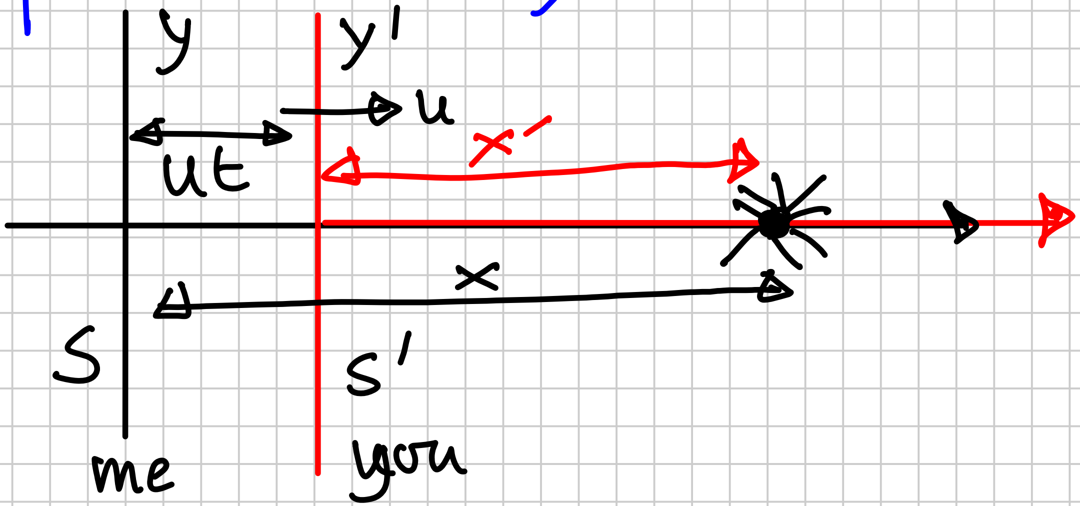
PH101 LECTURE 19 DKG

Note Title 16-Oct-12

Special Theory of relativity



$\left\{ \begin{array}{ll} x=0 \quad t=0 & \text{— me} \\ x'=0 \quad t'=0 & \text{— you} \end{array} \right.$

me — (x,t)

you — (x',t')

Then $\left. \begin{array}{l} x' = x - ut \\ t' = t \end{array} \right\}$ Gallian transformation

or $\left. \begin{array}{l} x = x' + ut \\ t = t' \end{array} \right\}$ u goes to $-u$

$$x' = x - ut$$

$$\frac{dx'}{dt} = \frac{dx}{dt} - u \Rightarrow w = v - u$$

Relative velocity!

$$\underset{w}{\overset{\shortparallel}{\frac{dx'}{dt}}} \qquad \underset{u}{\overset{\shortparallel}{\frac{dx}{dt}}}$$

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt} \longrightarrow$$ acceleration same!

Force does not change!

Newton's laws valid!

$x' = x - ut$
$x = x' + ut'$ } Does not Confirm velocity of light.

Speed of light c 'constant'

3×10^8 m/s

$$x' = (x - ut)\gamma \qquad \& \qquad x' = ct'$$
$$x = (x' + ut')\gamma \qquad\qquad x = ct$$

$$xx' = \gamma^2 [xx' + uxt' - ux't - u^2 tt']$$

$$1 = \gamma^2 \left[1 + u\cancel{\frac{t'}{x'}}_{\to \frac{1}{c}} - u\cancel{\frac{t}{x}}_{\to \frac{1}{c}} - u^2 \frac{t}{x}\frac{t'}{x'}\right]$$

$$= \gamma^2 [1 - u^2/c^2]$$

$$\Rightarrow \quad \gamma = 1 / [1 - u^2/c^2]^{\frac{1}{2}}$$

$$\Rightarrow \quad x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

Similarly

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

$\Bigg\}$ Lorentz transformation

Due to velocity of light behave strange way!

x and t are coupled!

