- The objective of this data structure is to maintain a balanced binary search tree (BBST). When there are n keys stored in the AVL[1] tree, its height is guaranteed to be $O(\lg n)$. This helps in accomplishing search, insert, delete, split, and join operations in $O(\lg n)$ time in the worst-case.

- The AVL tree is a BST on the keys stored in it. The balance condidtion states that for every node v in the tree, $-1 \leq (h(v.right) - h(v.left)) \leq 1$. At every node v, the AVL tree stores the balance information $h(v.right) - h(v.left)$ in a two-bit field $v.b$. It is immediate that $\forall_v v.b \in \{-1, 0, 1\}$ whenever the given BST is an AVL tree.

  Observation: Given $v.b$ for every node v of a tree T, the height of T can be determined in $O(height(T))$ time.

- For an AVL tree T of height h, let $n(h)$ be the minimum number of nodes that T can have.

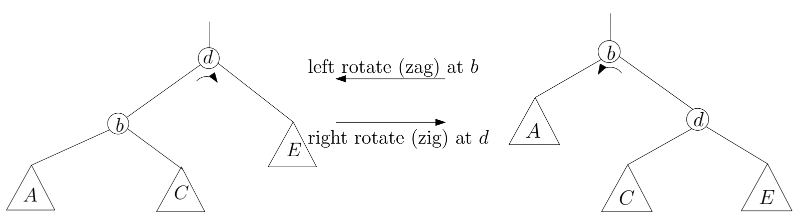
  It is obvious, $n(0) = 1, n(1) = 2$. Since the T has height h, either left or right subtree must have height $h - 1$. To minimize the number of nodes in T, without violating the balance condition, the height of the other subtree can be no smaller than $h - 2$. Hence,

  $$n(h) = n(h-1) + n(h-2) + 1 \text{ for } h \geq 2$$
  $$\Rightarrow n(h) > n(h-1) + n(h-2)$$
  $$\Rightarrow n(h) > 2n(h-2) > 2^2 n(h-4) > \ldots > 2^i n(h-2i)$$
  $$\Rightarrow n(h) > 2^i n(h-2i).$$

  Since $h - 2i \geq 2$, $i \leq \frac{h}{2} - 1$. Substituting $i = \frac{h}{2} - 1$ in the above, h is $O(\lg n)$.

  The insert, delete, split, and join algorithms ensure at no node balance condition is violated. Since no additional explicit algorithm is needed to balance the tree, AVL tree is a *self-balancing* binary search tree.

- Let T' be the tree resultant of right rotating at a node d of a BST T. Refer the below figure. Since the inorder traversals of T and T' yield the same ordering of keys, and since the relation of b, d, keys in subtrees A, C, and E are correct with respect to keys in $T - T_{d.parent}$, T' is a BST. The same holds good after any left rotate as well.



  Further, a left or a right rotation takes $O(1)$ time in the worst-case.

  Observation: With a right rotate at d, the $depth(d)$ increases by one and $depth(d.left)$ decreases by one. With a left rotate at b, the $depth(b)$ increases by one and $depth(b.right)$ decreases by one. However, the heights of d and b post a rotation depend on the heights of A, C, and E.

[1] named after its inventors Adelson-Velskii and Landis

- Algorithm for inserting node x into AVL tree T:

  First, node x is inserted into BST using the insertion algorithm for BSTs. Let P be the simple path from x to the root. While walking along P from x, at every node v, this algorithm updates $v.b$. Further, for the first node d along P at which the AVL balance condition is violated, if any, and its precededecessor b along P,

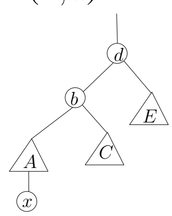
  if x is in a subtree which does not lie between d and b

  then //cases in which a single rotation suffices

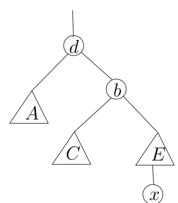
  if b is the left child of d //x is inserted into $T_{b.left}$

  then right-rotate(T, d) —————— (1)

  else left-rotate(T, d) —————— (2)



case (1)



