Assignment 4

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) Every orthonormal set in a Hilbert space H must be closed in H.
   (b) In any infinite dimensional Hilbert space, there exists a convergent series which is not absolutely convergent.
   (c) If (x_n) is a sequence in a Hilbert space H such that $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge in H.
   (d) If (u_n) is an orthonormal sequence in a Hilbert space H and if $x \in H$, then the series $\sum\limits_{n=1}^{\infty} \langle x, u_n\rangle u_n$ must converge in H but not necessarily to x.
   (e) If $\{u_n : n \in \mathbb{N}\}$ is a countably infinite orthonormal basis of a Hilbert space H and if $x \in H$, then the series $\sum\limits_{n=1}^{\infty} \langle x, u_n\rangle u_n$ in H must be absolutely convergent.
   (f) If in a Hilbert space H, every weakly convergent sequence is norm convergent, then H must be separable.

2. Let $(X, \langle ., .\rangle)$ be an inner product space. Prove the following generalized parallelogram law.

$$\sum_{\epsilon_k=\pm 1} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|^2 = 2^n \sum_{k=1}^{n} \|x_k\|^2 .$$

3. Let ω be a primitive nth root of unity and $n > 2$. Then Show that for x, y in an inner product space X following holds.

$$\langle x, y\rangle = \frac{1}{n} \sum \omega^p \|x + \omega^p y\|^2 .$$

4. Let X be an inner product space, let $x \in X$ and let (x_n) be a sequence in X such that $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$. Show that $x_n \to x$ in X.

5. Let X be an inner product space and (x_n) and (y_n) be sequences in $\overline{B(0,1)}$ such that $\|\frac{1}{2}(x_n + y_n)\| \to 1$ as $n \to \infty$. Show that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

6. Let X be an inner product space and let $y, z \in X$. If $Tx = \langle x, y\rangle z$ for all $x \in X$, then show that $\|T\| = \|y\|\|z\|$.

7. Consider $C_{\mathbb{R}}[0,1]$ with the usual inner product. Let $S = \{p_n : n = 0, 1, 2, \ldots\}$, where $p_n(t) = t^n$ for all $t \in [0,1]$ and for $n = 0, 1, 2, ...$ Prove that the orthogonal complement of S in $C_{\mathbb{R}}[0,1]$ is $\{0\}$.

8. Let M be a closed subspace of a Hilbert space H. If $x \in M$ and if (x_n) is a sequence in M, then show that $x_n \xrightarrow{w} x$ in H iff $x_n \xrightarrow{w} x$ in M.

9. Using Riesz representation theorem, show that $\{(x_n) \in l^2 : \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\}$ is not a closed subset of the Hilbert space l^2.

10. Determine $\|f\|$ for the linear functional $f : (l^2, \|\cdot\|_2) \to \mathbb{C}$, defined by $f(x) = \sum\limits_{n=1}^{\infty} \frac{x_n}{\sqrt{n(n+1)}}$, for all $x = (x_1, x_2, \ldots) \in l^2$.

11. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H. If $f(x) = \sum\limits_{n=1}^{\infty} \frac{1}{3^n}\langle x, e_n\rangle$ for all $x \in H$, then determine $\|f\|$.

12. Let H be a Hilbert space and let (T_n) be a sequence in $B(H, H)$ such that for each $x, y \in H$, $\lim\limits_{n\to\infty} \langle T_n x, y\rangle$ exists. Show that $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.

13. Let $(X, \|\cdot\|)$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. If $\|x\|_0 = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|\langle x, e_n\rangle|$ for all $x \in X$, then show that $\|\cdot\|_0$ is a norm on X which is not equivalent to $\|\cdot\|$.

14. Let $T : L^2[0, 1] \to L^2[0, 1]$ be a linear map which is defined by $(Tf)(x) = \int\limits_0^x f(t)dt$. Define $\langle Tf, g\rangle = \langle f, T^*\rangle$. Find the adjoint operator T^* of T.

15. Let T be linear operator on a Hilbert space H such that $\langle Tx, y\rangle = \langle x, Ty\rangle$. Show that T is continuous.

16. Let H be a Hilbert space and let $T : H \to H$ be linear. If $\langle T^2x, x\rangle \geq 0$ and $\langle Tx, x\rangle = 0$ for all $x \in H$, then show that $T = 0$.

17. If A is a subset of an inner product space such that $A^0 \neq \emptyset$, then show that $A^\perp = \{0\}$.

18. If A is a dense subset of an inner product space, then show that $A^\perp = \{0\}$.

19. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space H. Define a linear map $T : H \to H$ by $T(e_n) = a_n e_n,\ n = 1, 2, \ldots$. Show that T is bounded if and only if sequence $\{a_n\}$ is bounded.

20. Let T be a normal operator on a Hilbert space H. Show that $\|T^2\| = \|T\|^2$.

21. Let M be a closed subspace of a Hilbert space H and let $x \in H \setminus M$. Prove that $d(x, M) = \sup\{|\langle x, y\rangle| : y \in M^\perp,\ \|y\| \leq 1\}$.

22. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H and $(\alpha_1, \alpha_2, \ldots) \in l^\infty$. Define a linear map $T : H \to H$ by $T(x) = \sum\limits_{n=1}^{\infty} \langle x, e_n\rangle \alpha_n e_n$. Prove that T is continuous and $\|T\| = \sup |\alpha_n|$.

23. Let φ be a bounded function on $\mathbb{R}$. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T(f)(t) = \varphi(t)f(t)$. Show that T is a bounded operator. Find the adjoint operator T^* of T.

24. Suppose T is a bounded linear operator on a complex Hilbert space H such that $\langle Tx, x\rangle = 0$, for all $x \in H$. Show that $T = 0$.

25. Let T be a bounded and self-adjoint operator on a Hilbert space H. Suppose there exists $k > 0$ such that $\|Tx\| \geq k\|x\|$, for each $x \in H$. Prove that the equation $Tx = y$ has a unique solution for each $y \in H$.

26. Let $E = \{e_\alpha : \alpha \in I\}$ be an orthonormal basis of a Hilbert space H. For each $x \in H$, show that the set $\left\{e_\alpha \in E : |\langle x, e_\alpha\rangle|^2 > \left(\frac{2}{n} - e^{-n}\right)\|x\|\right\}$ is a finite set for each fixed $n \in \mathbb{N}$.

27. Let M be a proper closed subspace of a Hilbert space H. Define $\pi : H \to H/M$ by $\pi(x)\| = x + M$. Show that $\|\pi\| = 1$. Further, if there exists $0 \neq x \in H$ such that $\|\pi(x) = \|x\|$, then there exists a unique $0 \neq z \in M$ such that $\|z\|^2 = 2\mathrm{Re}\langle x, z\rangle$.