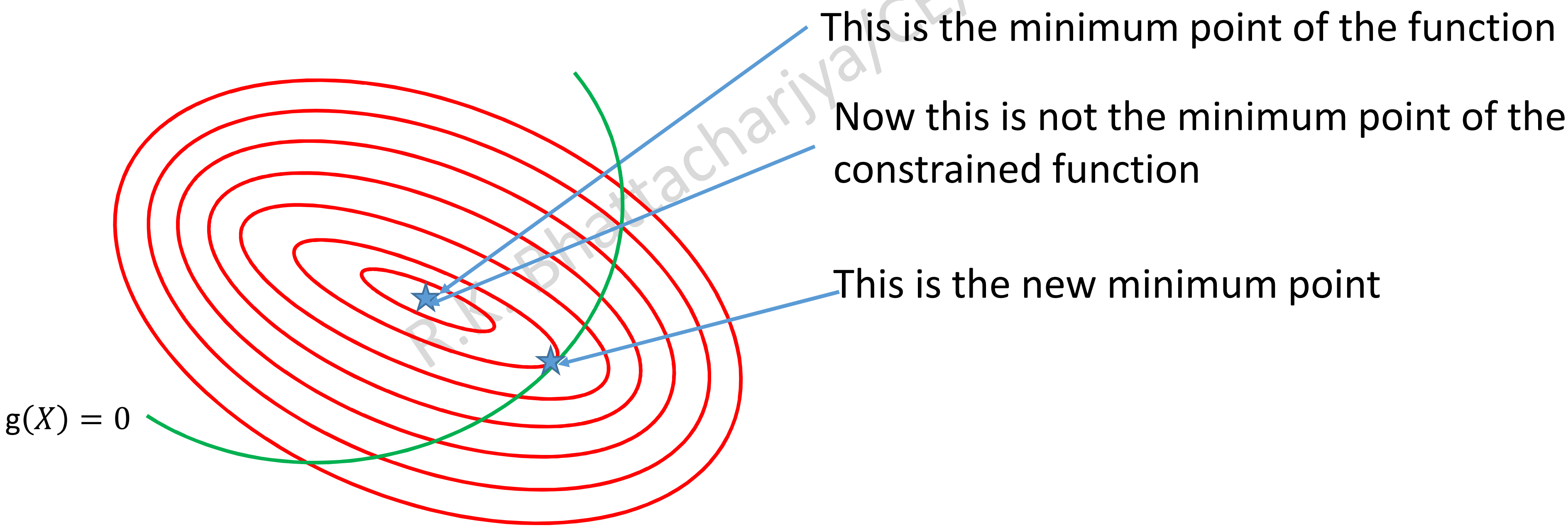
Multivariable problem with equality and inequality constraints

Rajib Bhattacharjya
Department of Civil Engineering
IIT Guwahati
Email: rkbc@iitg.ernet.in

General formulation

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \ldots, x_n]^T$

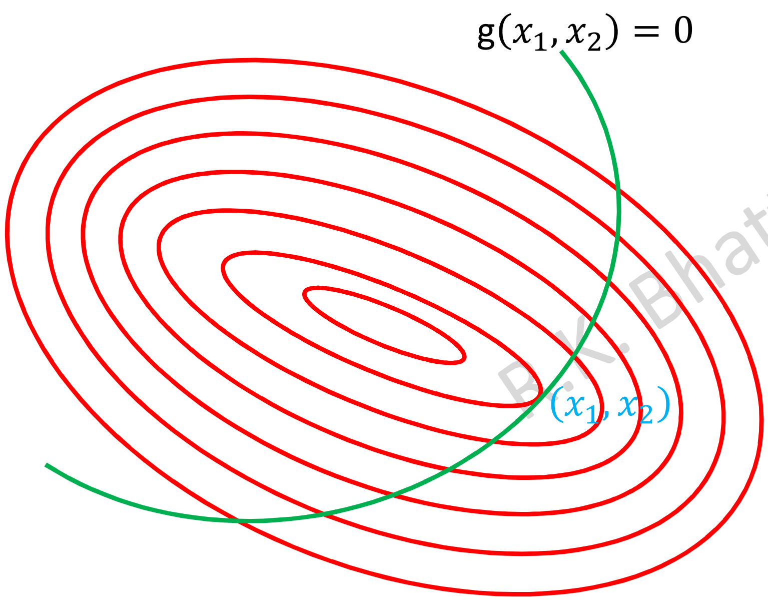
Subject to $g_j(X) = 0$ $j = 1,2,3, \ldots, m$



