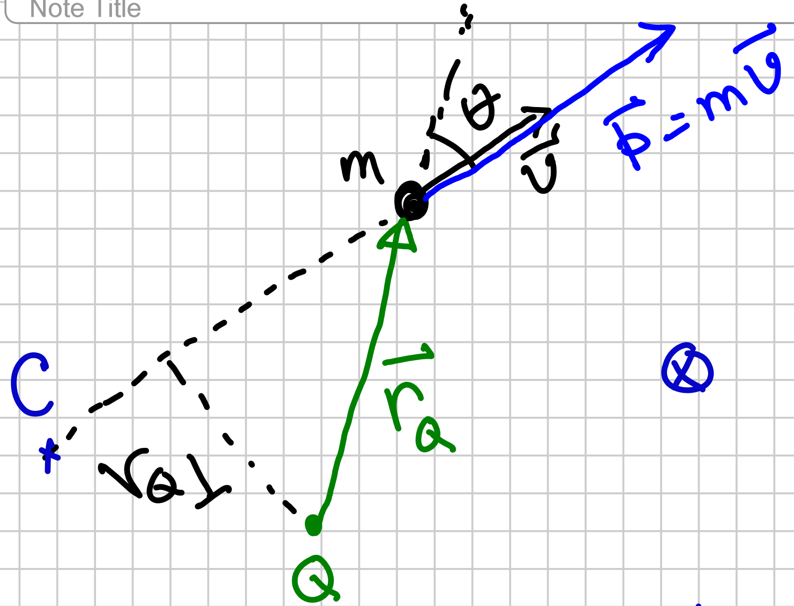
PH 101 LECTURE 10 DKG

Note Title 28-Aug-12



Angular Momentum

Angular momentum

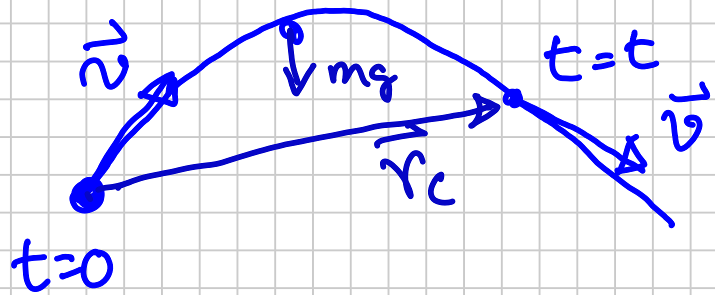
$$\vec{L}_Q = \vec{r}_Q \times \vec{p}$$

$$= (\vec{r}_Q \times \vec{v})\, m$$

$$|\vec{L}_Q| = m v \underbrace{r_Q \sin\theta}_{r_{Q\perp}}$$

Angular momentum is not intrinsic property

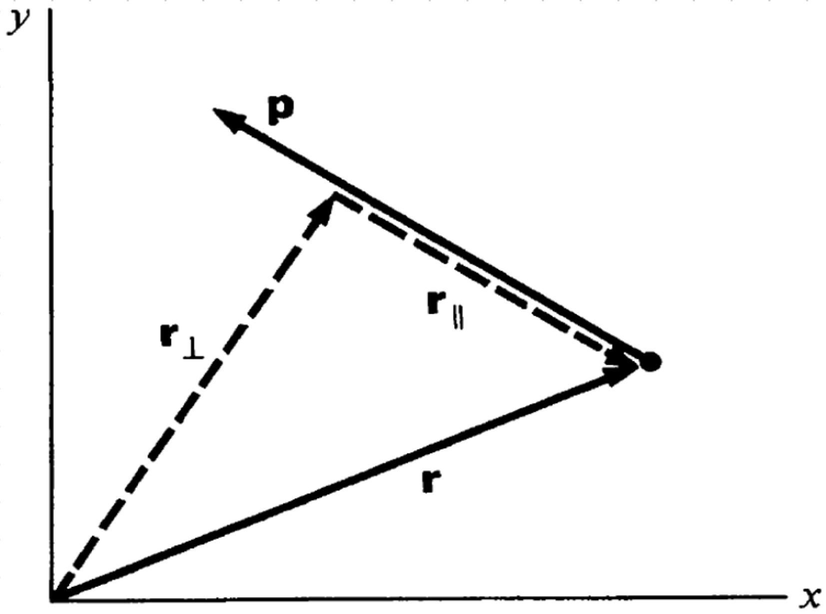
$$L_C = 0$$



at $t=0$ $\quad \vec{L}_C = 0$

at $t=t$ $\quad L_C \neq 0$

How to calculate $\vec{L}$:



Resolve into components

$$\mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_\parallel$$

$$\begin{aligned} \mathbf{L} = \mathbf{r} \times \mathbf{p} &= (\mathbf{r}_\perp + \mathbf{r}_\parallel) \times \mathbf{p} \\ &= (\mathbf{r}_\perp \times \mathbf{p}) + \underbrace{(\mathbf{r}_\parallel \times \mathbf{p})} \\ &= \mathbf{r}_\perp \times \mathbf{p}, \end{aligned}$$

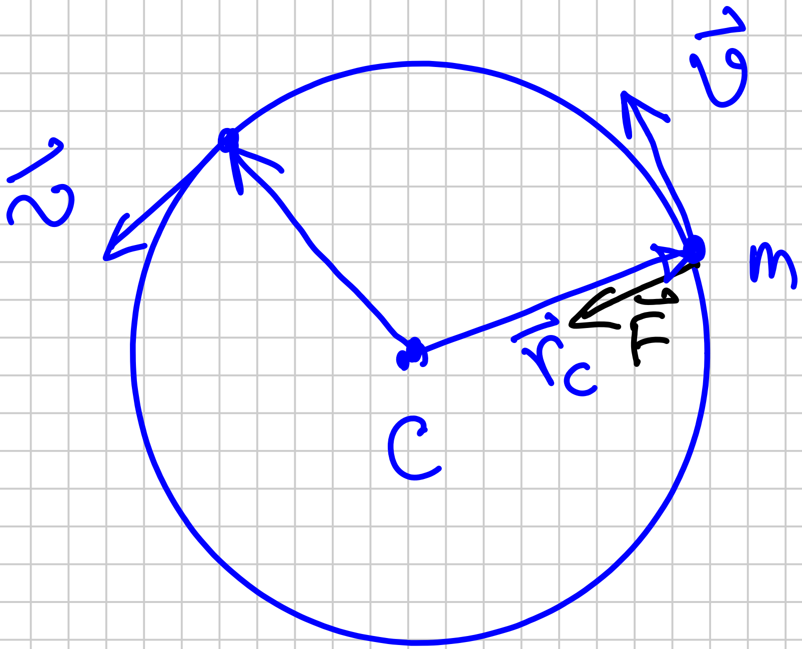
$$\mathbf{r}_\parallel \times \mathbf{p} = 0$$

$$L_z = |\mathbf{r}_\perp|\,|\mathbf{p}|$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \,?$$

Change in angular momentum!



$$|\vec{L}_c| = m\,|\vec{r}_c \times \vec{v}|$$

$$= m\, r_c\, v$$

$$\vec{L}_c = \vec{r}_c \times \vec{p}$$

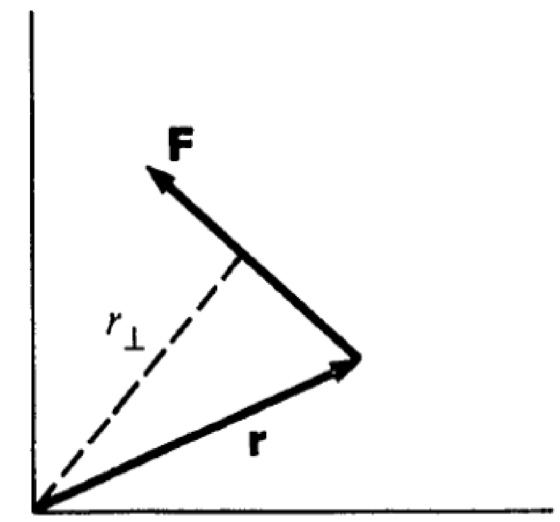
$$\frac{d\vec{L}_c}{dt} = \underbrace{\frac{d\vec{r}_c}{dt} \times \vec{p}}_{=0} + \vec{r}_c \times \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{L}_c}{dt} = \vec{r}_c \times \frac{d\vec{p}}{dt} = \vec{r}_c \times \vec{F}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \tau \quad \text{"}\underline{\text{TORQUE}}\text{"}$$

