

# Assignment 1

1. State TRUE or FALSE giving proper justification for each of the following statements.
  - (a) There exists an unbounded subset  $A$  of  $\mathbb{R}$  such that  $m^*(A) = 5$ .
  - (b) There exists an open subset  $A$  of  $\mathbb{R}$  such that  $[\frac{1}{2}, \frac{3}{4}] \subset A$  and  $m^*(A) = \frac{1}{4}$ .
  - (c) There exists an open subset  $A$  of  $\mathbb{R}$  such that  $m^*(A) < \frac{1}{5}$  but  $A \cap (a, b) \neq \emptyset$  for all  $a, b \in \mathbb{R}$  with  $a < b$ .
  - (d) If  $A$  and  $B$  are open subsets of  $\mathbb{R}$  such that  $A \subsetneq B$ , then it is necessary that  $m^*(A) < m^*(B)$ .
2. Examine whether  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ , where
  - (a)  $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$ .
  - (b)  $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) < +\infty \text{ or } m^*(\mathbb{R} \setminus A) < +\infty\}$ .
  - (c)  $\mathcal{A} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$ .
3. Let  $X$  be an uncountable set. Show that the class  $\{\{x\} : x \in X\}$  generates the  $\sigma$ -algebra  $\{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$ .
4. Let  $\mathcal{S}$  be a class of subsets of a nonempty set  $X$  and let  $A \subset X$ . Show that  $\sigma(\mathcal{S} \cap A) = \sigma(\mathcal{S}) \cap A$ , where for each class  $\mathcal{C}$  of subsets of  $X$ ,  $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$ .
5. Let  $X, Y$  be nonempty sets and let  $f : X \rightarrow Y$ . If  $\mathcal{S}$  is a class of subsets of  $Y$ , then show that  $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$ , where for each class  $\mathcal{C}$  of subsets of  $Y$ ,  $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$ .
6. If  $\mathcal{S}$  is a class of subsets of a nonempty set  $X$  and if  $A \in \sigma(\mathcal{S})$ , then show that there exists a countable subclass  $\mathcal{S}_0$  of  $\mathcal{S}$  such that  $A \in \sigma(\mathcal{S}_0)$ .
7. Prove that every infinite  $\sigma$ -algebra on an infinite set is uncountable.
8. Show that  $\mathcal{B}(\mathbb{R})$  is generated by each of the following classes.
 

(a) $\{(a, +\infty) : a \in \mathbb{R}\}$	(b) $\{(-\infty, a] : a \in \mathbb{Q}\}$
(c) $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$	(d) $\{A \subset \mathbb{R} : A \text{ is compact}\}$
9. Let  $E$  be a Borel subset of  $\mathbb{R}$  and let  $x \in \mathbb{R}$ . Show that  $x + E$  is a Borel subset of  $\mathbb{R}$ .
10. Examine whether  $\mu^*$  is an outer measure on  $\mathbb{R}$ , where for each  $A \subset \mathbb{R}$ ,
  - (a)  $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded.} \end{cases}$
  - (b)  $\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded.} \end{cases}$
11. Consider the outer measure  $\mu^*$  on  $\mathbb{R}$ , where for each  $A \subset \mathbb{R}$ ,  $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$   
Determine all the  $\mu^*$ -measurable subsets of  $\mathbb{R}$ .
12. If  $\mathcal{S} = \{\emptyset, [1, 2]\}$  and if  $\mu(\emptyset) = 0$ ,  $\mu([1, 2]) = 1$ , then determine the outer measure  $\mu^*$  on  $\mathbb{R}$  induced by the set function  $\mu : \mathcal{S} \rightarrow [0, +\infty)$ .  
Also, determine all the  $\mu^*$ -measurable subsets of  $\mathbb{R}$ .
13. Let  $(X, \mathcal{A})$  be a measurable space and let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  be finitely additive with  $\mu(\emptyset) = 0$ . Show that  $\mu$  is a measure on  $\mathcal{A}$  if either of the following conditions is satisfied.
  - (a) For every increasing sequence  $\{A_n\}_{n=1}^{\infty}$  of sets in  $\mathcal{A}$ ,  $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$ .
  - (b) For every decreasing sequence  $\{A_n\}_{n=1}^{\infty}$  of sets in  $\mathcal{A}$  with  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ ,  $\lim_{n \rightarrow \infty} \mu(A_n) = 0$ .

