Voronoi diagrams

Higher-order

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Given a set $S$ of $n$ sites, preprocess them to output $k$ closest sites among $S$ to any input query point.

Order among those $k$ closest sites does not matter.
Voronoi diagram of order $k$

- The locus of points $p$ for which each site in $T \subset S$ is closer to $p$ than to any site in $S - T$ is termed as the generalized Voronoi polygon w.r.t. set $T$, denoted with $VP|T|(T, S)$.

- Every such Voronoi polygon is convex.

- The collection of generalized Voronoi polygons of all $k$-subsets of $S$ is termed as the $k^{th}$-order Voronoi diagram, denoted with $VD_k(S)$ (for $1 \leq k < n$).

$$VD_1(S), VD_{n-1}(S)$$ are known respectively as the closest-point and farthest-point Voronoi diagrams.

(Higher-order Voronoi diagrams)
Life cycle of a Voronoi vertex

(c) close-type vertex in $VD_{k+1}(S)$

(d) far-type vertex in $VD_{k+2}(S)$

Every Voronoi vertex appears in diagrams of two successive orders:

if the centre of a circle $v$ is a close-type Voronoi vertex in $VD_{k+1}(S)$ then $v$ is a far-type Voronoi vertex in $VD_{k+2}(S)$. 
Combinatorial complexity

- Number of Voronoi vertices, edges and faces of all orders together $O(n^3)$.

- Number of cells and close-type vertices in $Vor_k(S)$ are $O(k(n - k))$ and $O(kn)$ respectively. – not proved in class; read the lecture note though
Let $S'$ be the set of points in $\mathbb{R}^3$ that are caused by projecting points in $xy$-plane onto paraboloid. Also, let $A(L)$ be the arrangement of planes that are caused by tangents to paraboloid at points in $S'$.

The points of intersection of $(k - 1)^{th}$- and $k^{th}$- levels in $A(L)$ define $VD_k(S)$. 

(Higher-order Voronoi diagrams)
Faces of an arrangement and Voronoi polygons

- Projection of every face $f$ in $A(L)$ to $xy$-plane defines a Voronoi polygon $P_f$ of some order.

- Sites that correspond to $P_f$ are determined by sites that correspond to half-spaces defined by bounding faces of $f$ that do not contain the paraboloid.
Incremental Algorithm to construct VDs of all orders

Construct arrangement in $R^3$ and analyze it using Zone Theorem.

Takes optimal $\Theta(n^3)$ time and $\Theta(n)$ space.
Iterative Algorithm to compute $VD_k(S)$

$v$ is close-type and far-type vertex resp. in $VD_k(S)$ and $VD_{k+1}(S)$

$VD_{k+1}(S)$ is obtained by intersecting each Voronoi polygon $VP_k(T, S)$ of $VD_k(S)$ with $VD_1(S - T)$.

compute $VD$ corresp. to vertices in $S - T$ that affect $VP_k(T, S)$

Takes $O(k^2n \lg n)$ time using $O(k^2(n - k))$ space.