Torque

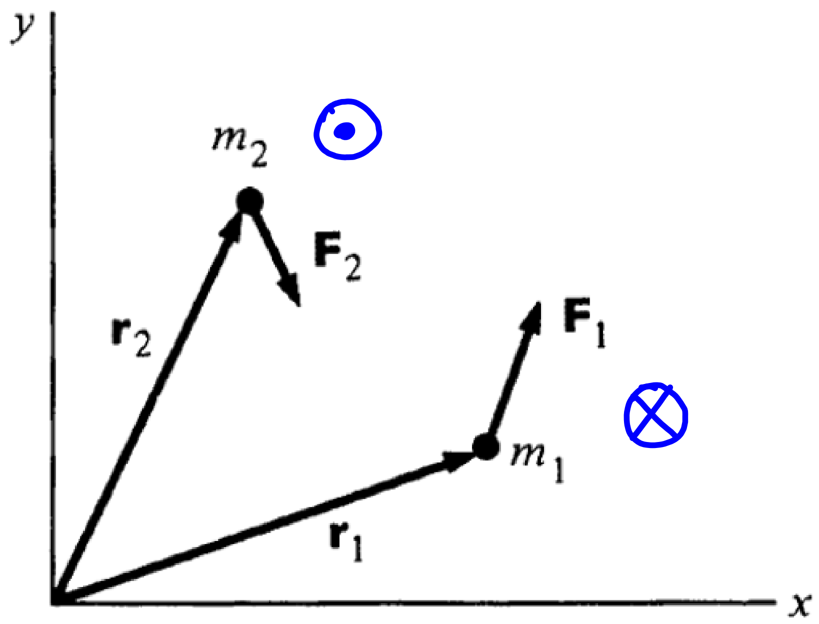
$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$|\boldsymbol{\tau}| = |\mathbf{r}_\perp|\,|\mathbf{F}|$$

$$\boldsymbol{\tau} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

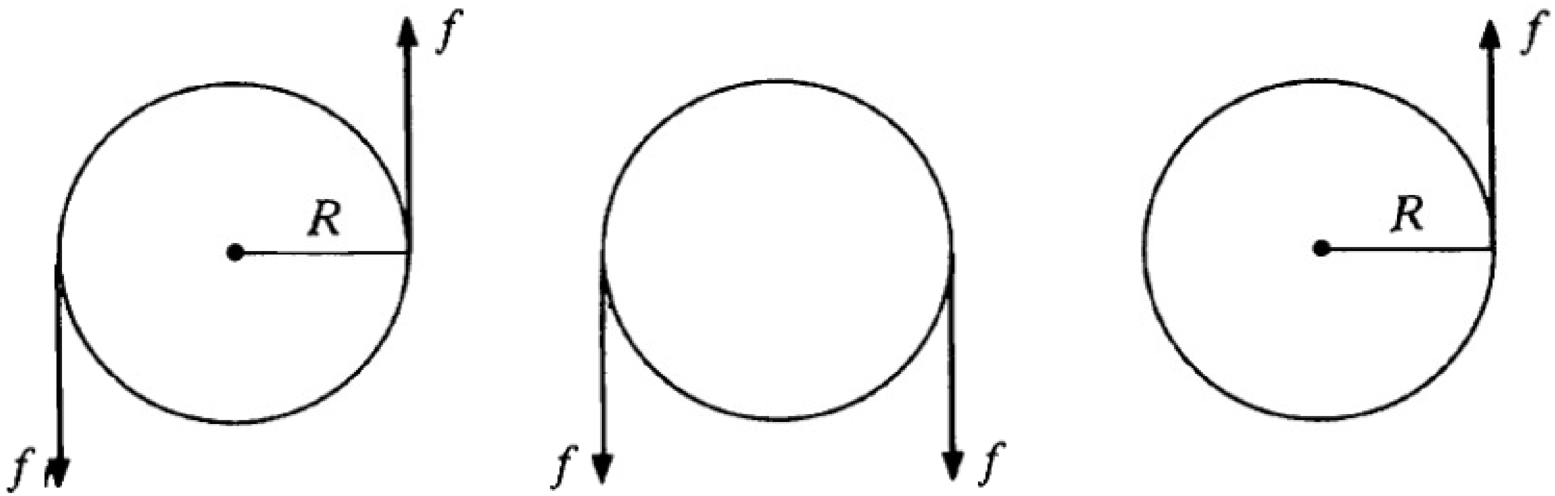
All forces in xy plane!



Torque (τ)
Force (F)
} Two different quantity.

Net force $= 0$ — Torque may not be zero

Net force $\neq 0$ — Torque can be zero



$\tau = 2Rf$, $F = 0$

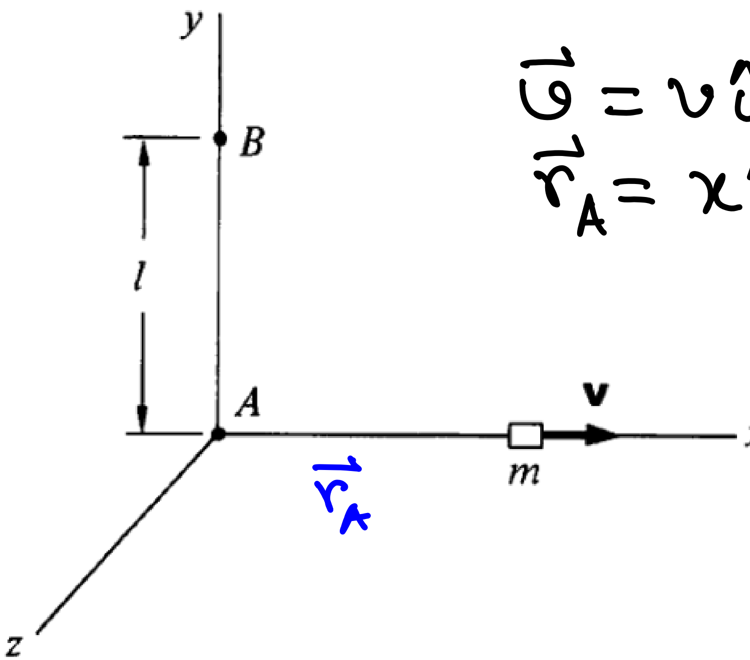
$\tau = 0$, $F = 2f$

$\tau = Rf$, $F = f$

Angular momentum of a sliding block: No friction!