$\Longrightarrow$ space-time (x, t)

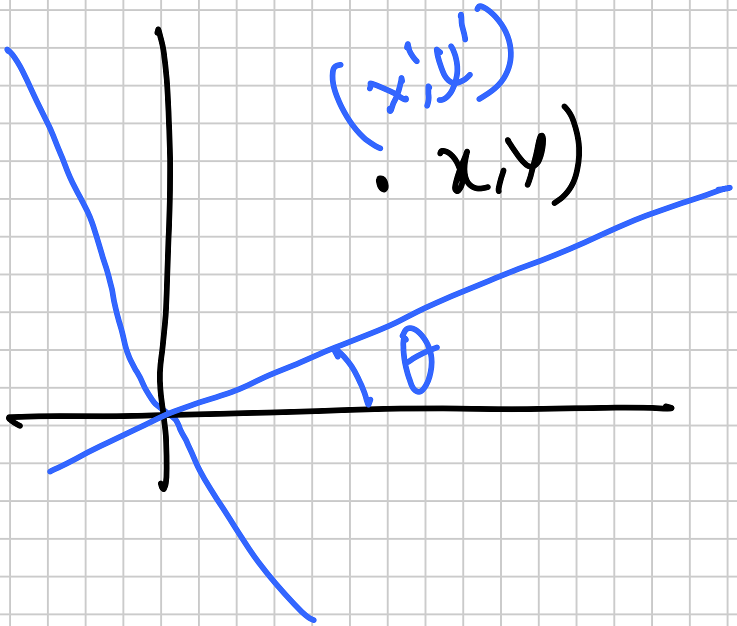
$\frac{u}{c} \ll 1$

$x' = x - ut$ & $t = t'$ $\rightarrow$ Gallian transformation

What are (x,t) & (x',t')

$\longrightarrow$ two events

Relates two events!



$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$\theta = \pi/4 \longrightarrow x' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$y' = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$(1,1) \to x' = \sqrt{2}$$

$$y' = 0$$

$$\Rightarrow \quad x' = \frac{x}{\sqrt{}} + \frac{ut}{\sqrt{}}$$

$\nleftarrow$ not ordinary transformation!

$$x = \frac{x' + ut}{\sqrt{1-u^2/c^2}} \quad , \qquad t = \frac{t' + ux'/c^2}{\sqrt{1-u^2/c^2}}$$

Reverse transformation.

Take a pair of events;

$$x_1' = \frac{x_1 - ut_1}{\sqrt{1-u^2/c^2}} \qquad t_1' = \frac{t_1 - ux_1/c^2}{\sqrt{1-u^2/c^2}}$$

$$x_2' = \frac{x_2 - ut_2}{\sqrt{1-u^2/c^2}} \qquad t_2' = \frac{t_2 - ux_2/c^2}{\sqrt{1-u^2/c^2}}$$

$$\Delta x' = x_2' - x_1' = \frac{\Delta x - u\,\Delta t}{\sqrt{1 - u^2/c^2}}$$

Seperation of events.

not infinitesimal

Similarly

$$\Delta t' = \frac{\Delta t - \frac{u\,\Delta x}{c^2}}{\sqrt{1 - u^2/c^2}}$$

$\Delta x = 10$ m $\quad \Delta x' =$ not 10 m.

Event 1 — fire gun

Event 2 — bullet hits wall.

$v \to$ velocity of bullet for me $= \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

w — " " you $= \frac{\Delta x'}{\Delta t'} = ?$

$$w = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x - u\,\Delta t}{\Delta t - \frac{u\,\Delta x}{c^2}} = \frac{\frac{\Delta x}{\Delta t} - u}{1 - \frac{u}{c^2}\frac{\Delta x}{\Delta t}}$$

$w = \frac{3}{4}c$

You $\rightarrow \frac{3}{4}c$

me $\rightarrow$?

$V = 2\cdot\frac{3}{4}c = 1.5c$

$$v = \frac{\frac{3}{4}c + \frac{3}{4}c}{1+\frac{9}{16}} = \frac{24}{25}c$$

$$\boxed{w = \frac{v-u}{1-\frac{uv}{c^2}}}$$

$$\boxed{v = \frac{w+u}{1+\frac{uw}{c^2}}}$$

Not like before!

