A simple experiment to estimate π

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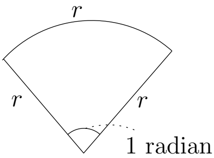
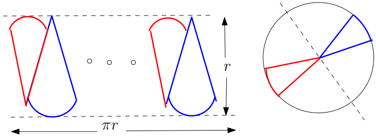
Significance of π



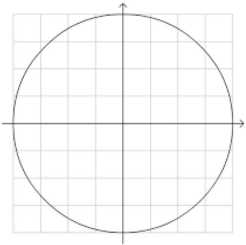
- 1 radian is defined as the angle subtended when the length of the arc is r; naturally, with 2π radians of angle at the center leads to perimeter being $2\pi r$.
- area of a circle is πr^2



Well known approximations of π are

- $\frac{22}{7}$ (accuracy 2.10^{-4})
- $\frac{355}{113}$ (accuracy 8.10^{-8})

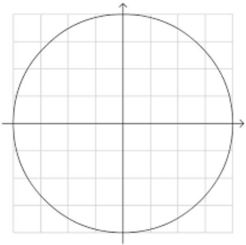
Unit grid vs area of circle



Let C be a circle of radius r centered at $(0,0)$; let the plane be tesselated with unit squares. Any such unit square can either be -

- interior to C
- exterior to C
- neither interior nor exterior to C $\rightarrow$ this leads to approximation

Unit grid vs area of circle



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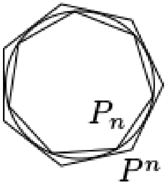
$$\pi = \lim_{r\to\infty} \sum_{x=-r}^{r} \sum_{y=-r}^{r} \begin{cases} 1 \text{ if } \sqrt{x^2+y^2} \leq r \\ 0 \text{ if } \sqrt{x^2+y^2} > r \end{cases}$$

Srinivasa Ramanujan's rapidly converging infinite series of π

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4(396)^{4k}}$$

* this computes a eight more decimal places of π with each term in the series

Polygon approximation to a circle

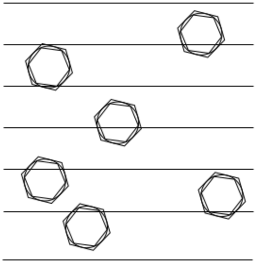


Let P_n and P^n respectively denote the perimeters of inscribed and circumscribed n-sided polygons with respect to circle C. Then,

- $P^{2n} = \frac{2p_n P_n}{p_n + P_n}$
- $P_{2n} = \sqrt{p_n P_{2n}}$

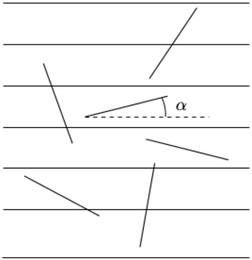
As $n \to \infty$, P^n or P_n approximates the perimter of C.

Buffon's needle: claim



- If a short needle of length ℓ is dropped on paper that is ruled with equally spaced lines of distance $d \geq \ell$, then the probability p that the needle comes to lie in a position where it crosses one of the lines is exactly $\frac{2\ell}{\pi d}$.

Buffon's needle: using calculus



- If α is the angle made by the needle with horizontal when it falls, then the probability that it crosses a horizontal line is $\frac{\ell \sin \alpha}{d}$.
- Hence, $p = \frac{1}{\pi/2} \int_0^{\frac{\pi}{2}} \frac{\ell \sin \alpha}{d} = \frac{2}{\pi} \frac{\ell}{d}$.

Buffon's needle: without using calculus

- Let p_i be the probability that the needle crosses exactly i lines. The probability that it crosses at least one line is $p_1 + p_2 + p_3 \ldots$.

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  - * since $\ell \leq d$, all terms except p_1 are 0.

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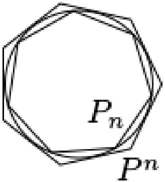
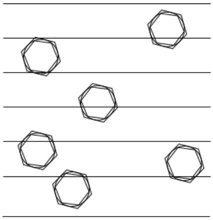
- Due to linearity of expectation $E[\ell] \propto \ell$, i.e., $E[\ell] = c\ell$ for some constant c.

Buffon's needle: without using calculus (cont)





- For any horizontal line ℓ', if ℓ' crosses P_n, then ℓ' crosses C; analogously, if ℓ' crosses C then ℓ' crosses P^n. Hence, $E[P_n] \leq E[C] \leq E[P^n]$.

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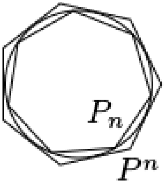

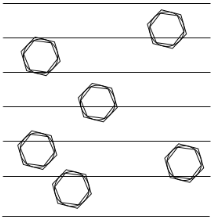

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- When circle C is chosen with diameter d, $E[C] = 2$; leading to $c.perimeter(P_n) \leq 2 \leq c.perimter(P^n)$.

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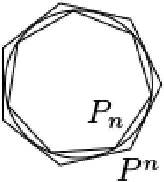

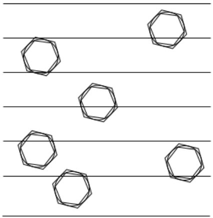

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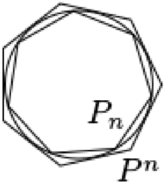

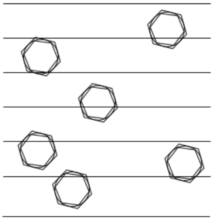

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- As $n \to \infty$, $perimeter(P_n) = perimeter(P^n) = \pi d$; therefore $c = \frac{2}{\pi d}$.

Hence, $E[\ell] = p = \frac{2}{\pi}\frac{\ell}{d}$.

Estimating π

Since p is proven to be equal to $\frac{2}{\pi}\frac{\ell}{d}$, to estimate π, drop a needle of length ℓ on paper that is ruled with equally spaced parallel lines of distance $d \geq \ell$ for n times (with n sufficiently large), leading to needle intersecting any of ruled lines be m times out of these n times, then π is $\frac{2\ell n}{dm}$.

References

Proofs from THE BOOK by Martin Aigner and Günter M. Ziegler. ← has a great collection of elegant proofs

Thanks!