$\vec{v} = v\hat{i}$

$\vec{r}_A = x\hat{i}$



$\vec{r}_A \parallel \vec{v}$

$\vec{L}_A = m\,\vec{r}_A \times \vec{v} = 0$

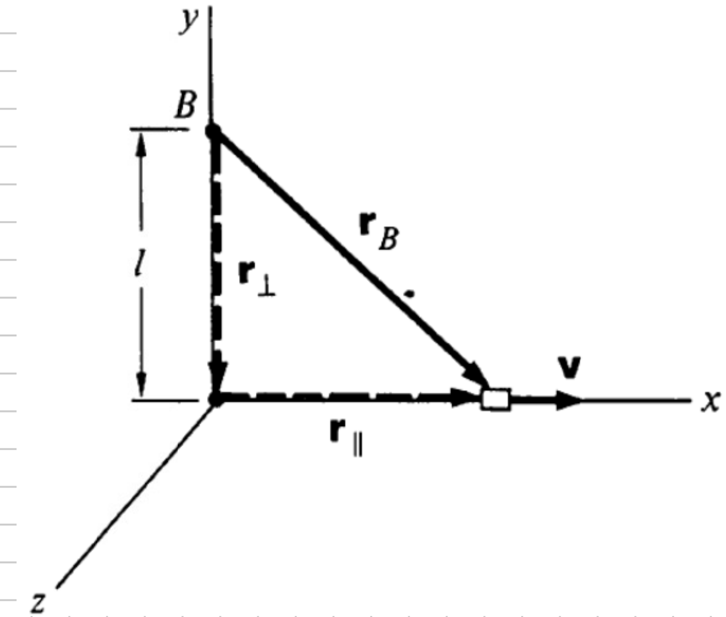
$$\begin{aligned} \mathbf{L}_B &= m\mathbf{r}_B \times \mathbf{v} \\ &= mlv\hat{\mathbf{k}}. \end{aligned}$$

$\mathbf{r}_\parallel \times \mathbf{v} = 0,$

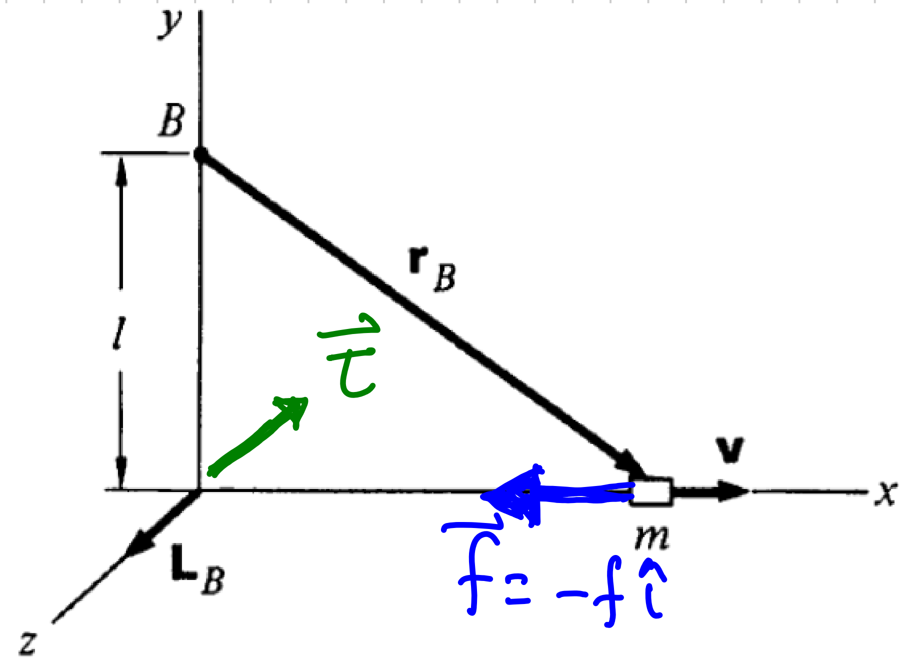
$|\mathbf{r}_\perp \times \mathbf{v}| = lv$

$$\mathbf{r}_B = x\hat{\mathbf{i}} - l\hat{\mathbf{j}}$$

$$\begin{aligned} \mathbf{L}_B &= m\mathbf{r}_B \times \mathbf{v} \\ &= m\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & -l & 0 \\ v & 0 & 0 \end{vmatrix} \\ &= mlv\hat{\mathbf{k}} \end{aligned}$$



Torque on a sliding block: Switch friction!



$$\vec{L}_B = m\vec{r}_B \times \vec{v}$$

$$= m l v \hat{k}$$

constant if $\vec{v}$ is constant

Consider $\vec{f} = -f\hat{\imath}$ Friction force!

Block experiences a torque!

$$\vec{\tau}_B = \vec{r}_B \times \vec{f}$$

$$= -lf\hat{k}$$

Due to friction force

$\vec{v} \Rightarrow$ Decreases

$\vec{L}_B = m v l \hat{k} \Rightarrow$ decreases

$\Delta L_B \Rightarrow$ along $-$ve z-direction!

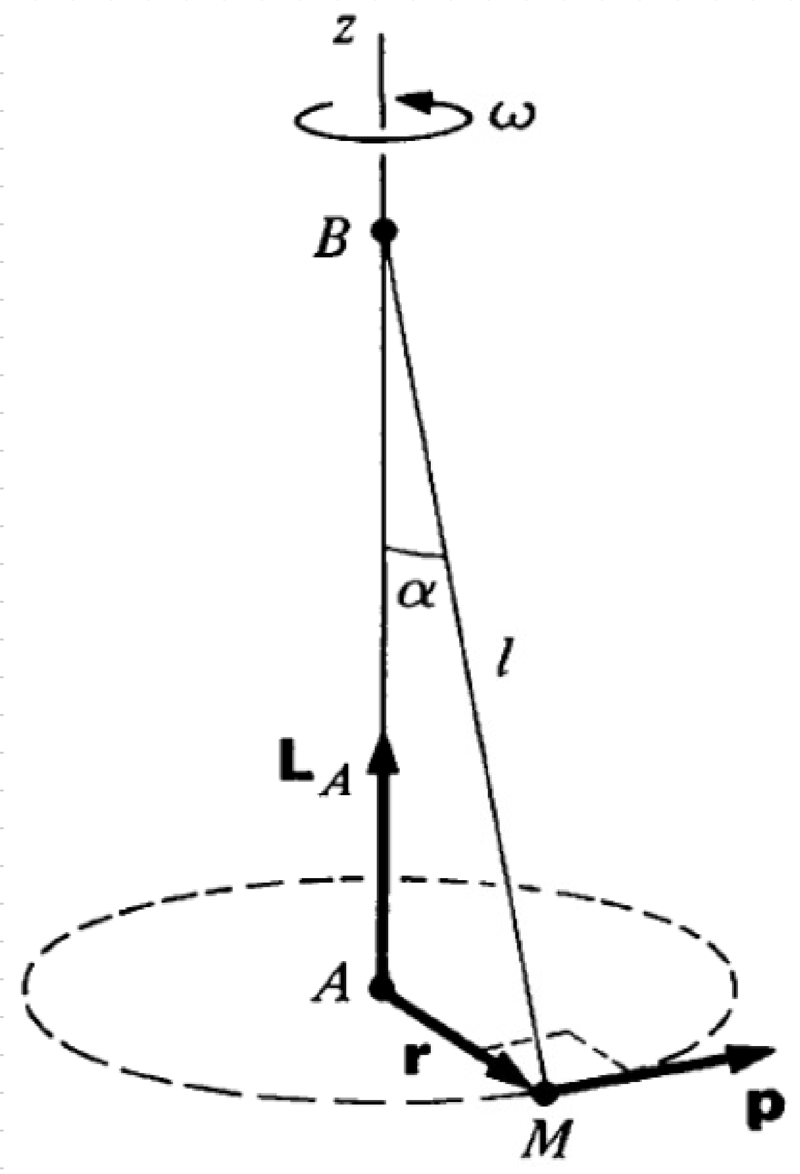
$$\Delta \vec{L}_B = m l \Delta v \hat{k}$$

$$\frac{\Delta \vec{L}_B}{\Delta t} = m l \frac{\Delta v}{\Delta t} \hat{k} \Rightarrow \frac{d\vec{L}_B}{dt} = m l \frac{dv}{dt} \hat{k}$$

external force is $\vec{f}$

$$m\frac{dv}{dt} = -f \Rightarrow \frac{dL_B}{dt} = -lf \hat{k}$$ As Before!

Angular Momentum of the Conical Pendulum



No gravity !

