Graph Theory

Introductory terminology

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Outline

1. Few basics
2. Covers etc.,
3. Matching
4. Connectivity
5. Shortest paths
6. Tours
7. Planar
8. Network flows
9. Coloring
10. Trees and forests
11. Tree decomposition
12. Miscellaneous

(Graphs: terminology)
Undirected graph

- vertex, edge
- adjacent vertices, or neighbors
- incidence
- vertex degree
- isolated vertex
- pendant vertex

- simple graph
- finite/infinite graph

(Graphs: terminology)
Multigraph

- **multigraph** has multiple edges between sets of two vertices
- **pseudograph** can have both self-loops and multiple-edges
• weights are associated with edges
Directed graph

- orientation of edges
- underlying undirected graph
- directed acyclic graph (DAG)

The transpose (converse) of a directed graph $G(V, E)$ is the directed graph $G(V, F)$ where the set of edges in $F$ are obtained by reversing the direction of each edge in $E$.

- Orienting every edge of a complete undirected graph results in a tournament.

(Graphs: terminology)
Consider two graphs \( G(V, E) \) and \( G'(V', E') \). If \( V' \subseteq V \) and \( E' \subseteq E \) then \( G' \) is said to be a subgraph of \( G \), or \( G \) as a supergraph of \( G' \).

A subgraph of \( G \) that has all the vertices of \( G \) is termed as a spanning subgraph of \( G \).
Consider two graphs $G(V, E)$ and $G'(V', E')$. If $G'$ is a subgraph of $G$ and for every edge $(v_i, v_j) \in E$ with $v_i, v_j \in V'$, $(v_i, v_j) \in E'$ then $G'$ is said to be *induced* by vertex set $V'$, and $G'$ is denoted with $G[V']$.
Induced subgraph

- Consider two graphs \( G(V, E) \) and \( G'(V', E') \). If \( G' \) is a subgraph of \( G \) and for every edge \( (v_i, v_j) \in E \) with \( v_i, v_j \in V' \), \( (v_i, v_j) \in E' \) then \( G' \) is said to be *induced* by vertex set \( V' \), and \( G' \) is denoted with \( G[V'] \).

- Let \( C \) be a simple cycle of graph \( G \). Then the *induced cycle* of \( C \) in \( G \) is an induced subgraph of vertices that belong to \( C \).
• Let $e = (u, v)$ be an edge of a graph $G = (V, E)$. By $G/e$ we denote the graph obtained from $G$ by contracting the edge $e$ into a new vertex $v_e$ which becomes adjacent to all the former neighbors of vertices $u$ and $v$.

• Similarly, two arbitrary vertices $v', v''$ of a graph $G$ can be contracted (essentially, merged) into one $v$ so that for any edge $e(v', v_i)$ in $G$, the corresponding edge has $v$ and $v_i$ as its vertices.

(Graphs: terminology)
Graph minors

• A graph $H$ is a *minor* of a graph $G$, denoted with $H \leq G$, if $H$ can be obtained from $G$ by deleting edges and/or vertices and/or by contracting edges of $G$.

• A family $\mathcal{G}$ of graphs is called minor closed, if for all $G \in \mathcal{G}$ and all $H \leq G$, $H \in \mathcal{G}$. Ex. forests

• For any graph $G$, if it is known that $G$ does not belong to $\mathcal{G}$ whenever $G$ has a graph from $\mathcal{G}'$ as a minor then $\mathcal{G}'$ is an *obstruction set* of family $\mathcal{G}$.  

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1The celebrated *Robertson & Seymour theorem*: Every minor closed graph family has a finite obstruction set.

(Graphs: terminology)
Complete graph

- Every vertex is connected to every other vertex with an edge; denoted with $K_{{|V|}}$. 

(Graphs: terminology)
A clique in a graph $G(V, E)$ is a subset $V' \subseteq V$ of vertices such that for every two vertices in $V'$ there exists an edge joining them.

The size of the maximum sized clique is known as the clique number, $\omega(G)$, of $G$.

The clique problem finds a clique of maximum size. – NP-complete
Regular graph

- If every vertex of a graph $G$ has the same degree, then $G$ is termed as a regular graph.