Consider a two variable problem

Min/Max $f(x_1, x_2)$

Subject to $g(x_1, x_2) = 0$



Take total derivative of the function at (x_1, x_2)

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

If (x_1, x_2) is the solution of the constrained problem, then

$$g(x_1, x_2) = 0$$

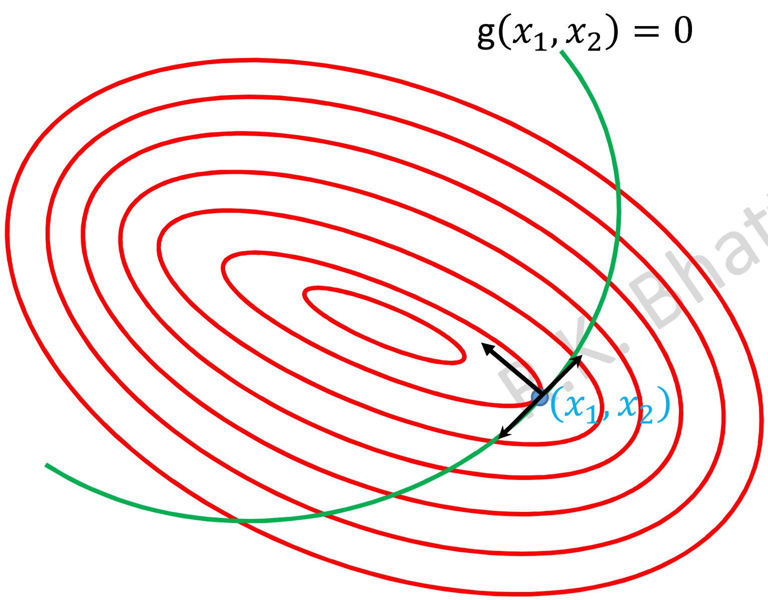
Now any variation dx_1 and dx_2 is admissible only when

$$g(x_1 + dx_1, x_2 + dx_2) = 0$$

Consider a two variable problem

Min/Max $\quad f(x_1, x_2)$

Subject to $\quad g(x_1, x_2) = 0$



$$g(x_1 + dx_1, x_2 + dx_2) = 0$$

This can be expanded as

$$g(x_1 + dx_1, x_2 + dx_2) = \cancelto{0}{g(x_1, x_2)} + \frac{\partial g(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial g(x_1, x_2)}{\partial x_2} dx_2 = 0$$

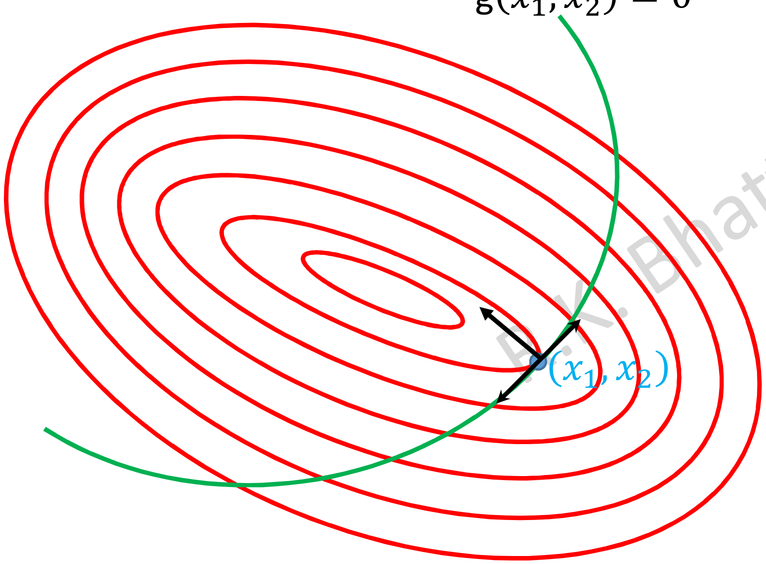
$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0$$

$$dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1$$

Consider a two variable problem

Min/Max $f(x_1, x_2)$

Subject to $g(x_1, x_2) = 0$



$$dx_2 = -\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1$$

Putting in $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$

$$df = \frac{\partial f}{\partial x_1} dx_1 - \frac{\partial f}{\partial x_2} \frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} dx_1 = 0$$