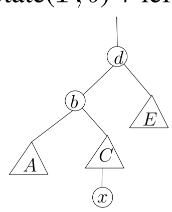
case (2)

  else //cases in which two rotations are needed

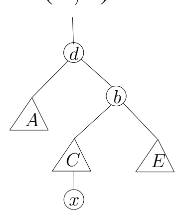
  if b is the left child of d

  then left-rotate(T, b) + rightRotate(T, d) —————— (3)

  else right-rotate(T, b) + leftRotate(T, d) —————— (4)



case (3)



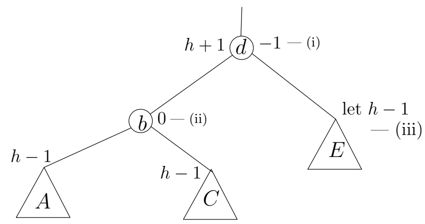
case (4)

  Observation: While walking along P from x, for any node v that occurs on this path, modified $v.b$ can be determined in constant time in the worst-case.

  Below, we present the correctness of (1) and (3). The analysis of cases (2) and (4) is symmetric.

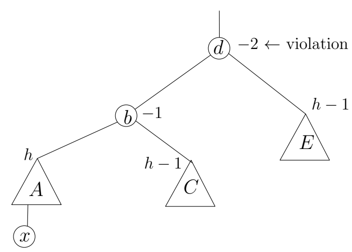
- — correctness of (1) —

* just before inserting x into A (while the insertion of x causing a balance violation at d):



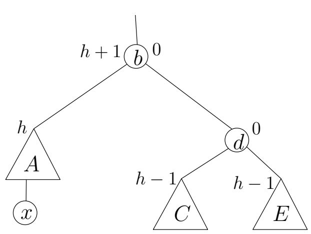
- since the insertion of x in T_b causing a violation at d, $d.b$ cannot be equal to 0 or $+1$
- since the first violation occurs at d after inserting x into A, $b.b \neq -1$; and, if $b.b = 1$, violation does not occur at d after insertion
- let $h - 1$ be the height of E; then, since $d.b = -1$, $h(b) = h$; since $b.b = 0$, $h(A) = h(C) = h - 1$; further, $h(d) = h + 1$

* just after inserting x into A:



- since $h(A \cup \{x\}) = h$, $h(d) = h + 2$; $b.b = -1$; $d.b = -2$

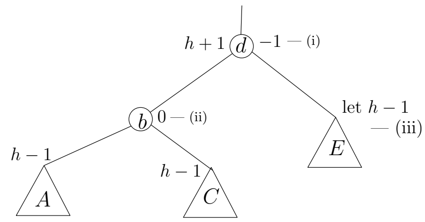
* just after right rotate at d:



- $d.b = h(E) - h(C) = (h - 1) - (h - 1) = 0$; $b.b = h(d) - h(A \cup \{x\}) = h - h = 0$;

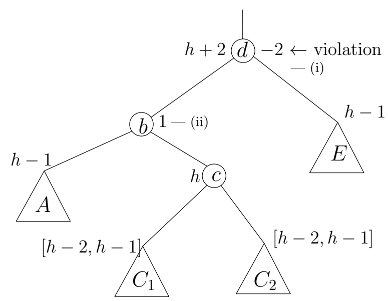
- — correctness of (3) —

* just before inserting x into C (while the insertion of x causing a balance violation at d):



- again, since the insertion of x in T_b causing a violation at d, $d.b$ cannot be equal to 0 or $+1$
- since the balance violation occurs first at d along x to d path after inserting x into C, $b.b \neq 1$; and, if $b.b = -1$, violation does not occur at d after insertion
- let $h-1$ be the height of E; then, since $d.b = -1$, $h(b) = h$; since $b.b = 0$, $h(A) = h(C) = h-1$; hence, $h(d) = h+1$

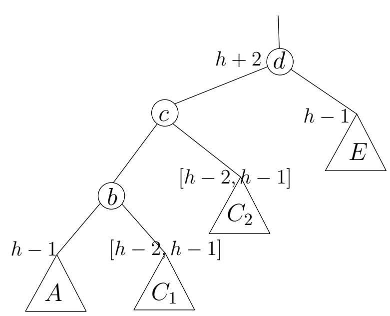
* just after inserting x into C:



- since $h(C \cup \{x\}) = h$, $b.b = 1$; since $h(b) = h+1$ and since $h(E) = h-1$, $d.b = -2$; further, $h(d) = h+2$
- since $h(c) = h(C \cup \{x\}) = h$, $h-2 \leq h(C_1) \leq h-1$: if $h(C_1) < h-2$, the first violation along x to d path does not occur at d; if $h(C_1) > h-1$, $h(c) > h$

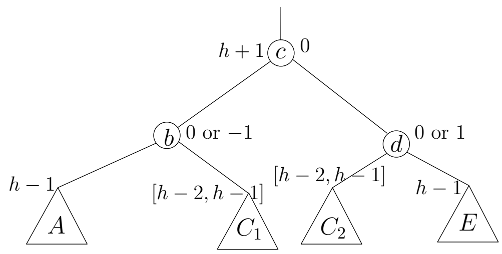
  analogously, since $h(C \cup \{x\}) = h$, $h-2 \leq h(C_2) \leq h-1$

* just after left rotate at b:



- height of T_d continues to be $h+2$; balance violation at d is not yet got fixed

* just after right rotate at d:



- since $max(h(C_1), h(C_2)) = h-1, h(A) = h-1$, and $h(E) = h-1$, $h(c) = h+1$;
- since $h(b) = h$ and $h(d) = h$, $c.b = 0$;
- since $h-2 \leq h(C_1) \leq h-1$, $b.b = -1$ or 0

  and, since $h-2 \leq h(C_2) \leq h-1$, $d.b = +1$ or 0

- Fixing imbalance at d ensures $v.b \in \{-1, 0, 1\}$ at every node v of T: $h(b)$ after insertion is made equal to $h(d)$ before insertion in single rotate case; $h(c)$ after insertion is made equal to $h(d)$ before insertion in double rotate case.

  The insertion involves traversing the simple path from x to d and one/two rotations. Since the height of an AVL tree is $O(\lg n)$ and since it takes $O(1)$ time for any one rotate, the time complexity of inserting a node into a AVL tree is $O(\lg n)$. Further, since every node obeys the balance condition in the updated tree T', the height of T' is $O(\lg n + 1)$, where $n+1$ is the nuber of nodes in T'.

- Algorithm to delete a node z from the AVL tree T:

  After deleting node z from T, let P be the path from the position where x was located to the root. (The significance of x was detailed in the algorithm for deleting a node from the BST.) For every node d along P at which the AVL balance condition is violated, if any:

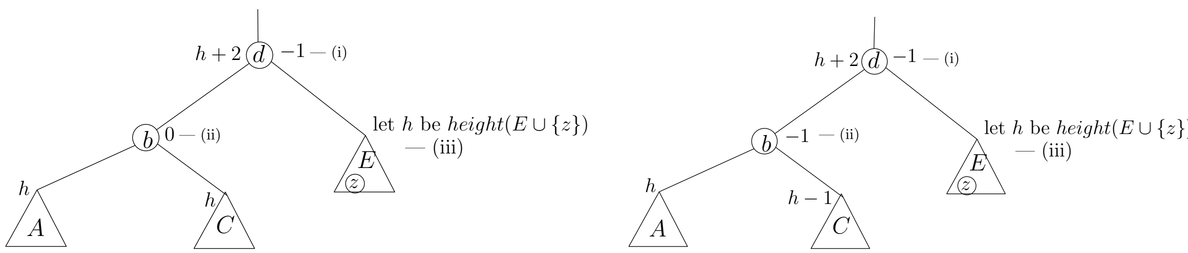
  if $d.b < -1$
      if $d.left.b \leq 0$
          then right-rotate(d) ———————— (5)
          else left-rotate($d.left$) + right-rotate(d) ———————— (6)
  if $d.b > 1$
      if $d.right.b \geq 0$
          then left-rotate(d) ———————— (7)
          else right-rotate($d.right$) + left-rotate(d) ———————— (8)

Essentially, (5) and (6) handle node getting removed from $T_{d.right}$; (7) and (8) handle node getting removed from $T_{d.left}$. Below, we present the correctness of (5) and (6). The analysis of cases (7) and (8) is symmetric.

Observation: Analogous to the case of insertion, while walking along P from x, for any node v that occurs on this path, modified $v.b$ can be determined in constant time in the worst-case.