- A $k$-regular spanning subgraph of a graph $G$ is called a $k$-factor of $G$. 
$k$-partite graph

- A graph $G(V, E)$ is called $k$-partite if $V$ admits a partition into $k$ classes such that every edge has its ends in different classes.
- Bipartite when $k = 2$; typically, denoted with $K_{|V_1|, |V_2|}$.
- Bipartite complete graph is termed as a biclique.
• A *decomposition* of a graph $G$ is a list of subgraphs such that the edges of $G$ are partitioned across these subgraphs.
The union of two simple graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted with $G_1 \cup G_2$, is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. 
Few more definitions

- The *complementary graph* \( \overline{G} \) of a simple graph \( G \) has the same set of vertices as \( G \). However, two vertices are adjacent in \( \overline{G} \) if and only if they are not adjacent in \( G \).

- The *line graph* \( L(G)(V', E') \) of a graph \( G(V, E) \) is the graph whose vertices are the edges of \( G \), with \( (e, f) \in E' \) whenever \( e \) and \( f \) incident to the same vertex.
Isomorphic graphs

Two simple graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are isomorphic if there is a one-to-one and onto function from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ iff $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$. Such a function $f$ is called an isomorphism. – NP, but yet to be determined whether it belongs to $P$ or NP-complete.

An automorphism of $G$ is an isomorphism from $G$ to $G$.

A graph is self-complementary if it is isomorphic to its complement.

$\begin{align*}
f(u_1) &= v_6, f(u_2) = v_3, f(u_3) = v_4, f(u_4) = v_5, f(u_5) = v_1, f(u_6) = v_2
\end{align*}$
Subgraph isomorphism

- Given two graphs $G_1$ and $G_2$ and finding whether $G_1$ is isomorphic to a subgraph of $G_2$ is known as the \textit{subgraph-isomorphism problem}. – NP-complete

- A graph $G$ is said to be \textit{H-free} if $G$ has no induced subgraph isomorphic to $H$. 

(Graphs: terminology)
Popular graphs

triangle

cycle (c₅)

wheel (W₆)

hypercube (Q₃)

paw

bowtie

(Graphs: terminology)
Popular graphs (cont): The Petersen graph

- The *Petersen graph* is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.
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(Graphs: terminology)
Vertex cover

- A vertex cover of an undirected graph $G(V, E)$ is a subset $V' \subseteq V$ such that if $(u, v) \in E$ then at least one of $u$ and $v$ belongs to $V'$. In other words, vertices in $V'$ cover $E$.

- The vertex cover problem is to find a vertex cover of minimum size in a given graph. – NP-complete
• An edge cover of an undirected graph $G(V, E)$ is a subset $E' \subseteq E$ such that if $v \in V$ then at least one edge that has $v$ as an endpoint belongs to $E'$. In other words, vertices in $E'$ cover $V$.

• The edge cover problem is to find an edge cover of minimum size in a given graph. – polynomial time using matching algorithms
A dominating set of a graph $G(V, E)$ is a subset $V' \subseteq V$ of vertices such that for every vertex not in $V'$ has a neighbor in $V'$.

The dominating set problem is to find a minimum-size dominating set in $G$. – NP-complete
An independent set of a graph $G(V, E)$ is a subset $V' \subseteq V$ of vertices such that each edge in $E$ is incident on at most one vertex in $V'$.

The independent set problem is to find a maximum-size independent set in $G$. – NP-complete

That maximum-size is known as the independence number $\alpha(G)$ of $G$. 

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(Graphs: terminology)
Maximum matching (a.k.a. independent edge set)

- Given a graph $G(V, E)$, a matching $M$ in $G$ is a set of pairwise non-adjacent edges. The problem of finding a matching $M$ that is of maximum cardinality (in case of weighted graph, it maximizes the total weight of edges in $M$) is termed as the maximum matching problem.

- If all vertices are covered with the edge set in $M$, then $M$ is a perfect matching.
If a matching $M$ is not extendable in size then we say $M$ is a maximal matching. – polynomial time

Maximal matching with the minimum cardinality is termed as the minimum maximal matching. – NP-complete
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*(Graphs: terminology)*
• A walk is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$. The walk is known as closed if $v_0 = v_k$; otherwise, it is open.

• A trail is a walk with no repeated edge. The trail is known as closed if its endpoints are same; otherwise, it is open.

• An open walk is known as a path. An open trail with no vertex repeated is known as simple path.

• A closed walk is known as a cycle. A closed trail with no vertex (except the first/last) repeated is known as a simple cycle.

The length of a walk/trail/(simple) path/(simple) cycle is the number of edges in it.