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right) dx_1 = 0$$

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} \right) = 0$$

This is the necessary condition for optimality for optimization problem with equality constraints

Lagrange Multipliers

Min/Max $\quad f(x_1, x_2)$

Subject to $\quad g(x_1, x_2) = 0$

We have already obtained the condition that

$$\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x_1} - \left(\frac{\frac{\partial f}{\partial x_2}}{\frac{\partial g}{\partial x_2}}\right)\frac{\partial g}{\partial x_1} = 0$$

By defining $\quad \lambda = -\dfrac{\frac{\partial f}{\partial x_2}}{\frac{\partial g}{\partial x_2}}$

$$\left.\begin{array}{ll}
\text{We have} & \dfrac{\partial f}{\partial x_1} + \lambda \dfrac{\partial g}{\partial x_1} = 0 \\[2ex]
\text{We can also write} & \dfrac{\partial f}{\partial x_2} + \lambda \dfrac{\partial g}{\partial x_2} = 0 \\[2ex]
\text{Also put} & g(x_1, x_2) = 0
\end{array}\right\} \text{Necessary conditions for optimality}$$

Lagrange Multipliers

Let us define

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

By applying necessary condition of optimality, we can obtain

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0 \\ \frac{\partial L}{\partial \lambda} &= g(x_1, x_2) = 0 \end{aligned}\right\} \quad \text{Necessary conditions for optimality}$$

Lagrange Multipliers

Sufficient condition for optimality of the Lagrange function can be written as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2 \partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} & \frac{\partial^2 L}{\partial \lambda \partial \lambda} \end{bmatrix} \Rightarrow H = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial g}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2 \partial x_2} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & 0 \end{bmatrix}$$

If H is positive definite , the optimal solution is a minimum point

If H is negative definite , the optimal solution is a maximum point

Else it is neither minima nor maxima

Lagrange Multipliers

Necessary conditions for general problem

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) = 0$ $j = 1,2,3, \dots, m$

$$L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \lambda_3 \dots, \lambda_m) = f(X) + \lambda_1 g_1(X) + \lambda_2 g_2(X), \dots, \lambda_m g_m(X)$$

Necessary conditions

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(X) = 0$$

Lagrange Multipliers

Sufficient condition for general problem

The hessian matrix is

$$H = \begin{bmatrix} L_{11} & L_{12} & L_{13} & \dots & L_{1n} & g_{11} & g_{21} & \dots & g_{m1} \\ L_{21} & L_{22} & L_{23} & \dots & L_{2n} & g_{12} & g_{22} & \dots & g_{m2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} & g_{1n} & g_{1n} & \dots & g_{mn} \\ g_{11} & g_{12} & g_{13} & \dots & g_{1n} & 0 & 0 & \dots & 0 \\ g_{21} & g_{22} & g_{23} & \vdots & g_{2n} & \vdots & \vdots & \vdots & \vdots \\ g_{31} & g_{32} & g_{33} & \dots & g_{3n} & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \dots & \dots \\ g_{m1} & g_{m2} & g_{m3} & \dots & g_{2n} & 0 & \dots & \dots & 0 \end{bmatrix}$$

Where,

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j}$$

$$g_{ij} = \frac{\partial g_i}{\partial x_j}$$

Lagrange Multipliers

Min/Max $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g(X) = b$ Or, $b - g(X) = 0$

Applying necessary conditions

$$\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0$$ Where, $i = 1,2,3, \dots, n$

$$g = 0$$

$$\boxed{\frac{\partial g}{\partial x_i} = \frac{\frac{\partial f}{\partial x_i}}{\lambda}}$$

