1 The relation for total energy (E) and momentum (p) for a relativistic particle is $E^2 = c^2p^2 + m^2c^4$, where m is the rest mass and c is the velocity of light. Using the relativistic relations $E = \hbar\omega$ and $p = \hbar k$, where ω is the angular frequency and k is the wave number, show that the product of group velocity (v_g) and the phase velocity (v_p) is equal to c^2, that is $v_p v_g = c^2$

2 Given $\psi(x) = \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4}} \exp\left(-\frac{\alpha^2 x^2}{2}\right)$, calculate $\text{Var}\, x$

3 If $\psi(x) = \frac{N}{x^2+a^2}$, calculate the normalization constant N.

4 The state of a free particle is described by the following wave function

$$\psi(x) = 0 \text{ for } x < -3a$$
$$= c \text{ for } -3a < x < a$$
$$= 0 \text{ for } x > a$$

(a) Determine c using the normalization condition

(b) Find the probability of finding the particle in the interval $[0, a]$

5 In the above problem,

(a) Compute $< x >$ and σ^2

(b) Calculate the momentum probability density.

6 Assuming that the radial wave function

$$U(r) = r\psi(r) = C\exp(-kr)$$

is valid for the deuteron from $r = 0$ to $r = \infty$ find the normalization constant C.

Hence if $k = 0.232\ \text{fm}^{-1}$ find the probability that the neutron – proton separation in the deuteron exceeds 2 fm. Find also the average distance of interaction for this wave function.

2.6 $E^2 = c^2p^2 + m^2c^4$

$$\hbar^2\omega^2 = c^2\hbar^2k^2 + m^2c^4$$

$$\omega = \left(c^2k^2 + \frac{m^2c^4}{\hbar^2}\right)^{1/2}$$

$$v_p = \frac{\omega}{k} = \left(c^2k^2 + \frac{m^2c^4}{\hbar^2}\right)^{1/2} / k$$

$$v_g = \frac{d\omega}{dk} = kc^2\left(c^2k^2 + \frac{m^2c^4}{\hbar^2}\right)^{-1/2}$$

$$\therefore v_p v_g = c^2$$

3.2 $\psi(x) = (\pi/\alpha)^{-1/4} \exp\left(-\frac{\alpha^2}{2}x^2\right)$

Var $x = < x^2 > - < x >^2$

The expectation value

$$< x > = \int_{-\infty}^{\infty} \psi^* x \, \psi \, dx = 0$$

because ψ and also ψ^* are even functions while x is an odd function. Therefore the integrand is an odd function

$$< x^2 > = \left(\frac{\pi}{\alpha}\right)^{-1/2} \int_{-\infty}^{\infty} x^2 \exp(-\alpha^2 x^2) dx$$

Put $\alpha^2 x^2 = y$; $dx = {}^1/_2 \, \alpha \sqrt{y}$

$$< x^2 > = (\pi\alpha^5)^{-1/2} \int_0^{\infty} y^{1/2} e^{-y} \, dy$$

But $\int_0^{\infty} y^{1/2} e^{-y} dy = \Gamma(3/2) = \sqrt{\pi}/2$

Var $x = < x^2 > = (4\,\alpha^5)^{-1/2}$

3.3 Normalization condition is

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$N^2 \int_{-\infty}^{\infty} (x^2 + a^2)^{-2} dx = 1$$

Put $x = a \tan\theta$; $dx = \sec^2\theta \, d\theta$

$$\left(\frac{2N^2}{\alpha^3}\right) \int_0^{\pi/2} \cos^2\theta \, d\theta = N^2\pi/2a^3 = 1$$

Therefore $N = \left(\frac{2a^3}{\pi}\right)^{1/2}$

3.10 (a) The normalization condition requires

$$\int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}x = \int_{-3a}^{a} |c|^2 \mathrm{d}x = 1 = 4a|c|^2$$

Therefore $c = 1/2\sqrt{a}$

(b) $\int_0^a |\psi|^2 \mathrm{d}x = \int_0^\alpha c^2\, \mathrm{d}x = 1/4$

3.11 (a) The expectation values are

$$< x >= \int_{-\infty}^{\infty} \psi^* \, x \, \psi \, \mathrm{d}x = \int_{-3a}^{a} x \frac{\mathrm{d}x}{4a} = -a$$

$$< x^2 >= \int_{-\infty}^{\infty} \psi^* x^2 \, \psi \, \mathrm{d}x = \int_{-3a}^{a} (1/4a)\, x^2 \mathrm{d}x = \left(\frac{7}{3}\right) a^2$$

$$x\sigma^2 =< x^2 > - < x >^2 = \left(\frac{7}{3}\right) a^2 - (-a)^2 = \frac{4}{3} a^2$$

(b) Momentum probability density is $|\varphi(p)|^2$

$$\varphi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}x \, \psi\,(x) e^{-ipx/\hbar}$$

$$= (2\pi\hbar)^{-1/2} \int_{-3a}^{a} \mathrm{d}x c e^{-ipx/\hbar}$$

$$= \left(\frac{ic}{p}\right)\left(\frac{\hbar}{2\pi}\right)^{1/2} \left[e^{-\frac{ipa}{\hbar}} - e^{\frac{3ipa}{\hbar}} \right]$$

$$= \left(-\frac{ic}{p}\right)\left(\frac{\hbar}{2\pi}\right)^{1/2} e^{ipa/\hbar} \left[e^{\frac{2ipa}{\hbar}} - e^{\frac{-2ipa}{\hbar}} \right]$$

$$= \left(\frac{2c}{p}\right)\left(\frac{\hbar}{2\pi}\right)^{\frac{1}{2}} e^{\frac{ipa}{\hbar}} \sin\left(\frac{2pa}{\hbar}\right)$$

Therefore $|\varphi(p)|^2 = \frac{\hbar}{2\pi\, a p^2} \sin^2\left(\frac{2pa}{\hbar}\right)$

3.21 $u = C\,e^{-kr}$

$\int_0^\infty |u|^2 \mathrm{d}r = c^2 \int_0^\infty e^{-2kr} = \frac{c^2}{2k} = 1$

$C = \sqrt{2k}$

The probability that the neutron – proton separation in the deuteron exceeds R is

$$P = \int_R^\infty |u|^2 \mathrm{d}r = 2k \int_R^\infty e^{-2kr} \mathrm{d}r$$

$$= e^{-2kR} = e^{-(2\times 0.232\times 2)} \approx 0.4$$

Average distance of interaction

$$< r >= \int_0^\infty r|u^2|\mathrm{d}r = 2k \int_0^\infty r e^{-2kr} \mathrm{d}r$$

$$= \frac{1}{2k} = \frac{1}{2\times 0.232} = 2.16 \text{ fm}$$