- — correctness of (5) —

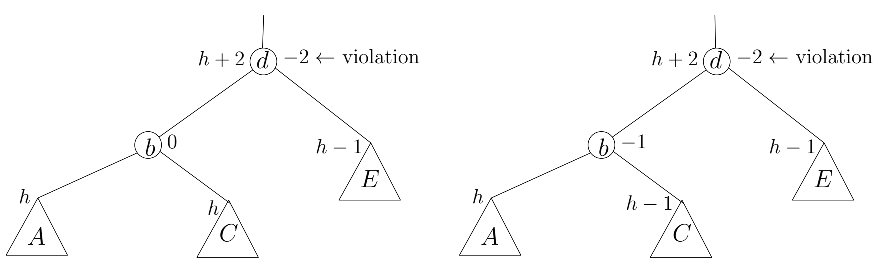
* just before deleting z from $E \cup \{z\}$ (while that deletion causing a balance violation at d):



- since deleting z from $E \cup \{z\}$ causing a violation at d, $d.b \neq 1$ and $d.b \neq 0$
- (5) occurs only if $b.b = -1$ or 0 ((6) handles the case of $b.b = 1$)
- let $h(E \cup \{z\}) = h$; then, since $d.b = -1$, $h(b) = h + 1$
- if $b.b = 0$, $h(A) = h(C) = h$ — (I)

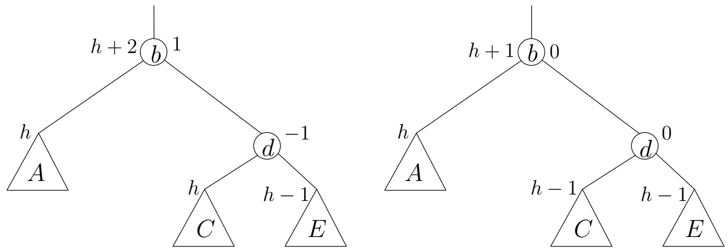
  if $b.b$ is -1, $h(C) = h - 1$ and $h(A) = h$ — (II)

* just after deleting z from $E \cup \{z\}$:



- since there is a violation at d upon deleting z from $E \cup \{z\}$, $d.b = -2$; since $d.b$ is getting changed due to deletion, $h(E) = h - 1$

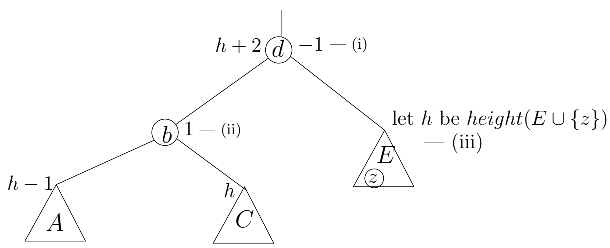
* just after right rotate at d:



- in case (I), $h(A) = h(C) = h$ and $h(E) = h - 1$; hence, $h(d) = h + 1, h(b) = h + 2, d.b = -1, b.b = 1$
- in case (II), $h(A) = h, h(C) = h-1$, and $h(E) = h-1$; hence, $h(d) = h, h(b) = h+1, d.b = 0, b.b = 0$

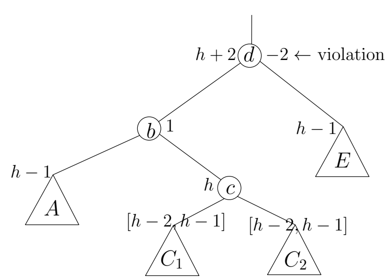
- — correctness of (6) —

* just before deleting z from $E \cup \{z\}$ (while that deletion causing a balance violation at d):



- again, since deleting z from $E \cup \{z\}$ causing a violation at d, $d.b \neq 1$ and $d.b \neq 0$
- let, $h(E \cup \{z\}) = h$; since $d.b = -1$, $h(b) = h + 1$; hence, $h(d) = h + 2$
- $b.b = 1$ (since the case of $b.b$ equal to either 0 or -1 was handled in (5))
- since $b.b = 1$, $h(C) = h$ and $h(A) = h - 1$

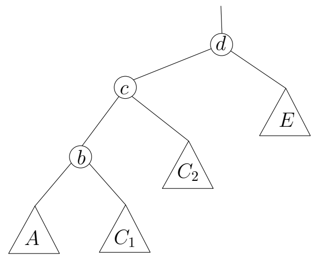
* just after deleting z from $E \cup \{z\}$:



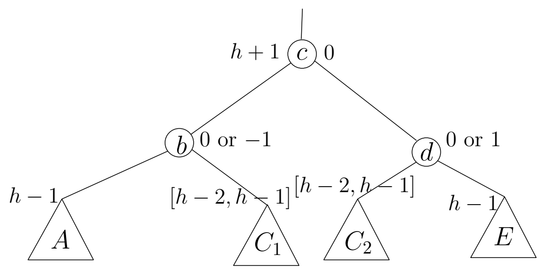
- since $d.b$ was -1, the violation occurs at d only if $d.b$ changes to -2
- since there is a violation at d, the $h(E)$ must be changing from h; hence, $h(E) = h - 1$

- since $h(C) = h$ and since there is no violation at c, various possible values for $(h(C_1), h(C_2))$ tuple are $(h-1, h-1), (h-1, h-2), (h-2, h-1)$

* after left rotate at b:



* after right rotate at d:



- $h(c) = max(h(A), h(C_1), h(C_2), h(E)) + 2 = max(h-1, max(h(C_1), h(C_2)), h-1) + 2 = h+1$
- since $h(E) = h - 1$ and $h - 2 \leq h(C_2) \leq h - 1$, $d.b$ is 0 or +1; analogously, $b.b$ is 0 or -1

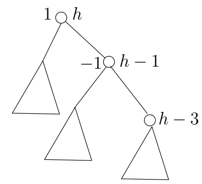
- Since the $h(d)$ before deletion is $h + 2$ and $h(c)$ after two rotations is $h + 1$, further ancestors need to be checked for the balancing condition.

  Since the $h(d)$ before deletion could differ from $h(b)$ post rotation(s) in some of the cases of deletions, unlike in insert, further ancestors need to be checked for possible balance condition violation.

- The deletion algorithm takes $O(\lg n)$ time: the work involves, in the worst-case, at every node v on the simple path from x to the root, at most a couple of rotations and updating the balance information at a constant number of nodes in the vicinity of v.

- The following observation is useful in joining AVL trees:

  For any AVL tree T of height h, there always exists a node on the right spine of T whose height is either i or $i + 1$, for any $0 \leq i \leq h$. However, there is no guarantee that the right spine of T to have a node of height i or a node of height $i + 1$. (Refer to below figure.) Specifically, the last node on the right spine of T is guaranteed to have height either 0 or 1.



no node on the right spine has height $h-2$

* Given two AVL trees T_1 and T_2 with every key in T_1 less than or equal to every key in T_2, the following algorithm joins (merges) T_1 and T_2:

    (a) using the balance information stored at nodes, determine $height(T_1)$ and $height(T_2)$ $\leftarrow$ takes $O(height(T_1) + height(T_2))$ time

    w.l.o.g., suppose $height(T_1) \geq height(T_2)$; the other case is analogous

    (b) find a node x in T_2 such that $x.key$ is the minimum among keys stored in T_2 $\leftarrow$ takes $O(height(T_2))$ time

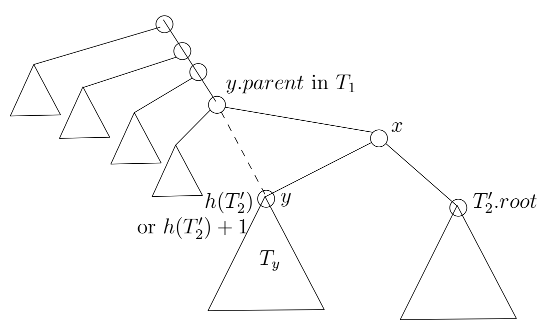
    - using the algorithm detailed above, delete x from AVL tree T_2 $\leftarrow$ takes $O(height(T_2))$ time
    - let T_2' be T_2 sans node x

    (c) using the balance information stored at nodes along the right spine and $height(T_1)$, starting from $root(T_1)$, walk along the right spine of T_1 to find a node y whose height equals to either $height(T_2')$ or $height(T_2') + 1$ $\leftarrow$ takes $O(1 + height(T_1) - height(T_2'))$ time

    (d) hang T_y as the left child of x and T_2' as the right child of x; make $y.parent$ in T_1 as the parent of x $\leftarrow$ takes $O(1)$ time

    (e) while walking along the simple path from x to root of the resultant tree, at every node v, if the balance condition is violated at v, fix with one/two left/right rotations as in insert algorithm for AVL trees [2] $\leftarrow$ takes $O(1 + height(T_1) - height(T_2'))$ time

    - let this tree be T



* Correctness: Since keys in T_y are $\leq x.key$, since keys in T_2' are $> x.key$, and since $x.key > y.parent.key$, the T is a BST. The violation of balance condition due to deletion of x from T_2 is taken care of by the delete algorithm for AVL trees. The violations are fixed while traversing along the simple path from x to root. Hence, the resultant tree is an AVL tree.