Further $db - dg = 0$

$$db = dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$$

$$db = \sum_{i=1}^{n} \frac{1}{\lambda} \frac{\partial f}{\partial x_i} dx_i$$

$$db = \frac{df}{\lambda}$$

$$\lambda = \frac{df}{db}$$

$$\boxed{df = \lambda db}$$

<u>There may be three conditions</u>

$\lambda^* > 0$
$\lambda^* < 0$
$\lambda^* = 0$

Multivariable problem with inequality constraints

Minimize $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) \le 0$ $j = 1,2,3, \dots, m$

We can write $g_j(X) + y_j^2 = 0$

Thus the problem can be written as

Minimize $f(X)$

Subject to $G_j(X, Y) = g_j(X) + y_j^2 = 0$ $j = 1,2,3, \dots, m$

Where Y= $[y_1, y_2, y_3, \dots, y_m]^T$

Multivariable problem with inequality constraints

Minimize $\quad f(X) \quad$ Where $X = [x_1, x_2, x_3, \ldots, x_n]^T$

Subject to $\quad G_j(X,Y) = g_j(X) + y_j^2 = 0 \qquad j = 1,2,3,\ldots,m$

The Lagrange function can be written as

$$L(X,Y,\lambda) = f(X) + \sum_{j=1}^{m} \lambda_j G_j(X,Y)$$

The necessary conditions of optimality can be written as

$$\frac{\partial L(X,Y,\lambda)}{\partial x_i} = \frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\ldots,n$$

$$\frac{\partial L(X,Y,\lambda)}{\partial \lambda_j} = G_j(X,Y) = g_j(X) + y_j^2 = 0 \qquad j = 1,2,3,\ldots,m$$

$$\frac{\partial L(X,Y,\lambda)}{\partial y_j} = 2\lambda_j y_j = 0 \qquad j = 1,2,3,\ldots,m$$

Multivariable problem with inequality constraints

From equation $\frac{\partial L(X,Y,\lambda)}{\partial y_j} = 2\lambda_j y_j = 0$

Either $\lambda_j = 0$ Or, $y_j = 0$

If $\lambda_j = 0$, the constraint is not active, hence can be ignored

If $y_j = 0$, the constraint is active, hence have to consider

Now, consider all the active constraints,
Say set J_1 is the active constraints
And set J_2 is the active constraints

The optimality condition can be written as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\dots,n$$

$$g_j(X) = 0 \qquad j \in J_1$$

$$g_j(X) + y_j^2 = 0 \qquad j \in J_2$$

Multivariable problem with inequality constraints

$$-\frac{\partial f}{\partial x_i} = \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \lambda_3 \frac{\partial g_3}{\partial x_i} + \cdots + \lambda_p \frac{\partial g_p}{\partial x_i} \qquad i = 1,2,3,\ldots,n$$

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla g_3 + \cdots + \lambda_m \nabla g_m$$

$$\nabla f = \begin{Bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_n \end{Bmatrix} \qquad \nabla g_j = \begin{Bmatrix} \partial g_j / \partial x_1 \\ \partial g_j / \partial x_2 \\ \vdots \\ \partial g_j / \partial x_n \end{Bmatrix}$$

This indicates that negative of the gradient of the objective function can be expressed as a linear combination of the gradients of the active constraints at optimal point.