Angular momentum about A: $\vec{L}_A$

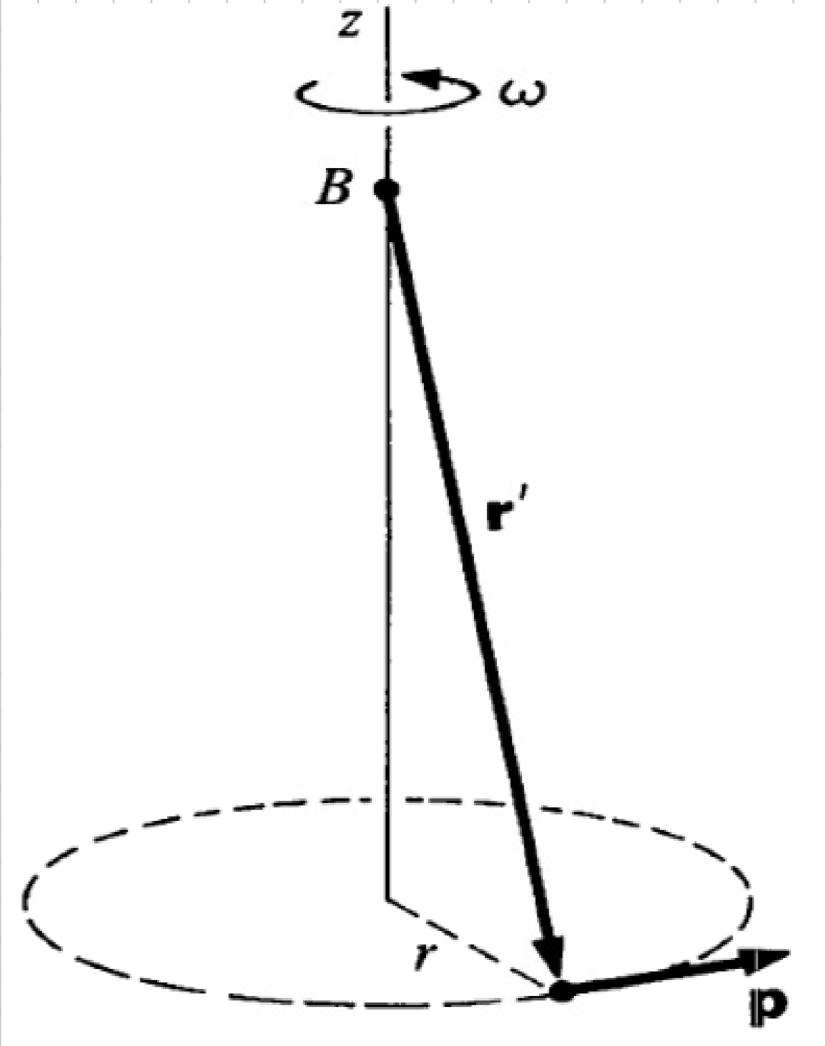
$$\vec{p} = m\vec{v}$$

$$|\vec{p}| = mv = mr\omega$$

$$\vec{L}_A = mr\omega^2 \hat{k}$$

! Constant in magnitude & direction

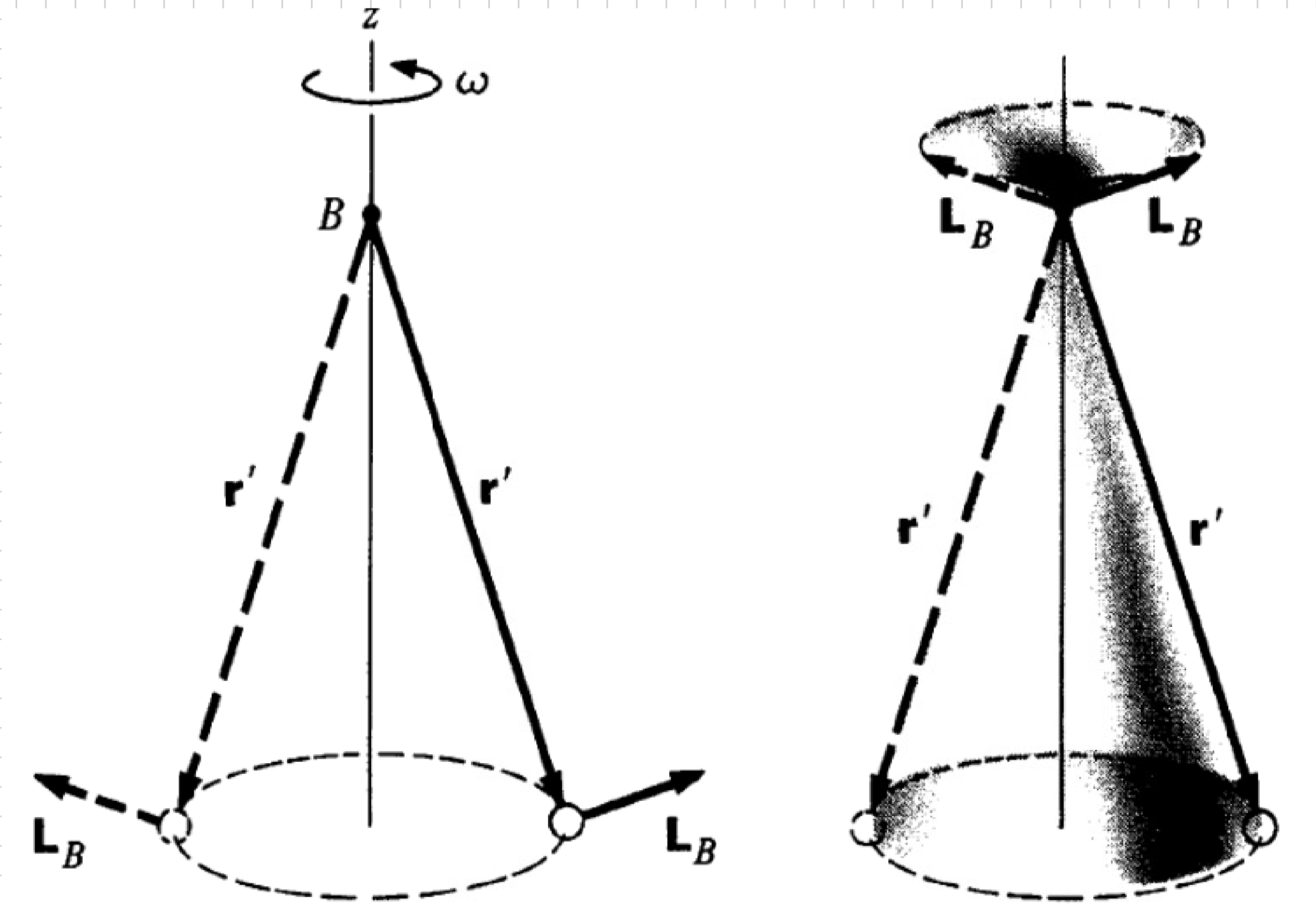
Angular momentum about point B



$$|\vec{L}_B| = |\vec{r}\,' \times \vec{p}|$$
$$= |\vec{r}\,'|\,|\vec{p}|$$
$$= l\,|\vec{p}|$$
$$= M l r \omega$$