(Graphs: terminology)
Vertex-disjoint paths and edge-disjoint paths

• Two paths are (internally) *vertex-disjoint* if they do not have any internal vertex in common.

• Two paths are *edge-disjoint* if they do not have any edge in common.

A set of paths that are vertex-disjoint are obviously edge-disjoint. However, the converse is not true.
Maximum path vs maximal path

- A simple path whose length is maximum in $G$ is termed as a *maximum path* (a.k.a. *longest path*).

- If no larger simple path contains a simple path $P$, then $P$ is termed as a *maximal path*.

Every maximum path is a maximal path. However, not every maximal path is a maximum path.
• The *length of a walk, trail, path, or a cycle* is its number of edges.

• The *girth* of a graph with a cycle is the length of its shortest simple cycle. A graph with no cycle has infinite girth. – $O(VE)$

• The *circumference* of a graph is the length of a simple cycle that has the maximum length among all the simple cycles. – NP-complete
A maximal connected subgraph of $G$ is called a component of $G$. – polynomial time

A cut-vertex (or, articular point) (resp. cut-edge (or, bridge)) of a graph is a vertex (resp. edge) whose deletion increases the number of components. – polynomial time

(Graphs: terminology)
A biconnected component (a.k.a. block) is a maximal biconnected subgraph. – $O(|V| + |E|)$ time

Another common way of characterizing the same: a biconnected component is a maximal subgraph of an undirected graph such that any two edges in the subgraph lie on a common simple cycle.
A *strongly connected component* is a maximal subgraph of a directed graph such that for every pair of vertices $u, v$ in the subgraph, there is a directed path from $u$ to $v$ and a directed path from $v$ to $u$.  \(-O(V + E)\)

A *strongly connected digraph* is a directed graph in which it is possible to reach any node starting from any other node.
• Let $G$ be the underlying undirected graph of digraph $D$. The connected components of $G$ are termed \textit{weakly connected components} of $D$. And, $D$ is said to be \textit{weakly connected} whenever $G$ has one connected component.
(Node-)connectivity

- The (node-)connectivity of $G$, denoted with $\kappa(G)$, is the minimum number of nodes that need to be removed from $G$ so that the leftover graph is either disconnected or has only vertex.
- A graph is $k$-(node-)connected if its (node-)connectivity is at least $k$.

$\kappa(K_n) = n - 1$; (node-)connectivity of a graph that is not complete is $\leq n - 2$
The *edge-connectivity of* $G$, denoted with $\lambda(G)$, is the minimum number of edges that need to be removed from $G$ so that the leftover graph is disconnected.

- A graph is $k$-edge-connected if its edge-connectivity is at least $k$.

$$\lambda(K_n) = n - 1$$
A cut of a graph

• A cut $C(S, V - S)$ partitions the vertex set $V$ of the given graph $G(V, E)$ into two sets $S$ and $V - S$ such that $|S| \neq 0$. The set of edges crossing the cut are termed as a cut-set. The cardinality/weight of cut-set is the cut-value of $C$.

  how many cuts are possible?

• A $s$-$t$ cut $C(S, V - S)$ of a graph $G(V, E)$ is a cut of $G$ such that $s \in S$ and $t \in V - S$.

• If a set of edges $E' \subset E$ does not cross a cut $C$, then $C$ is said to respect $E'$.

• A bond is a cut-set which does not contain any other cut-set.

(Graphs: terminology)
• Given a weighted graph $G(V, E)$ and integers $k$, partition $V$ into $k$ roughly equal-sized subsets such that these subsets have approximately equal weight while the size of the edge-cut is minimum. – NP-complete
• Given an undirected or directed graph $G(V, E)$, the *feedback vertex set* of $G$ is a set of vertices whose removal leaves $G$ acyclic. The *feedback vertex set problem* finds a feedback vertex set with minimum cardinality.
  – NP-complete

• Given an undirected (resp. directed) graph $G(V, E)$, the *feedback edge (resp. arc) set* of $G$ is a set of edges (resp. arcs) whose removal leaves $G$ acyclic. The *feedback edge (resp. arc) set problem* finds a feedback edge (resp. arc) set with minimum cardinality.  – NP-complete for directed graphs;
  polynomial for undirected graphs

(Graphs: terminology)
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(Graphs: terminology)
Diameter, eccentricity, radius, and center

- If $G$ has a $u, v$-path, then the *distance from $u$ to $v$*, denoted with $d(u, v)$, is the least length of a $u, v$-path. If $G$ does not have a path from $u$ to $v$, then $d(u, v) = \infty$.