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

Let S be a feasible direction, then we can write

$$-S^T \nabla f = \lambda_1 S^T \nabla g_1 + \lambda_2 S^T \nabla g_2$$

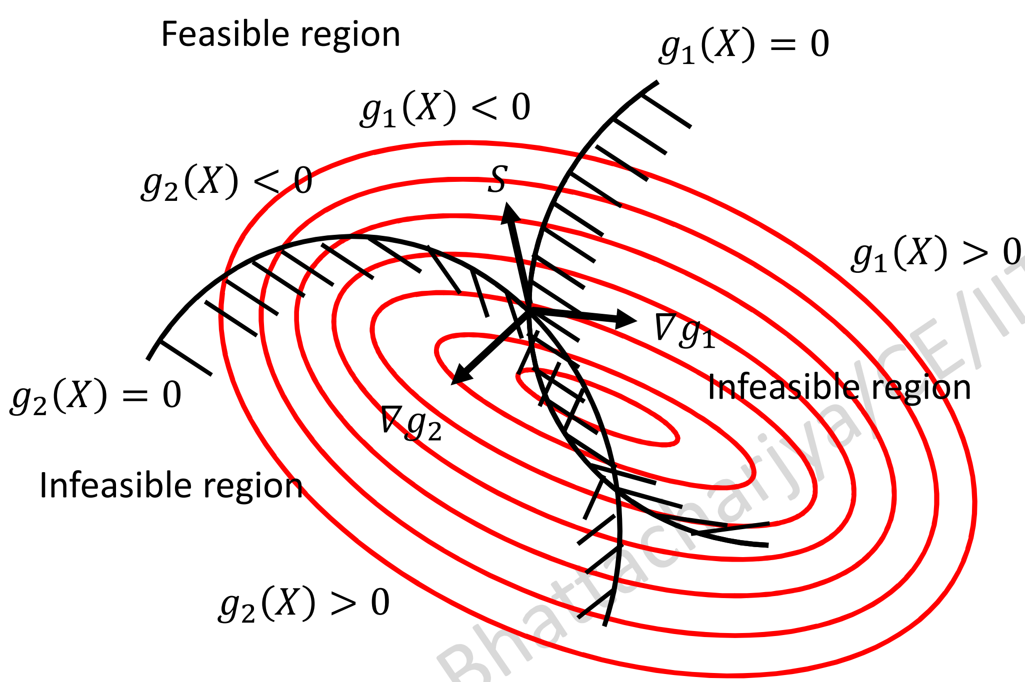
Since S is a feasible direction

$$\boxed{S^T \nabla g_1 < 0 \quad \text{and} \quad S^T \nabla g_2 < 0}$$

If $\lambda_1, \lambda_2 > 0$
Then the term $S^T \nabla f$ is +ve

This indicates that S is a direction of increasing function value

Thus we can conclude that if $\lambda_1, \lambda_2 > 0$, we will not get any better solution than the current solution



Feasible region
$g_1(X) = 0$
$g_1(X) < 0$
$g_2(X) < 0$
S
$g_1(X) > 0$
∇g_1
$g_2(X) = 0$
Infeasible region
∇g_2
Infeasible region
$g_2(X) > 0$

Multivariable problem with inequality constraints

The necessary conditions to be satisfied at constrained minimum points X^* are

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\ldots,n$$

$$\lambda_j \geq 0 \qquad j \in J_1$$

These conditions are called **Kuhn-Tucker conditions,** the necessary conditions to be satisfied at a relative minimum of $f(X)$.

These conditions are in general not sufficient to ensure a relative minimum, However, in case of a convex problem, these conditions are the necessary and sufficient conditions for global minimum.

Multivariable problem with inequality constraints

If the set of active constraints are not known, the Kuhn-Tucker conditions can be stated as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\ldots,n$$

$$\left.\begin{aligned} \lambda_j g_j &= 0 \\ g_j &\le 0 \\ \lambda_j &\ge 0 \end{aligned}\right\} \qquad j = 1,2,3,\ldots,m$$

Multivariable problem with equality and inequality constraints

For the problem

Minimize $f(X)$ Where $X = [x_1, x_2, x_3, \dots, x_n]^T$

Subject to $g_j(X) = 0$ $\quad j = 1,2,3,\dots,m$

$k_k(X) = 0$ $\quad k = 1,2,3,\dots,p$

The Kuhn-Tucker conditions can be written as

$$\frac{\partial f(X)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(X)}{\partial x_i} + \sum_{k=1}^{p} \beta_k \frac{\partial h_k(X)}{\partial x_i} = 0 \qquad i = 1,2,3,\dots,n$$

$$\lambda_j g_j = 0 \qquad j = 1,2,3,\dots,m$$

$$g_j \le 0 \qquad j = 1,2,3,\dots,m$$

$$h_k = 0 \qquad k = 1,2,3,\dots,p$$

$$\lambda_j \ge 0 \qquad j = 1,2,3,\dots,m$$

Thanks for your attention