$\frac{u}{c}\cdot\frac{w}{c} \ll 1$

$v = w + u$

Light pulse!

$$w = \frac{c-u}{1-\frac{uc}{c^2}} = c$$

Consider two events !

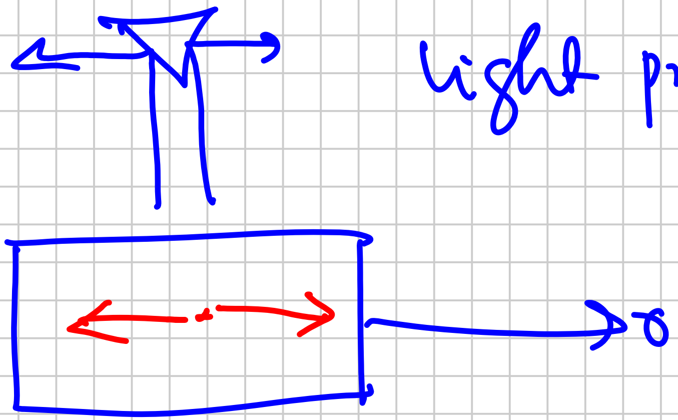
$$\Delta t' = \frac{\Delta t - \frac{u \Delta x}{c^2}}{\sqrt{1 - u^2/c^2}} \rightarrow \left.\begin{array}{l} \text{if } \Delta t = 0 \\ \Delta t' \neq 0 \end{array}\right]$$

?? How??

Simultaneity lost !

If $c \to \infty$ then $\Delta t' = \Delta t$

! Two events occur same time and same plane →



light pulse splited simultaneously !

Velocity same for every one !

$\Delta t = 0$ but $\Delta t' = -ve$!

clock !

will not run with the same rate

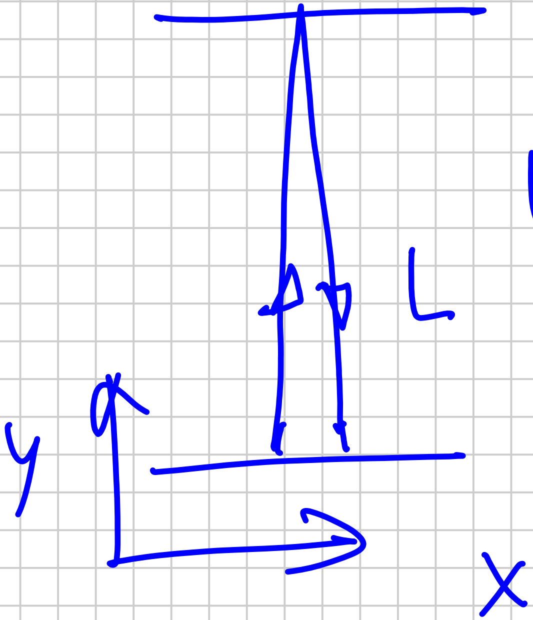
think about pair of events!

$(0,0)$ — tick
$(0,\tau_0)$ — tock
$\Big\}$ me

$\Delta x = 0$, $\Delta t = \tau_0$

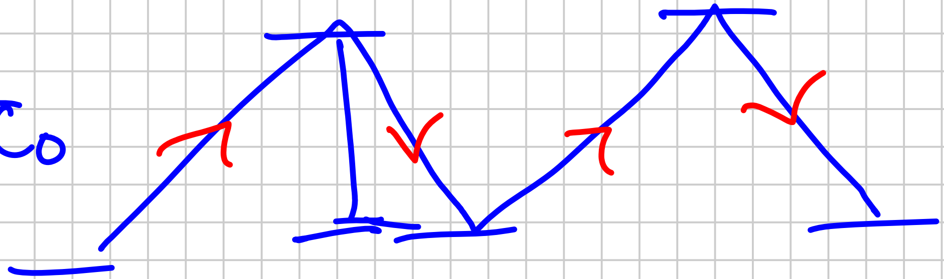
you $\quad \Delta t' = \dfrac{\tau_0 - 0}{\sqrt{1-u^2/c^2}} = \dfrac{\tau_0}{\sqrt{1-u^2/c^2}}$ — your clock is slow

$$\Delta t = \frac{\Delta t' + u\,\Delta x'}{\sqrt{1-u^2/c^2}} \longrightarrow$$ your clock is slow!