14. Let  $\mu^*$  be an outer measure generated by a premeasure  $\mu_o$  on an algebra  $\mathcal{A}$  on a nonempty set  $X$ . Show that  $E \subseteq X$  is  $\mu^*$ -measurable iff for each  $\varepsilon > 0$ , there exists a  $\mu^*$ -measurable set  $F$  contained in  $E$  such that  $\mu^*(E \setminus F) < \varepsilon$ .
15. Let  $f : [0, 2) \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$   
Find  $m^*(A)$ , where  $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$ .
16. Let  $B \subset A \subset \mathbb{R}$  such that  $m^*(B) = 0$ . Show that  $m^*(A \setminus B) = m^*(A)$ .
17. Let  $A \subset \mathbb{R}$  such that  $m^*(A) > 0$ . Show that there exists  $B \subset A$  such that  $B$  is bounded and  $m^*(B) > 0$ .
18. If  $G$  is a nonempty open subset of  $\mathbb{R}$ , then show that  $m^*(G) > 0$ .
19. Let  $A$  be a countable subset of  $\mathbb{R}$  and let  $B \subset \mathbb{R}$  such that  $m^*(B) = 0$ . Show that  $m^*(A+B) = 0$ .
20. Prove or disprove: A subset  $E$  of  $\mathbb{R}$  is Lebesgue measurable iff  $m^*(A \cup B) = m^*(A) + m^*(B)$  for each  $A \subset E$  and for each  $B \subset \mathbb{R} \setminus E$ .
21. Let  $E = \{x \in [0, 1] : \text{The decimal representation of } x \text{ does not contain the digit } 5\}$ . Show that  $m(E) = 0$ .
22. Let  $A_n \subset \mathbb{R}$  for  $n \in \mathbb{N}$  such that  $\sum_{n=1}^{\infty} m^*(A_n) < \infty$ . If  $E = \{x \in \mathbb{R} : x \in A_n \text{ for infinitely many } n\}$ , then show that  $m(E) = 0$ .
23. Show that a subset  $E$  of  $\mathbb{R}$  is Lebesgue measurable iff  $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$  for every bounded open interval  $I$  of  $\mathbb{R}$ .
24. Let  $A \subset E \subset B \subset \mathbb{R}$  such that  $A, B$  are Lebesgue measurable and  $m(A) = m(B) < \infty$ . Show that  $E$  is Lebesgue measurable.  
More generally, let  $A \subset B \subset \mathbb{R}$  such that  $A$  is Lebesgue measurable and  $m^*(B) = m(A) < \infty$ . Show that  $B$  is Lebesgue measurable.
25. Let  $A, B \subset \mathbb{R}$  such that  $m^*(A) = 0$  and  $A \cup B$  is Lebesgue measurable. Show that  $B$  is Lebesgue measurable.
26. Let  $A, B \subset \mathbb{R}$  such that  $A$  is Lebesgue measurable and  $m^*(A \Delta B) = 0$ . Show that  $B$  is Lebesgue measurable.
27. Let  $A, B \subset \mathbb{R}$  be such that  $A \cap B$  is Lebesgue measurable and  $m^*(A \Delta B) = 0$ . Show that  $A$  and  $B$  are Lebesgue measurable and  $m(A) = m(B)$ .
28. Let  $A \subset \mathbb{R}$  such that  $A \cap B$  is Lebesgue measurable for every bounded subset  $B$  of  $\mathbb{R}$ . Show that  $A$  is Lebesgue measurable.
29. Let  $E$  be a Lebesgue measurable subset of  $\mathbb{R}$  and let  $A \subset \mathbb{R}$ . Show that  $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$ .
30. Let  $I$  and  $J$  be disjoint open intervals in  $\mathbb{R}$  and let  $A \subset I, B \subset J$ . Show that  $m^*(A \cup B) = m^*(A) + m^*(B)$ .