- The *eccentricity of a vertex $u$*, denoted with $\epsilon(u)$, is $\max_{v \in V(G)} d(u, v)$.

- The *diameter* of a graph $G$ is $\max_{u \in V(G)} \epsilon(u)$.

- The *radius* of a graph $G$ is $\min_{u \in V(G)} \epsilon(u)$.
  
  For any node $u$, if $\epsilon(u) = \text{rad}(G)$, then $u$ is a *central node* of $G$.
  
  We shall prove later: $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.

- The *center* of a graph $G$ is the subgraph induced by the vertices of minimum eccentricity.
• Reachability

• Single-source shortest path (SSSP)

• All-pair shortest path (APSP)

• Counting the number of (shortest) paths
  
  – all of these are solvable in polynomial time
Transitive closure and transitive reduction

- The **transitive closure** of a directed graph $G(V, E)$ is defined as the graph $G'(V, E')$ with edge $(v_i, v_j) \in E'$ whenever there is a directed path from $v_i$ to $v_j$ in $G$. – polynomial time

- The **transitive reduction** of a directed graph $G(V, E)$ is defined as a smallest (in terms of number of edges) graph $G'(V, E')$ with a directed path from any two vertices $v_i$ and $v_j$ in $G'$ whenever there is a directed path from $v_i$ to $v_j$ in $G$. – polynomial time
Longest path

- Given a weighted undirected/directed graph, finding a simple path with maximum weight. – NP-complete

- Given a weighted DAG, finding a longest path with maximum weight. – polynomial time
Widest path (a.k.a. bottleneck shortest path)

- Given a weighted undirected/directed graph, finding a path between two given vertices so that to maximize the weight of the minimum weighted in the path. – polynomial time
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(Graphs: terminology)
A simple cycle that contains every vertex is a Hamiltonian tour. – NP-complete

A graph that has a Hamiltonian cycle is said to be Hamiltonian.

A Hamiltonian tour with least weight is a travelling salesman tour. Finding the same is known as the traveling salesman problem (TSP). – NP-complete

In the Euclidean space (wherein a complete graph on vertices exists), finding a traveling salesman tour is known as an Euclidean traveling salesman tour. – NP-complete

A closed trail containing all the edges is a Eulerian tour. – $O(E)$ time

A graph that has a Eulerian tour is said to be Eulerian.

A closed walk of minimum total weight wherein every edge appears in that walk at least once is a Chinese postman tour. – polynomial time

(Graphs: terminology)
• A *drawing* of a graph is a pictorial representation of a graph. Useful for the understandability, aesthetics, etc.,

• A drawing is *planar* if no two distinct edges intersect except at their endpoints. A graph is *planar* if it admits a planar drawing.

Given a simple graph $G$, *Hopcroft & Tarjan’74* devises an algorithm to determine whether $G$ is planar and computes a planar drawing of $G$ in linear time.
Embedding

- Given a planar drawing, the (clockwise) circular order of the edges incident to each vertex is fixed. Two planar drawings are equivalent if they determine the same circular orderings of the edges incident to each vertex.

- A (planar) embedding is an equivalence class of planar drawings and is described by the (clockwise) circular order of the edges incident to each vertex. A graph together with one of its planar embedding is sometimes referred as a plane graph.
• The *dual graph* of a planar graph $G$ is defined as the set of vertices corresponding to plane regions of $G$ and an edge for each edge in $G$ joining two neighboring regions. Note that the dual graph of a planar graph is a planar pseudograph.
Benefits of planar graphs

• sparser graph  (proved later)

• four colorability  (5-colorability will be proved later)

• computing a 3-coloring of every triangle-free graph in linear time

• determining whether the graph contains a $k$-clique in polynomial time

• graph isomorphism in linear time

• planar separator theorem leading to an array of divide-and-conquer algorithms  (done later)
Crossing number

- The *crossing number* of a graph $G$ is the minimum possible number of edge crossings when all possible planar embeddings of $G$ are considered.
  - NP-complete

(Graphs: terminology)
Thickness

- The minimum number of planar graphs needed to cover a graph $G$ is the thickness of $G$. – NP-hard
A graph is *outerplanar* if it has an embedding in which every vertex lies on the boundary of the outer face.
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(Graphs: terminology)
A flow network is a directed graph $G(V, E)$ in which each arc is associated with a nonnegative capacity; also, two distinguished vertices source $s$ (no incoming arcs to $s$) and sink $t$ (no arcs go out from $t$) belong to $V$. 
An s-t flow is a function $f : E \to \mathbb{R}^+$ that denotes the flow, say $f_e$, carried along each arc, say $e$, satisfying the following properties:

- **capacity constraint**: For each arc in $E$, $0 \leq f_e \leq c_e$.