light pulse!

$$\frac{2L}{c} = \tau_0$$

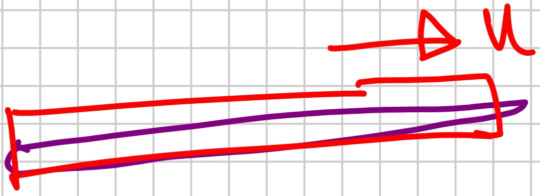


Me

you (light travel longer length)

Twin paradox

Length contraction:



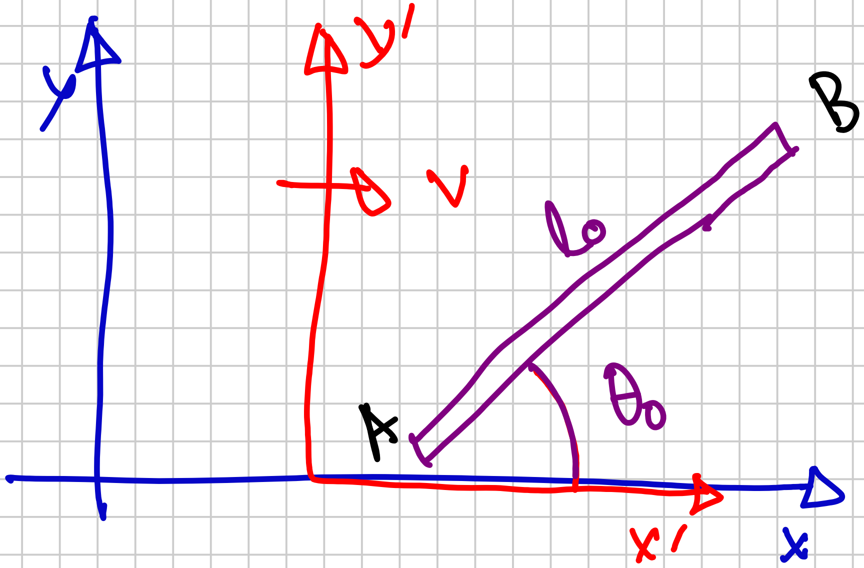
$$\Delta x' = \frac{\gamma x - u\,\delta t}{\sqrt{1-u^2/c^2}}$$

Same time $\delta t = 0$
measure two ends of the rod

$$\Rightarrow \quad L_0 = \frac{L}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \quad L = L_0\sqrt{1-u^2/c^2}$$

The orientation of a moving rod!



What is the length and orientation of the rod in S frame!

End of the rod A & B

$$A: \quad x'_A = 0 \qquad y'_A = 0$$

$$x_B' = l_0 \cos\theta \qquad y'_B = l_0 \sin\theta$$

$$x' = \gamma (x - vt) \quad \& \quad y' = y$$

$$A: \quad x'_A = 0 = (x_A - vt)\gamma \qquad y'_A = 0 = y_A$$

B:

$$x'_B = l_0 \cos\theta = \gamma(x_B - vt) \qquad y'_B = l_0 \sin\theta$$
$$= y_B$$

Hence

$$x_B - x_A = \frac{l_0 \cos\theta}{\gamma}$$

$$y_B - y_A = l_0 \sin\theta$$

Length

$$l = \left[(x_B - x_A)^2 + (y_B - y_A)^2 \right]^{\frac{1}{2}}$$

$$= l_0 \left[\left(1 - \frac{v^2}{c^2}\right) \cos^2\theta + \sin^2\theta \right]^{\frac{1}{2}}$$

$$= l_0 \left[1 - \frac{v^2}{c^2} \cos^2\theta \right]^{\frac{1}{2}}$$

$$\theta_c = \arctan\left(\frac{y_B - y_A}{x_B - x_A}\right) = \arctan\left(r\,\frac{\sin\theta}{\cos\theta}\right)$$

$$= \arctan\left(r \tan\theta\right)$$

Moving rod is contracted and rotated!

