Assignment 3

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) If $f((x_n)) = \sum\limits_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$ for all $(x_n) \in \ell^1$, then the linear functional $f : (\ell^1, \|\cdot\|_2) \to \mathbb{K}$ is continuous.

   (b) If X is a normed linear space and $f \in X^*$, then $\{x \in X : f(x) \neq 1\}$ must be dense in X.

2. Let f be a linear functional on a normed linear space X. Then f is bounded if and only if $\ker f$ is closed.

3. Let X^* denote the dual space of a normed linear space X. For $x \in X$, show that $\|x\| = \sup\{|f(x)| : \ f \in X^* \text{ and } \|f\| = 1\}$.

4. Let $1 \leq p < \infty$. Define a linear map $T : l^p \to l^p$ by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Find the adjoint operator T^* of T.

5. Show that the linear map $T : (C^1[0,1], \|.\|) \to (C[0,1], \|.\|)$ defined by $(Tf)(t) = f'(t)$ does not have the continuous adjoint.

6. Let X and Y be two normed linear spaces. Suppose $T \in B(X, Y)$. Show that $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

7. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\mathbb{R})$, prove that

$$\|f\|_p = \sup\left\{\left|\int_{\mathbb{R}} f(x)g(x)dx\right| : g \in L^q(\mathbb{R}) \text{ and } \|g\|_q = 1\right\}.$$

8. Define a family of linear functionals $f_n : c_o(\mathbb{N}) \to \mathbb{C}$ by $f_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j$. Show that $\lim\limits_{n\to\infty} f_n(x) = 0$ but $\|f_n\| = 1$.

9. Let c_o be the space of all sequences on $\mathbb{C}$ that converges to 0. Show that the dual of $(c_o, \|\cdot\|_\infty)$ is isomorphic to $(l^1, \|\cdot\|_1)$

10. Let Y be a proper dense subspace of a normed linear space X. Show that the identity operator on Y cannot be extended as a continuous operator from Y to X.

11. Let M be a proper subspace of a normed linear space X. Suppose $\text{dist}(x_o, M) = \delta > 0$ for some $x_o \notin M$. Prove that there exists $f \in X^*$ such that $\|f\| = 1, \ f(x_o) = \delta$ and $f(x) = 0$ for all $x \in M$. Does such f exist uniquely?

12. Let $T : l^2 \to l^2$ be a linear map such that $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Find the adjoint T^* of T.

13. Let $M = \{(y_1, y_2, \ldots) \in l^2 : 2y_1 - y_2 = 0\}$ and $x_o = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. Find $y_o \in M$ such that $\text{dist}(x_o, M) = \|x_o - y_o\|_2$.

14. Let $\{e_1, e_2 \ldots, e_n\}$ be a linearly independent set in an infinite dimentional normed linear space X. For $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, prove that there exists $f \in X^*$ such that $f(e_j) = a_j$, for $j = 1, 2, \ldots, n$.

15. Suppose the sequence $g_n \in L^2[0,1]$ is defined by

$$g_n(t) = \begin{cases} \sqrt{n} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

Show that $\|g_n\|_2 = 1$ and g_n converges weakly to 0.

16. For $f \in L^2[-\pi, \pi]$, define a sequence (φ_n) of linear functionals by $\varphi_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$. Show that $\|\varphi_n\| = 1$ and $\varphi_n(f) \to 0$.

17. Let c_o be the space of all sequences converging to zero. Show that $(c_o)^* = l^1$ and $(c_o)^{**} = l^\infty$. Further, for $x = (x_n) \in c_o$, show that $x \mapsto \sum_1^\infty x_n$ is weakly continuous but not weak* continuous.

18. Let X and Y be two Banach spaces and $T_n, T \in B(X, Y)$. If $T_n \to T$ weakly. Show that $\sup \|T_n\| < \infty$.

19. Let $X = (C[0,1], \|\cdot\|_\infty)$. Define a map $T : X \to \mathbb{C}$ by $T(f) = \int_0^1 tf(t)dt$, for all $f \in X$. Find a vector $f \in X$ such that $T(f) = \|T\|$.

20. Suppose X and Y be two Banach spaces and $T : X \to Y$ such that $f \circ T \in X^*$, for all $f \in Y^*$. Show that T is continuous.

21. Let X and Y be two normed linear spaces. For $T \in B(X, Y)$, define $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$, for all $f \in Y^*$. Show that
(a) $\ker T^* = (\text{Im}T)^\perp$.
(b) T is bijective then T^* is bijective.

22. Let X and Y be two Banach spaces. Suppose $S : X \to Y$ and $T : Y^* \to X^*$ be linear maps satisfying $f \circ S = T(f)$, for all $f \in Y^*$. Show that S is continuous. (Hint: use close graph theorem).

23. Let X and Y be two Banach spaces and $T \in B(X, Y)$ be such that range $\mathcal{R}(T)$ is closed. Prove that $\mathcal{R}(T^*) = (\ker T)^\perp$, where $M^\perp = \{f \in X^* : \ f(x) = 0, \ \forall x \in M\}$, for $M \subseteq X$.

24. Let X be a reflexive Banach space and $f \in X^*$. Show that there exists $x \in \overline{B(0,1)}$ such that $f(x) = \|f\|$.

25. Let K be a closed bounded convex subset of a reflexive Banach space X. Prove that K is weakly compact.

26. Suppose M is a subspace of a Banach space X. Then $M^\perp$ is weak* closed subspace of X^*.

27. Let X be a normed linear space and let $f(\neq 0) \in X^*$. If $x_0 \in X$ and if $\alpha \in \mathbb{K}$, then show that $d(x_0, \{x \in X : f(x) = \alpha\}) = \frac{|f(x_0) - \alpha|}{\|f\|}$.

28. Let $Y = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\}$ and let $g(x, y) = x$ for all $(x, y) \in Y$. Determine all the Hahn-Banach extensions of g to $(\mathbb{R}^2, \|\cdot\|_2)$.

29. Let (x_n) be a sequence in a Banach space X and let (f_n) be a sequence in X^*. Prove the following:
(a) If $x_n \xrightarrow{w} x \in X$, then (x_n) is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.
(b) If $f_n \xrightarrow{w^*} f \in X^*$, then (f_n) is bounded and $\|f\| \leq \liminf_{n\to\infty} \|f_n\|$.

30. Let X be a Banach space, let $x \in X$ and let $f \in X^*$. If (x_n) is a sequence in X such that $x_n \to x$ and if (f_n) is a sequence in X^* such that $f_n \xrightarrow{w^*} f$, then show that $f_n(x_n) \to f(x)$.