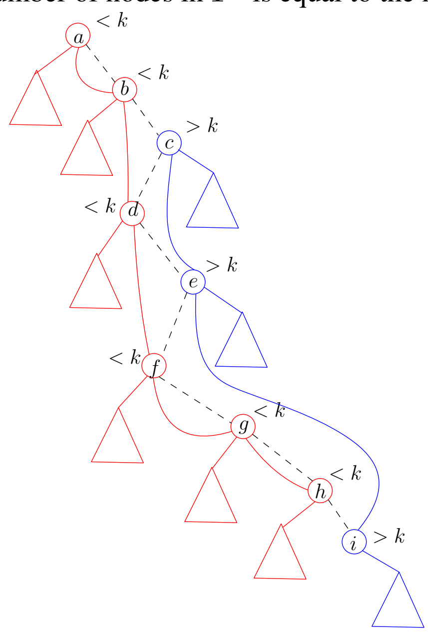
[2] like in insert algorithm, it suffices to fix the first violation along this path

* This algorithm takes $O(height(T_1) + height(T_2))$ time to join two AVL trees.

- Corollary: Given AVL trees T_1 and T_2 with their respective heights, and a key k such that every key in T_1 is less than or equal to k and every key in T_2 is greater than k, the algorithm to join T_1, T_2 and k into an AVL tree takes $O(1 + |height(T_1) - height(T_2)|)$ time.

  Proof: Since step (a) and step (b) of the algorithm listed above are avoided.

- Given an AVL tree T with n keys and a key k, split T into two AVL trees T', T'' such that (i) T' has every key of T less than or equal to k and T'' has every key of T strictly greater than k, and (ii) the sum of the number of nodes in T' and the number of nodes in T'' is equal to the number of nodes in T.



dashed line path is the search path for k in T; T' is in red and T'' is in blue

* For easier understanding, the algorithm below is adapted to the above example:

  (a) Using the balance information stored at nodes, compute the height of T ← takes $O(\lg n)$ time

  (b) $T_1 = AVLJoin(T_{a.left}, h(T_{a.left}), a, T_{b.left}, h(T_{b.left}))$
  $T_2 = AVLJoin(T_1, h(T_1), b, T_{d.left}, h(T_{d.left}))$
  $T_3 = AVLJoin(T_2, h(T_2), d, T_{f.left}, h(T_{f.left}))$
  $T_4 = AVLJoin(T_3, h(T_3), f, T_{g.left}, h(T_{g.left}))$
  $T_5 = AVLJoin(T_4, h(T_4), g, T_{h.left}, h(T_{h.left}))$
  $AVLInsert(T_5, h)$

  ← This algorithm takes $O(\lg n)$ time, due to the following: heights of subtrees $T_1, T_2, T_3, \ldots$ can be determined as part of AVLJoin, heights of $T_{a.left}, T_{d.left}, \ldots$ are determined when roots of those subtrees are encounted while exploring the search path for k, above corollary, and telescoping of terms involved[3] in summing time complexities of joins.

(c) $T_1 = AVLJoin(T_{e.right}, h(T_{e.right}), c, T_{c.right}, h(T_{c.right}))$
$T_2 = AVLJoin(T_{i.right}, h(T_{i.right}), e, T_1, h(T_1))$
$AVLInsert(T_2, i)$

$\leftarrow$ This part of the algorithm also takes $O(\lg n)$ time.

* The correctness of this AVLSplit algorithm is immediate given the correctness of BST split algorithm. An algorithm for the latter was detailed and a proof of correctness for the same was provided in an earlier lecture.

[3] since the height of T_1 is at most $\max(h(T_{a.left}, T_{b.left})) + 2$, since the height of T_2 is at most $\max(h(T_1), h(T_{d.left})) + 2$, etc.