- **flow conservation constraint**: For every $v \in V - \{s, t\}$, the flow coming to $v$ equals to the flow going out of $v$.

The *s-t flow value* is the amount of flow from going out of $s$. The *s-t max-flow* problem finds a flow vector that maximizes the s-t flow value.
Few variants

- single-commodity maximum flow – polynomial time
- multi-commodity maximum flow – NP-complete

(Graphs: terminology)
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(Graphs: terminology)
Vertex coloring

- A *proper k-coloring* of a graph $G(V, E)$ is a function $c : V \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. A graph $G$ is said to be *k-colorable* if $k$ colors suffice to properly color $G$.

- The minimum number of colors needed for a proper coloring is termed as the *chromatic number*, $\chi(G)$, of graph $G$.

- The *graph coloring problem* is to determine the $\chi(G)$ of a given graph $G$. – NP-complete; however, 2-coloring is in P
A proper $k$-edge coloring of a graph $G(V, E)$ is a function $c : E \rightarrow \{1, 2, \ldots, k\}$ such that $c(e) \neq c(f)$ for adjacent edges $e, f \in E$. A graph $G$ is said to be $k$-edge-colorable if $k$ colors suffice to properly edge-color $G$.

The minimum number of colors used in edge coloring problem is termed as the edge-chromatic number or chromatic index, $\chi'(G)$ of the graph.

The edge coloring problem is to determine the $\chi'(G)$ of a given graph $G$. – NP-complete
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Definition

An undirected graph $G$ is a tree whenever $G$ is both:

- connected: $\exists$ a path between every two nodes of $G$
- acyclic: $\not\exists$ a cycle in $G$

If a graph $G$ is a union of zero or more trees, then $G$ is termed as a forest.
Tree terminology

- rooted tree
- root, leaf, internal vertex
- ancestor, descendent, siblings
- height, level
- ordered tree, balanced tree
- forest
Tree terminology (cont)

- A rooted tree is said to be a *k-ary tree* whenever no node of $T$ has more than $k$ children.
  
  A *k*-ary tree is said to be a *binary tree* whenever it is a $k$-tree with $k$ equals to 2.

- A rooted tree is said to be a *full/strict $k$-ary tree* whenever every node of $T$ has exactly 0 or $k$ children.

- A rooted tree $T$ is said to be a *balanced tree* if the height difference between any two leaf nodes of $T$ is upper bounded by 1.

- A rooted $k$-ary tree $T$ is said to be *almost complete $k$-ary tree* whenever $T$ is a full and balanced $k$-ary tree together with leaves in the last level are as left as possible in $T$.

- A rooted $k$-ary tree $T$ is said to be *complete $k$-ary tree* whenever $T$ is a full $k$-ary tree together with all the leaf nodes are at the same height.
Popular trees

chain

star

m(3)–Star

banana tree (from 2–star, 2–star, 1–star)

k(3)–olive tree

(Graphs: terminology)
A spanning subgraph $G'$ of graph $G$ is said to be a *spanning tree* when $G'$ is a tree; $G'$ is a *spanning forest* when it is a collection of (zero or more) trees.
Let $F(V, E')$ be a spanning forest of a connected undirected graph $G(V, E)$. For every non-tree edge $e(u, v) \in E - E'$, the simple cycle formed with the edges participating in the unique simple path between $u$ and $v$ in $F$ together with edge $e$ is known as the fundamental cycle of $e$ w.r.t. spanning forest $F$.

Note that every non-tree edge in $G$ defines a unique fundamental cycle w.r.t. $F$. 

(Graphs: terminology)
• A directed graph $H$ is an *(out-*)arborescence *(a.k.a. out-tree, or (out-*)branching)* rooted at a node $r$ if the number of incoming arcs to each node in $H - r$ has one incoming arc and zero or more outgoing arcs; further, $r$ has no incoming arcs.