Order of events and causality :

Cause — event

Event 1 — Event 2

Lorentz transformation !

$$\Delta t' = \frac{\Delta t - \frac{u}{c^2}\Delta x}{\sqrt{1 - u^2/c^2}}$$

$\Delta t'$ can be +ve or -ve

depends on $\frac{u}{c}\frac{\Delta x}{c}$

$\Delta t = t_2 - t_1 > 0$ — 2 occurs after 1

If $\frac{u}{c^2}\Delta x > \Delta t \longrightarrow \Delta t' < 0$ — 2 before 1

$$\Rightarrow \frac{u}{c} > c\,\frac{\Delta t}{\Delta x}$$

$c\Delta t > \Delta x$

Then $\frac{u}{c} > 1$ —— **IMPOSSIBLE**

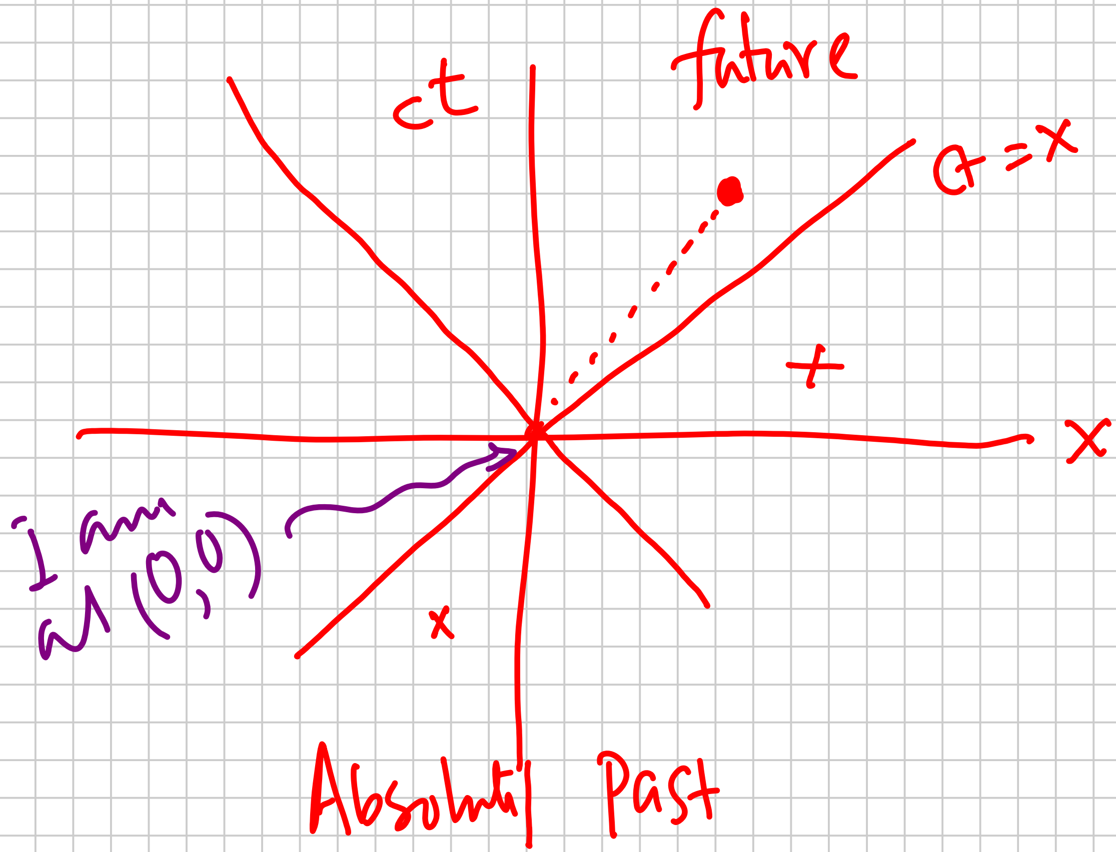
$c\Delta t$

Δx

$c\Delta t$ → Even light cannot connect the events!

