- A subset of a poset such that every two elements of this subset are comparable is called a *chain*. A *maximal chain* is a chain that is not a proper subset of any other chain. A *maximum chain* is a chain that has cardinality at least as large as every other chain. The *height of a poset* is the cardinality of a maximum chain.

- A subset of a poset is called an *antichain* if every two elements of this subset are incomparable is called an *antichain*. A *maximal antichain* is an antichain that is not a proper subset of any other antichain. A *maximum antichain* is an antichain that has cardinality at least as large as every other antichain. The *width of a poset* is the cardinality of a maximum antichain.

- *Dilworth's theorem*: If $w$ is the width of a poset $(S, \preccurlyeq)$, then there exists a partition $S = \cup_{i=1}^{w} C_i$, where $C_i$ is a chain.

  [The following is a proof by induction on $|S|$.]

  * Basis: Consider a set $S$ with one element, say $S = \{a\}$. In $(S, \preccurlyeq)$, the only maximum antichain is $\{a\}$, its size is 1, and $C_1 = \{a\}$ with $C_1 = S$.

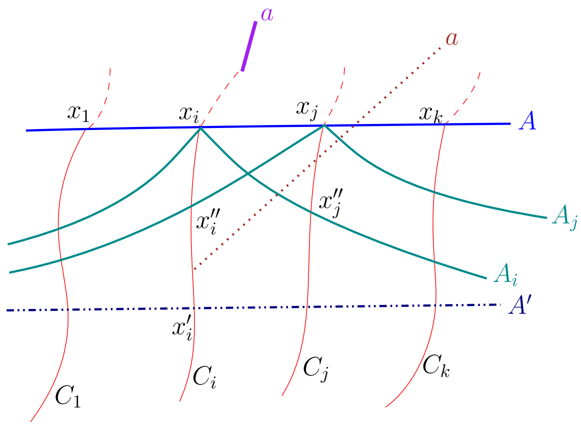
  * IH: For every $k' \leq k$, if $k'$ is the width of poset $(S - \{a\}, \preccurlyeq)$, then there exists a partition $\mathcal{C} = C_1 \cup \ldots \cup C_{k'}$ of $S - \{a\}$, where $a$ is a maximal element along a chain of $(S, \preccurlyeq)$.
  Let $A'$ be a maximum antichain of $(S - \{a\}, \preccurlyeq)$.

  \- Lemma 1: For every $C_i \in \mathcal{C}$, $C_i \cap A' \neq \phi$. Specifically, $A'$ has exactly one element form each $C_i \in \mathcal{C}$.
  Proof: Suppose no element of a chain in this decomposition belongs to $A'$, then $A'$ cannot be an antichain of size $k$: If $A'$ has no element from a $C_{j'} \in \mathcal{C}$, then two elements in $A'$ belong to a chain $C_j$ for $j \neq j'$. However, if two elements in $A'$ belong to a chain $C_j \in \mathcal{C}$, then those elements are comparable w.r.t. $\preccurlyeq$; and, hence $A'$ cannot be an antichain in that case.
  Since IH only says that there exists an $A'$ and the chain decomposition, via the following lemma, we determine a maximum antichain $A$ of $(S - \{a\}, \preccurlyeq)$ given a chain decomposition comprising $k$ chains.

  \- Lemma 2: For every $C_i \in \mathcal{C}$, let $x_i$ be the maximal element in $C_i$ that belongs to a maximum antichain $A_i$ of $(S - \{a\}, \preccurlyeq)$. Then, $A = \{x_1, \ldots, x_k\}$ is an antichain of $(S - \{a\}, \preccurlyeq)$.
  Proof: For every $i$, $A_i$ always exists, since an element $x_i'$ of $C_i$ belongs to antichain $A'$. (For example, such an element can be found by walking along $C_i$ from top to bottom.)
  Suppose $x_{j''} \in A_i \cap C_j$. From the definition of $x_j$, we know $x_{j''} \preccurlyeq x_j$. Suppose $x_j \preccurlyeq x_i$. Then, from transitivity, $x_{j''} \preccurlyeq x_i$. However, $x_{j''}$ and $x_i$ are part of an antichain; therefore, $x_j \not\preccurlyeq x_i$.
  Analogously, suppose $x_{i''} \in A_j \cap C_i$. From the definition of $x_i$, we know $x_{i''} \preccurlyeq x_i$. Suppose $x_i \preccurlyeq x_j$. Then, from transitivity, $x_{i''} \preccurlyeq x_j$. However, $x_{i''}$ and $x_j$ are part of an antichain; therefore, $x_i \not\preccurlyeq x_j$.
  For every $x_i, x_j \in A$, since $x_j \not\preccurlyeq x_i$ and $x_i \not\preccurlyeq x_j$, $A$ is an antichain.

  * IS: Since $a$ is a maximal element of $(S, \preccurlyeq)$, there are two possibilities: $x_i \preccurlyeq a$ for some $C_i \in \mathcal{C}$ (via maximal element along $C_i$) or $x_i \not\preccurlyeq a$ for every $C_i \in \mathcal{C}$.
  In the latter case, $a$ is not related to any element in $A$. Hence, using induction hypothesis, $C_1 \cup \ldots \cup C_k \cup \{a\}$ is a partition of $S$ into $k + 1$ chains. Further, due to Lemma 2 and since no $x_i$ is related to $a$, it is immediate to note $A \cup \{a\}$ is an antichain of size is $k + 1$.

In the former case, consider $(S - C_i - \{a\}, \preccurlyeq)$. From the induction hypothesis, $S - C_i - \{a\}$ is partitioned into $\mathcal{C}' = \{C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_k\}$ of chains. (Note that $S - C_i - \{a\}$ has size smaller than $|S|$; hence, we were able to apply IH, by the means of strong induction.) And, since $A$ is an antichain (Lemma 2), $(S - C_i - \{a\}, \preccurlyeq)$ has an antichain $A - \{x_i\}$, which is of size $k - 1$. The chains in $\mathcal{C}'$ together with the chain formed by the subpart of $C_i$ underneath $a$ (including $a$) is a partition of $S$ into $k$ chains, while $A$ is an antichain of size $k$.




[Illustrating the conventions in the above proof. In the first case of IS, $a$ is above the maximal element of a chain, say $C_i$. In the second case of IS, $a$ is located on its own chain. The dashed lines indicate elements beyond elements of $A$.]

- *Sperner's lemma*: The size of a largest antichain of any poset $(\mathcal{P}(S), \subseteq)$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, where $S = \{1, 2, \ldots, n\}$.

* for any fixed $k$, all $k$-sets together form an antichain;
for $k = \lfloor \frac{n}{2} \rfloor$, there exists an antichain of size $\lfloor \frac{n}{2} \rfloor$

* no antichain of size $> \binom{n}{\lfloor \frac{n}{2} \rfloor}$ is possible:
consider adding one by one of the elements in $S$ along each chain, leading to, number of chains being $n!$;
for any element $A$ of any antichain $\mathcal{F}$ with $|A| = k$, there are $k!(n-k)!$ chains that contain $A$ (each chain comprising monotonically increasing sized sets from $\phi$ to $S$ containing $A$);
denoting the number of $k$-sets $\mathcal{F}$ contains with $m_k$,
since no chain can pass through two different sets $A$ and $B$ of $\mathcal{F}$, number of chains passing through all the members of $\mathcal{F}$ is $\sum_{k=0}^{n} m_k k!(n-k)!$, which is $\leq n! \Rightarrow \sum_{k=0}^{n} \frac{m_k}{\binom{n}{k}} \leq 1 \Rightarrow \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} m_k \leq 1$