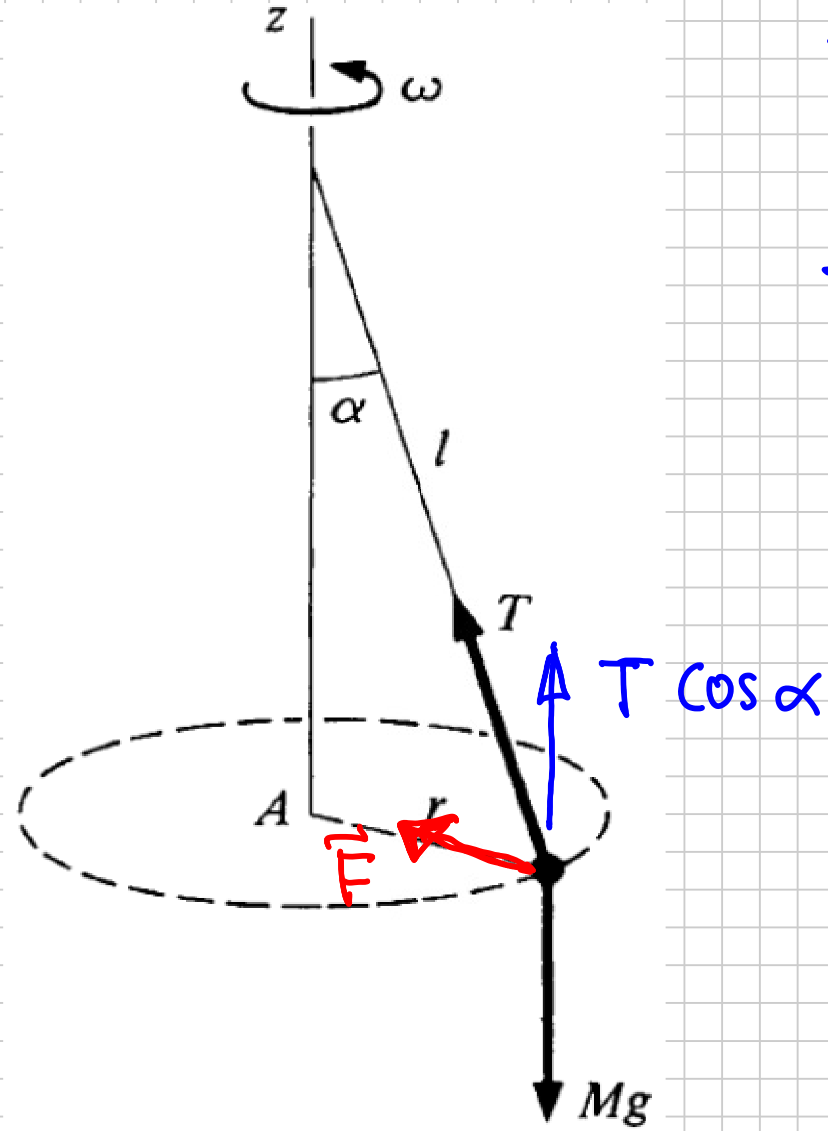
Direction ⊥ to both $\vec{r}\,'$ and $\vec{p}$.

Constant in magnitude but DIRECTION changes always!



$\vec{L}_B$ sweeps out the shaded cone !

Switch Gravity! $\Rightarrow$ Torque!



$$T\cos\alpha - mg = 0$$

No net force along z direction

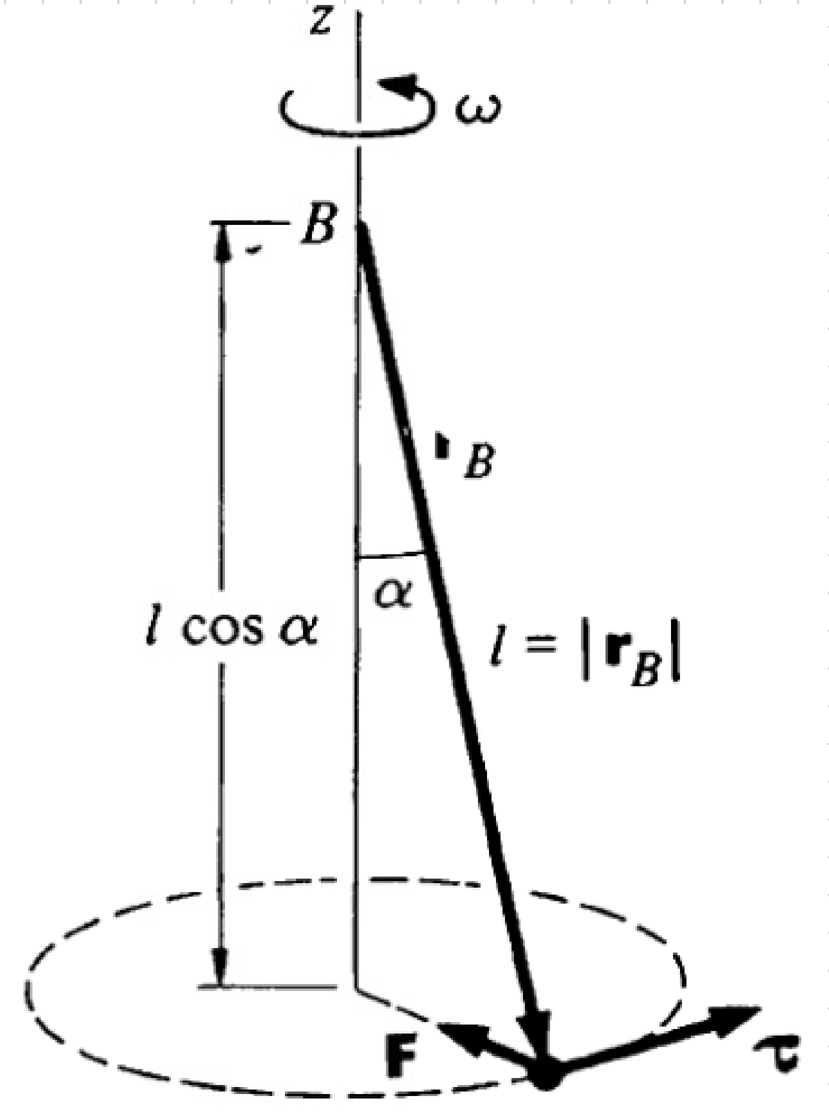
Total net force along $-\hat{r}$

$$\vec{F} = -T\sin\alpha\,\hat{r}$$

Torque $\vec{\tau} = \vec{r}_A \times \vec{F} = 0$

$$\frac{dL_A}{dt} = 0 \Rightarrow L_A = \text{constant}$$

Switch origin at B!



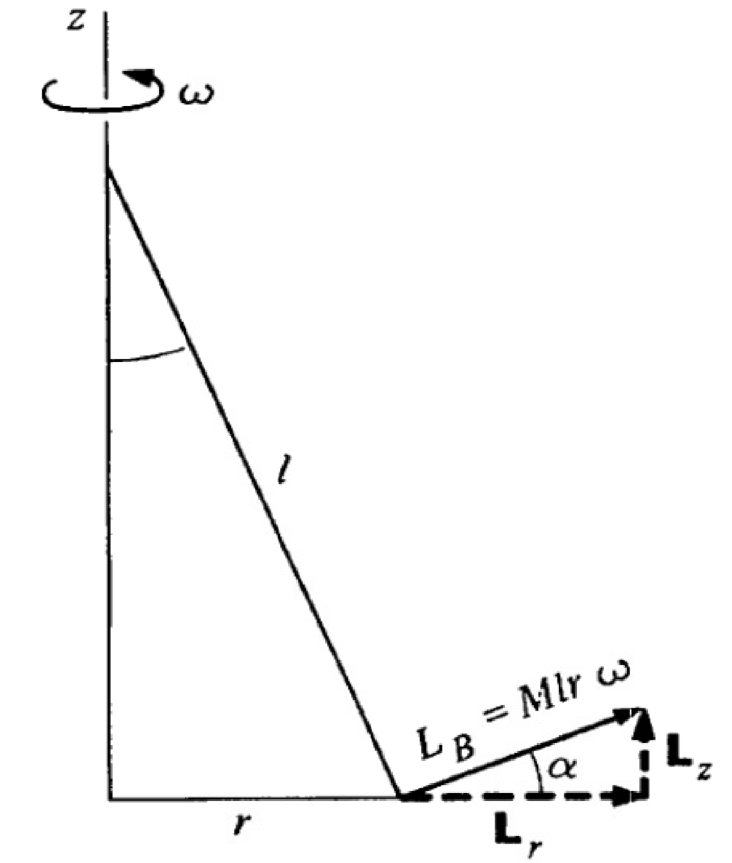
Torque $\tau_B = \vec{r}_B \times \vec{F}$

$$|\tau_B| = l \cos\alpha \, F$$
$$= l \cos\alpha \, T \sin\alpha$$
$$= Mgl \sin\alpha$$

$$\vec{\tau}_B = Mgl \sin\alpha \, \hat{\theta}$$

We need to show

$$\vec{\tau}_B = \frac{dL_B}{dt}$$



$$|\vec{L}_B| = M l r \omega$$

Component wise:

$$L_z = M l r \omega \sin\alpha$$

$$L_r = M l r \omega \cos\alpha$$

$$\vec{L}_B = \vec{L}_z + \vec{L}_r$$

L_z = constant !