$\frac{u}{c} > 1$

No signal travel faster than light!



Four vectors!

Space time (x, t)

$x^2 + t^2 = ?$ Can we write

$X = (x_0, x_1)$

$x_0 \to ct$, $x_1 \to x$

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$x'^2 + y'^2 = x^2 + y^2$

invariant!

$(x_0, x_1, x_2, x_3) \equiv (ct, x, y, z) \equiv (x_0, \vec{r})$

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \frac{x - \frac{u}{c}ct}{\sqrt{1 - u^2/c^2}} = \frac{x - \beta x_0}{\sqrt{1 - \beta^2}}$$

$$t' = \frac{t - \frac{u}{c}\frac{x}{c}}{\sqrt{1 - u^2/c^2}} \Rightarrow ct' = \frac{x_0 - \beta x}{''}$$

$$x_1' = \frac{(x_1 - \beta x_0)}{\sqrt{1-\beta^2}} \qquad x_2' = x_2$$

$$x_0' = \frac{(x_0 - \beta x_1)}{\sqrt{1-\beta^2}} \qquad x_3' = x_3$$

$$\beta = \frac{v}{c} \qquad u \equiv V$$

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$$x_1' = x \frac{1}{\sqrt{1-\beta^2}} - x_0 \frac{\beta}{\sqrt{1-\beta^2}}$$

$$x_0' = x_0 \frac{1}{\sqrt{1-\beta^2}} - x_1 \frac{\beta}{\sqrt{1-\beta^2}}$$

compare with (x, y) rotation.

$$x^2 + y^2 = x'^2 + y'^2$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = A_x' B_x' + A_y' B_y'$$

$$\cos\theta = \frac{1}{\sqrt{1-\beta^2}} , \qquad \sin\theta = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\cos^2\theta + \sin^2\theta \neq 1$$

$$x_0'^2 - x_1'^2 = \frac{x_0^2 + \beta^2 x_1 - 2\cancel{\beta x_0} x_1 - x_1^2 - \beta^2 x_0^2 + 2\cancel{\beta x_0} x_1}{1-\beta^2}$$

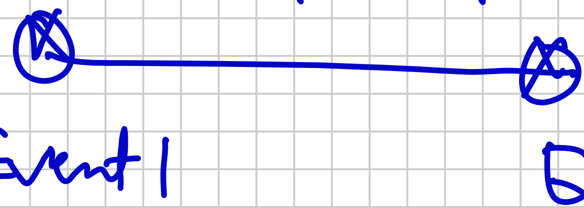
$$= \frac{(x_0^2 - x_1^2)(1-\beta^2)}{(1-\beta^2)} = x_0^2 - x_1^2$$

Space time interval $\quad s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$

$$= x_0'^2 - x_1'^2 - x_2^2 - x_3'^2$$

More generally:

Δx_1 – space, Δx_0 – time

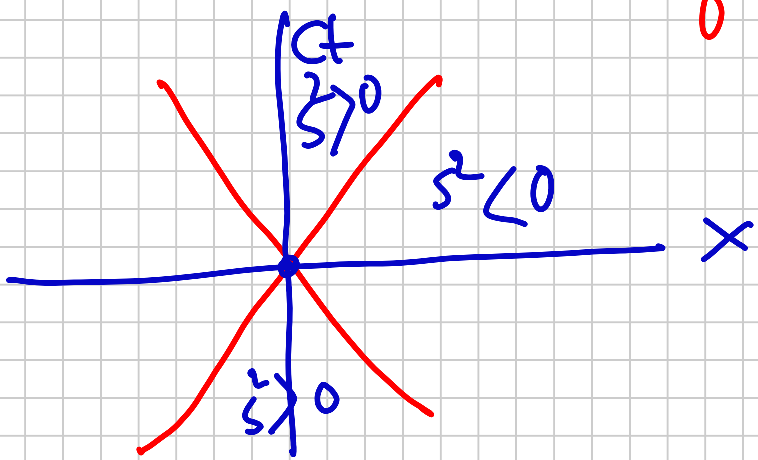


$$(\Delta s)^2 = (\Delta x_0)^2 - (\Delta x_1)^2$$

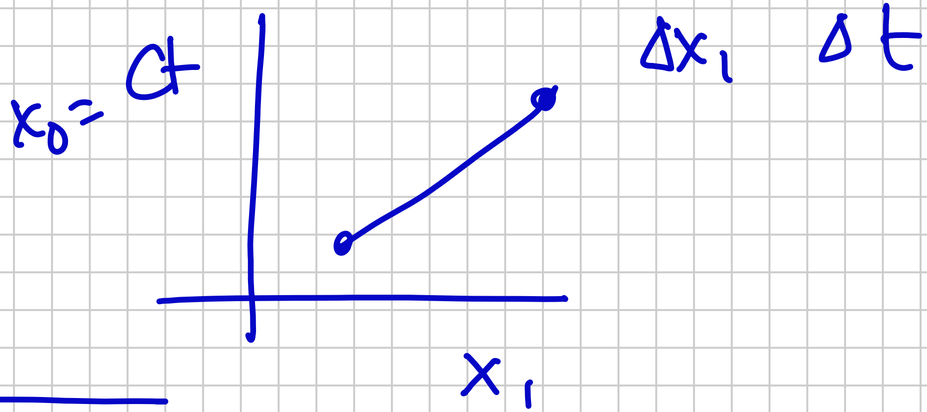
$= +ve \to$ time-like $\quad \Delta x_0 > \Delta x_1$

$= -ve \to$ Space-like $\quad \Delta x_0 < \Delta x_1$

$= 0 \to$ light-like $\quad \Delta x_0 = \Delta x_1$



Momentum $\quad \vec{p} = m \frac{\Delta \vec{r}}{\Delta t} = m \vec{v}$



$$\Delta s = \sqrt{(c\Delta t)^2 - (\Delta x)^2}$$

$$= c\Delta t \sqrt{1 - \left(\frac{\Delta x}{\Delta t}\right)^2 \frac{1}{c^2}} = c\, \underline{d\tau}$$

$$d\tau = dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

$$= dt \sqrt{1 - u^2/c^2}$$

$$d\tau = \sqrt{dt^2 - dx^2/c^2}$$

Four momentum

$$\vec{P} = m\left(\frac{\overset{ct}{dx_0}}{d\tau}, \frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}\right)$$

$$= m\left(c\frac{dt}{d\tau}, \frac{d\bar{r}}{d\tau}\right)$$

$$\frac{df}{d\tau} = \frac{df}{dt}\frac{dt}{d\tau} = \frac{df}{dt} \cdot \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\vec{p} = \left(\frac{mc}{\sqrt{1 - u^2/c^2}} , \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} \right)$$

$$= \left(E/c , \vec{p} \right)$$

$$P_0 = mc\left(1 + \frac{u^2}{2c^2} + \cdots\right)$$

$$= mc + \frac{1}{2} m u^2 \frac{1}{c} + \cdots$$

$$\underbrace{cP_0}_{\uparrow \text{ Energy}} = mc^2 + \underbrace{\frac{1}{2} m u^2}_{\text{K.E}} + \cdots = E$$

$$P_0 = E/c$$

$X = (ct, \vec{r}) \longrightarrow$ one four vector

We got

$$p = \left(\frac{E}{c}, \vec{p}\right) = (p_0, p_1, p_2 \cdots)$$

Energy momentum four vector!

Conservation of energy & momentum.

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

$$P \cdot P = p_0^2 - p^2 = \left(\frac{mc}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{mv}{\sqrt{1-v^2/c^2}}\right)^2$$

$$= m^2c^2$$

$$\left(\frac{E}{c}\right)^2 - p^2 = m^2c^2$$

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

$$p = 0 \Rightarrow \boxed{E = mc^2}$$

for photon $\boxed{E^2 = p^2c^2}$