frac{2GMm}{r^3}$$

at $r = r_0$ → Effective Spring constant $k = \frac{d^2U'}{dr^2}$

$$k = \frac{GMm}{r_0^3}$$

$$T_r = 2\pi \left(\frac{m}{k}\right)^{\frac{1}{2}} = \frac{2\pi}{(GM)^{1/2}} \cdot r_0^{3/2}$$