L_r = constant, changes its direction

$$\vec{L}_B = L_z \hat{k} + L_r \hat{r}$$

$$\frac{d\vec{L}_B}{dt} = M l r \omega \cos\alpha \frac{d\hat{r}}{dt} = M l r \omega^2 \cos\alpha \; \hat{\theta} \qquad \frac{d\hat{r}}{dt} = \omega \hat{\theta}$$

$Mr\omega^2 \rightarrow$ radial force $= T \sin\alpha$

$T \cos\alpha = Mg$

$$\frac{dL_B}{dt} = Mgl \sin\alpha \, \hat{\theta}$$

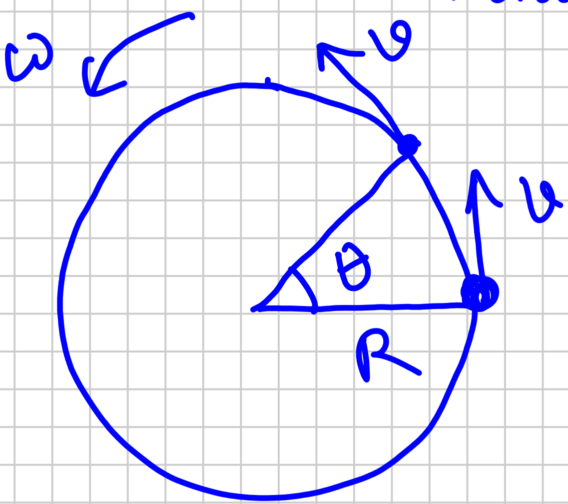
$$= \tau_B \quad \text{[Proved]}$$

Angular Momentum and Fixed Axis Rotation

Rigid body motion $\Rightarrow$ Free rotation about any axis!

[Next chapter]

Rotation about fixed axis!



$$v = \omega R = \dot{\theta} R$$

tangential accl^n

$$a_{tan} = \dot{\omega} R = \ddot{\theta} R = \alpha R$$

$\alpha \rightarrow$ angular accl^n

rotating object

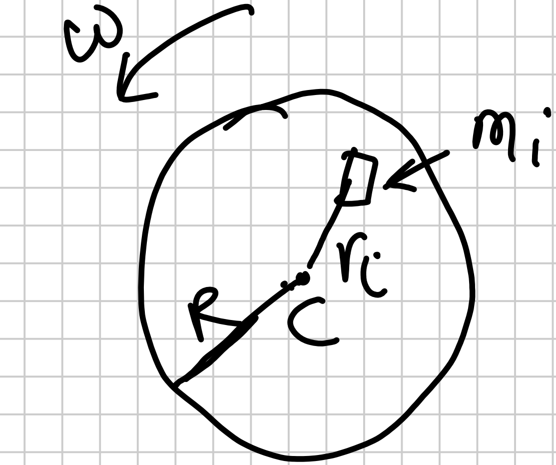
$x \rightarrow \theta$, $v \rightarrow \omega$, $a \rightarrow \alpha$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \Rightarrow \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$v = v_0 + at \Rightarrow \omega = \omega_0 + \alpha t$$

Rotational Kinetic Energy!

Disk



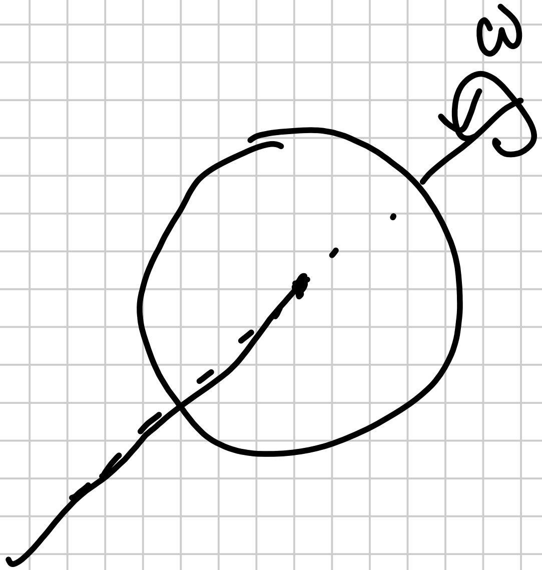
$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \omega^2 r_i^2$$

$$K_{Disk} = \frac{\omega^2}{2} \underbrace{\sum m_i r_i^2}_{\text{moment of inertia}} = \frac{1}{2} \omega^2 I_C$$

$$\downarrow$$

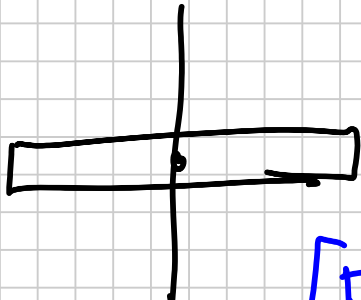
Depends on object & axis of rotation

$m \to I$



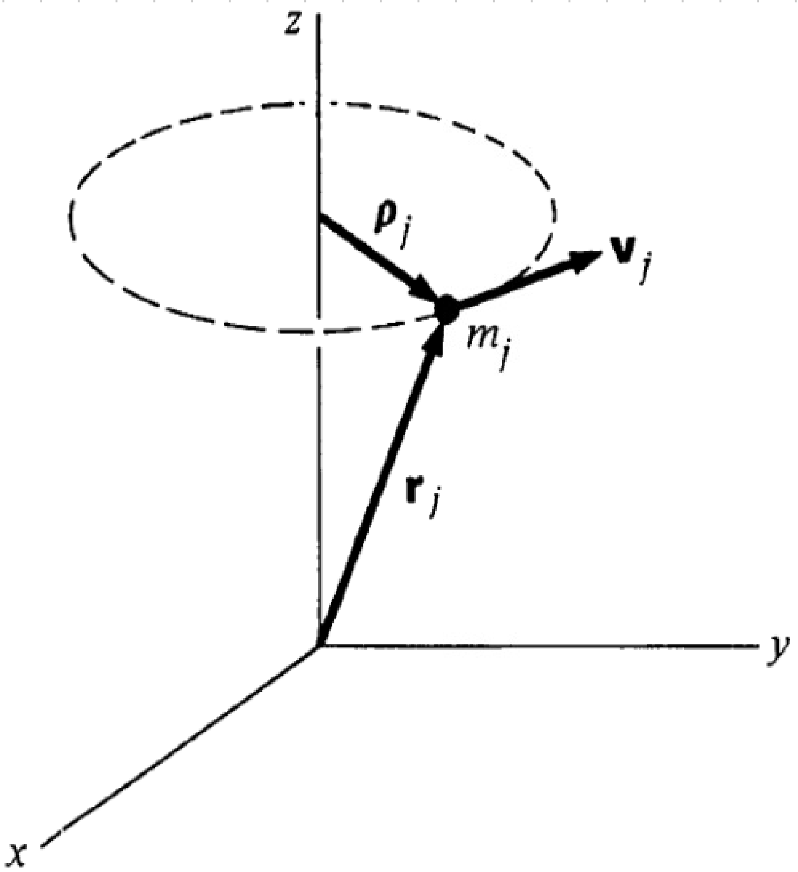
$$I_C = \frac{1}{2} m R^2$$

Rod



$$I = \frac{1}{12} m l^2$$

[Find in the table]