• In case of *in-arborescence* *(a.k.a. in-tree, or in-branching)*, each node in $H - r$ has one outgoing arc and zero or more incoming arcs; $r$ has no outgoing arcs.

• A directed graph $H$ is a *directed tree* if its underlying undirected graph is a tree.
Minimum spanning tree (MST)

- A spanning tree of an undirected weighted graph with least cost is known as a minimum spanning tree (MST). – polynomial time

- An arborescence of a directed weighted graph with least cost is known as a minimum cost arborescence. – polynomial time
Given a weighted graph $G(V, E)$ with $S \subset V$, a tree of least cost that spans $S$ together with some of the (Steiner) vertices from $V - S$ is said to be a Steiner tree. – NP-complete
A graceful labeling of a graph $G(V, E)$ is a function $f : V \to \{0, \ldots, |E|\}$ such that distinct vertices receive distinct numbers and $\{f(u) - f(v) : (u, v) \in E\} = \{1, \ldots, |E|\}$. A graph is graceful if it has a graceful labeling.

The graceful tree conjecture (Ringel-Kotzig conjecture): Every tree has a graceful labeling.
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(Graphs: terminology)
A tree decomposition of $G(V,E)$ consists of a tree $T$ wherein a subset, termed as bag, $V_t \in V$ is associated with each node $t$ of $T$ while ensuring:

- **Node coverage**: Every node of $G$ belongs to at least one bag.

- **Edge coverage**: For every edge $e$ of $G$, there is some bag that contains both ends of $e$.

- **Coherence**: The nodes corresp. to all the bags to which any $v \in V$ belongs together form a connected component in $T$. 

(Graphs: terminology)
The *treewidth* of a tree decomposition is defined as the maximum bag size minus one. – NP-complete

Tree decomposition with minimum treewidth – NP-hard; $O(f(w)|V||E|)$

parameterized algo with tree width $4w$

If $T$ happens to be a path, then so obtained decomposition is called as a *path decomposition*. 
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(Graphs: terminology)
Two popular models for generating (undirected simple) random graphs

- $G_{n,p}$ model: Start with a graph $G$ with $n$ isolated nodes. Consider each of the $\binom{n}{2}$ possible edges in some order and then independently add each edge to $G$ with probability $p$.

- $G_{n,m}$ (a.k.a. $G^m$) model: Start with a graph $G$ with $n$ isolated nodes. Choose one of the $\binom{n}{2}$ possible edges uniformly at random and add it to $G$. Then, choose one of the $\binom{n}{2} - 1$ possible edges uniformly at random and add it to $G$ etc., continue doing the same until there are $m$ edges are chosen.
Few other categories of graphs

- The *intersection graph* of a collection of sets is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection.
  
  ex. interval graph (in $R^1$), unit disk graph

- A graph is *chordal (or triangulated)* if each of its cycles of length at least four has a chord, i.e. if it contains no induced cycles other than triangles.

- A graph is *perfect* if the chromatic number $X(H)$ of every induced subgraph $H \subseteq G$ equals to the clique number $w(H)$ of $H$. 

(Graphs: terminology)
Few more geometric graphs

• Planar straight-line graph (PSLG)

• Euclidean graph

• Visibility graph
Commonly used symbols

- min-degree $\delta(G)$
- max-degree $\Delta(G)$
- eccentricity of $u \in V$: $\epsilon(u)$
- (node-)connectivity $\kappa(G)$
- edge-connectivity $\lambda(G)$
- complete graph $K_n$
- biclique: $K_{m,n}$
- girth: $g(G)$
- neighbors of $U \subseteq V$: $N(U)$
- induced subgraph $G[V']$ for $V' \subseteq V$
- tree width $tw(G)$
• independence number $\alpha(G) = \max\{|C| \mid C \text{ is an independent set}\}$

• vertex cover number $\tau(G) = \min\{|W| \mid W \text{ is a vertex cover}\}$

• matching number $\nu(G) = \max\{|M| \mid M \text{ is a matching}\}$

• edge cover number $\rho(G) = \min\{|F| \mid F \text{ is an edge cover}\}$

• clique number $\omega(G) = \max\{|C| \mid C \text{ is a clique}\}$

• chromatic number $\chi(G) = \text{minimum number of colors needed to color vertices of } G$

• chromatic index $\chi'(G) = \text{minimum number of colors needed to color edges of } G$