Angular momentum

$$\vec{L}(j) = \vec{r}_j \times m_j \vec{v}_j$$

$$L_z(j) = \rho_j m_j v_j$$

$$v_j = \omega \rho_j$$

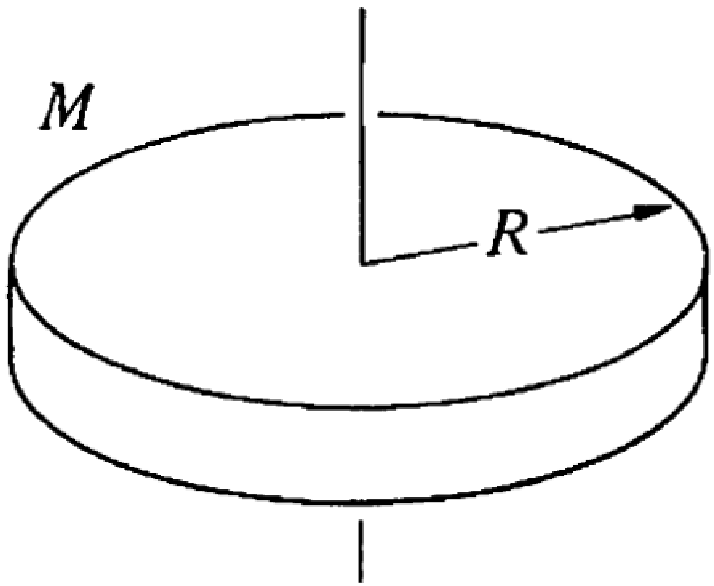
$$L_z(j) = m_j \rho_j^2 \omega$$

$$L_z = \sum_j L_z(j) = \sum_j m_j \rho_j^2 \omega = I\omega$$

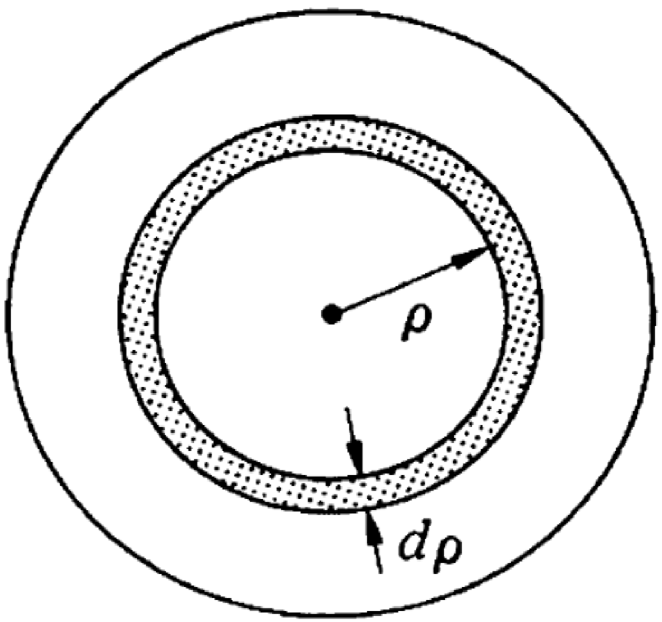
where, $I = \sum_j m_j \rho_j^2 \equiv \int \rho^2 \, dm$

Uniform Disk

$$I = \int dI$$



$$dA = 2\pi\rho\, d\rho$$



$$dm = M\frac{dA}{A} = \frac{M 2\pi\rho\, d\rho}{\pi R^2}$$

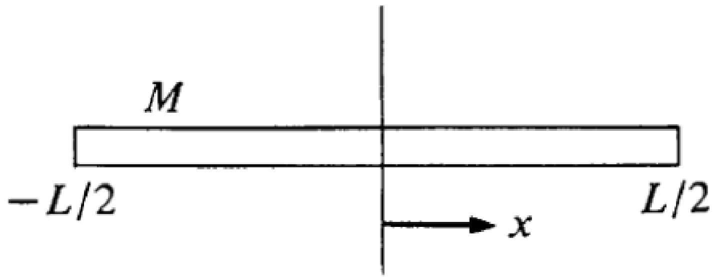
$$= \frac{2M\rho\, d\rho}{R^2}.$$

$$dI = \rho^2\, dm = \frac{2M\rho^3\, d\rho}{R^2}$$

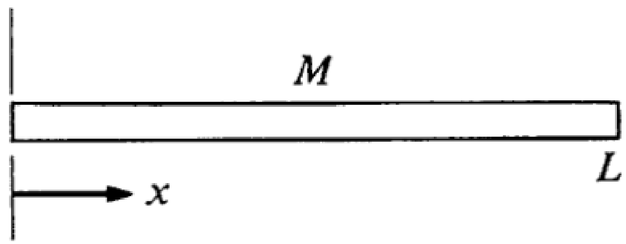
$$I = \int_0^R \frac{2M\rho^3\, d\rho}{R^2}$$

$$= \frac{1}{2} MR^2.$$

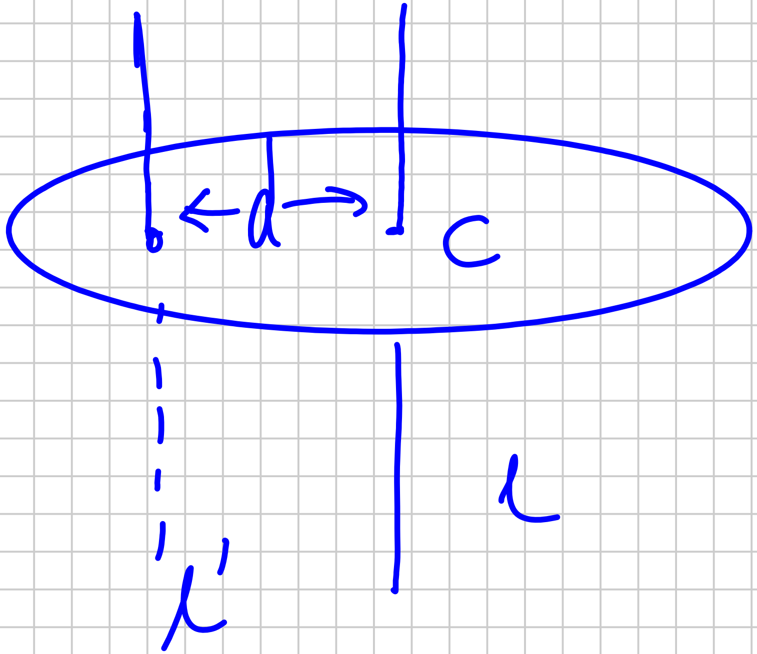
Uniform Thin Stick:



$$
\begin{aligned}
I &= \int_{-L/2}^{+L/2} x^2\, dm \\
&= \frac{M}{L} \int_{-L/2}^{+L/2} x^2\, dx \\
&= \frac{M}{L} \frac{1}{3} x^3 \Big|_{-L/2}^{+L/2} \\
&= \tfrac{1}{12} M L^2
\end{aligned}
$$



$$
\begin{aligned}
I &= \frac{M}{L} \int_0^L x^2\, dx \\
&= \tfrac{1}{3} M L^2 .
\end{aligned}
$$

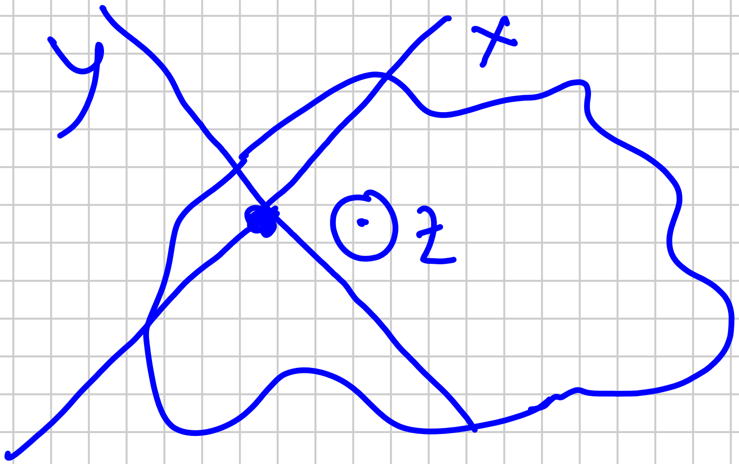


Parallel axis theorem!

$l \parallel l'$

$$I_{l'} = I_l + md^2 > I_l$$

Perpendicular axis Theorem!



$$I_z = I_x + I_y$$

Newton's Law $\rightarrow$ Force [internal & external]

Internal forces - cancel

$\rightarrow$ Conservation of momentum

Rotational Motion $\rightarrow$ Torque [internal torque add to zero!]

Only external torque changes to angular momentum!

Consider a body rotating with ω about z-axis

$L_z = I\omega \rightarrow z$ component

$\vec{\tau} = \frac{d\vec{L}}{dt}$ external torque

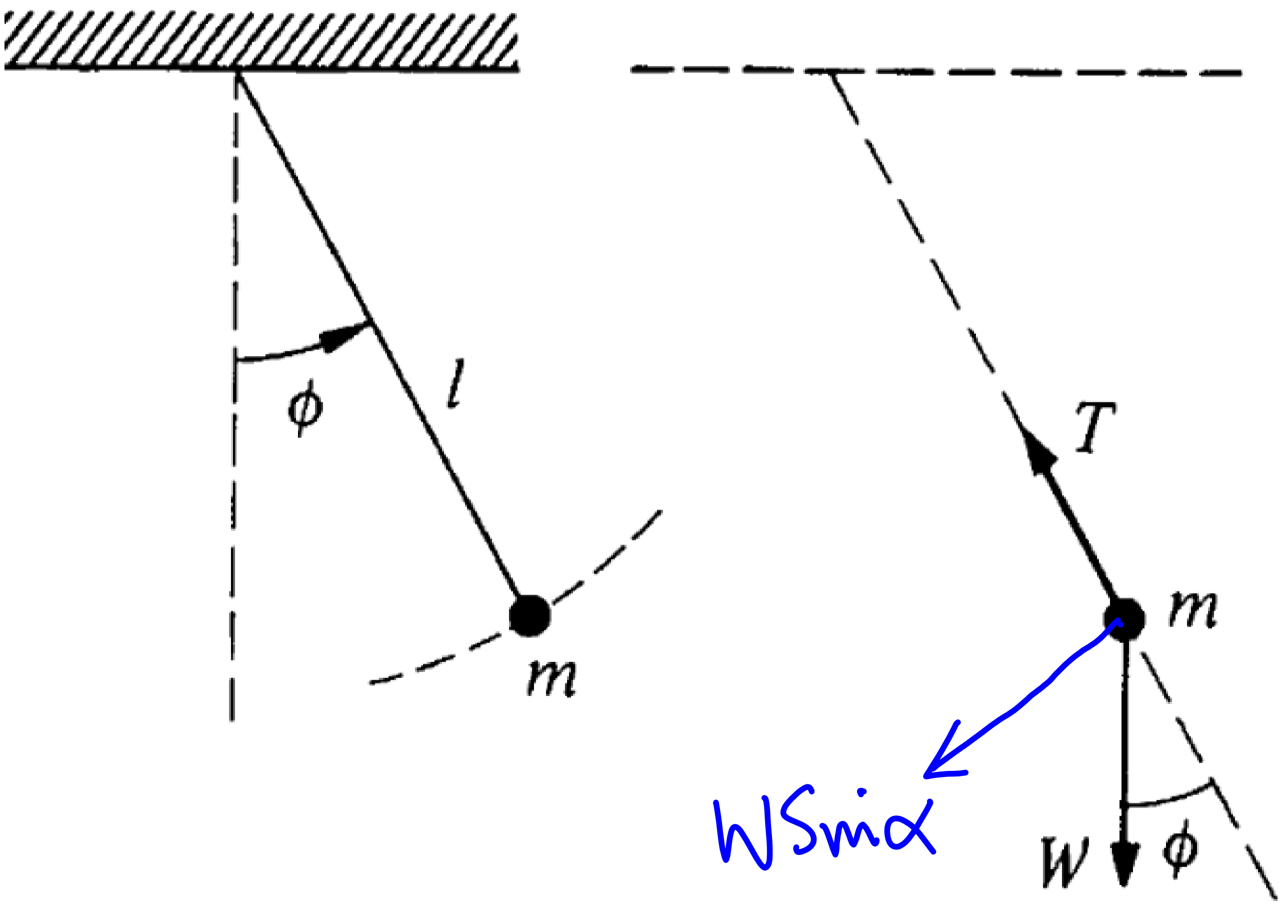
$\tau_z = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha$

$\alpha \rightarrow$ angular accleration!

Angular kinetic energy

$K_i = \frac{1}{2} I \omega^2$

Simple Pendulum:



$$I = ml^2$$

$$\alpha = \ddot{\phi}$$

$$\tau = -Wl\sin\phi = I\alpha$$

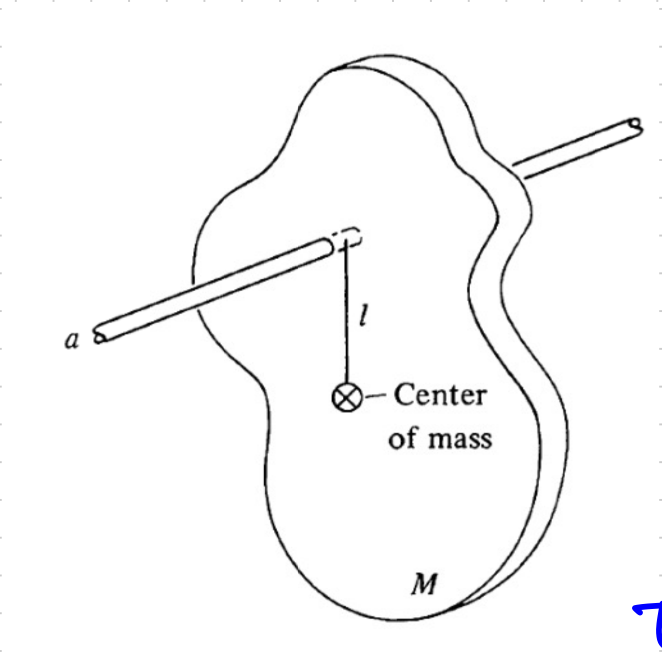
$$ml^2\ddot{\phi} = -Wl\sin\phi$$

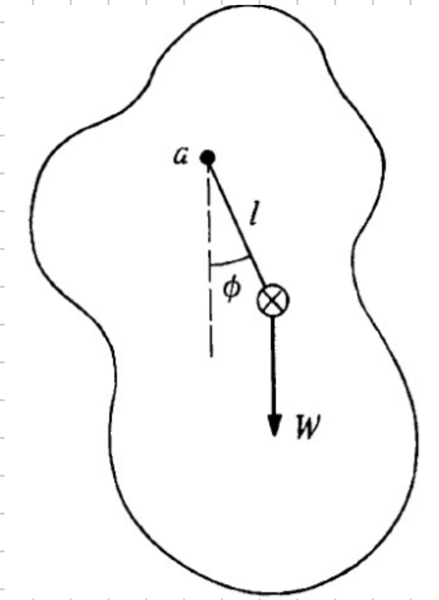
$$= -mgl\sin\phi$$

$$\ddot{\phi} + \frac{g}{l}\phi = 0$$

$$\omega = \sqrt{g/l} \qquad T = 2\pi\sqrt{l/g}$$

Physical Pendulum :





$$\tau = I\alpha$$

$$-lW\sin\phi = I_a\ddot{\phi}$$

$$\Rightarrow \quad I_a\ddot{\phi} + Mgl\,\phi = 0$$

$$\omega = \sqrt{Mlg/I_a}$$