Assignment 4

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) Whether $L^1(X, S, \mu)$ has an almost non-zero function for every measure space (X, S, μ)?
   (b) Let $f : (X, S, \mu) \to [0, \infty]$ be such that $\|f\|_1 > 0$. Does there exist some $n \in \mathbb{N}$ such that $\mu\{x \in \mathbb{X} : |f(x)| < n\} > 0$?
   (c) There exists a Lebesgue measurable function f on $(\mathbb{R}, M, m)$ such that $\int_E f dm$ is finite for every $E \in M$ but $f \notin L^1(\mathbb{R}, M, m)$.
   (d) For $n \in \mathbb{N}$, define $f_n = \chi_{(n,n+1)}$. Then there exists a measurable set $E \in M(\mathbb{R})$ with $m(E) = \infty$ such that f_n converges to 0 uniformly on E.
   (e) Suppose $f_n \in L^+(\mathbb{R}, M, m)$ be converges to f point-wise. If $\int_{\mathbb{R}} f_n dm \leq M < \infty$, $\forall\ n \leq 1$. Then $\int_{\mathbb{R}} f dm = \lim \int_{\mathbb{R}} f_n dm$.
2. Let μ be the counting measure on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let $f : \mathbb{N} \to [0, +\infty]$. Show that $\int\limits_E f\, d\mu = \sum\limits_{n\in E} f(n)$ for every $E \subset \mathbb{N}$ and hence, in particular, $\int\limits_{\mathbb{N}} f\, d\mu = \sum\limits_{n=1}^{\infty} f(n)$.
3. Let δ_x be the Dirac measure at $x \in X$ on the measurable space $(X, \mathcal{P}(X))$. If $f : X \to [0, +\infty]$ and $E \subset X$, then show that $\int\limits_E f\, d\delta_x = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$
   (Hence, in particular, $\int\limits_X f\, d\delta_x = f(x)$.)
4. Let μ_n be a sequence of measures on (X, S). For $E \in S$, define $\mu(E) = \sum\limits_{n=1}^{\infty} \mu_n(E)$. If $f \in L^+(X, S, \mu)$, then prove that $\int\limits_X f d\mu = \sum\limits_{n=1}^{\infty} \int\limits_X f d\mu_n$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f = \frac{1}{\sqrt{x}}\chi_{(0,1)}$. Let $g(x) = \sum\limits_{r_n\in\mathbb{Q}} 2^{-n} f(x - r_n)$, then show that the function g belongs to $L^1(\mathbb{R}, M, m)$.
6. Let $f_n = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}$. Construct an increasing sequence $\{g_n\}$ of measurable functions on $(\mathbb{R}, M, m,)$ in terms of f_n such that $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} g_n dm < \infty$.
7. For each $x \in [0, 1]$, let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ for some } k, n \in \mathbb{N} \text{ with g.c.d.}(k, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$
   Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$.
8. Let $f, g : (X, S, \mu) \to [0, +\infty]$ be measurable. If $\lambda(E) = \int\limits_E f\, d\mu$ for all $E \in \mathcal{S}$, then show that λ is a measure on $(X, \mathcal{S})$ and that $\int\limits_X g\, d\lambda = \int\limits_X gf\, d\mu$. Does $\lambda(E) = 0$ imply $\mu(E) = 0$?
9. For each $x \in [0, 1]$, let $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$
   Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$.
10. Let $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$
    Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$, where C denotes the Cantor ternary set in $[0, 1]$.

11. Evaluate the Lebesgue integrals: (a) $\int\limits_{[0,+\infty)} e^{-[x]}\, dm(x)$ (b) $\int\limits_{(0,1]} \frac{1}{\sqrt[3]{x}}\, dm(x)$

12. Let $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{\mathbb{R}} f\, dm$.

13. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{(0,+\infty)} f\, dm$.

14. Evaluate the following: (a) $\lim\limits_{n\to\infty} \int\limits_{-2}^{2} \frac{x^{2n}}{1+x^{2n}}\, dx$ (b) $\lim\limits_{n\to\infty} \int\limits_{[0,1]} \frac{1+nx}{(1+x)^n}\, dx$ (c) $\int\limits_{0}^{1} (\sum\limits_{n=1}^{\infty} \frac{x^n}{n})\, dx$
(d) $\lim\limits_{n\to\infty} \int\limits_{1}^{\infty} \frac{1}{1+x^{2n}}\, dx$ (e) $\sum\limits_{n=0}^{\infty} \int\limits_{0}^{1} \frac{x^2}{(1+x^2)^n}\, dx$ (f) $\lim\limits_{n\to\infty} \int\limits_{[0,\infty)} \dfrac{n^2 x e^{-x^2}}{n^2 + x^2}\, dx$

15. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Define a set function $\nu : S \to \overline{\mathbb{R}}$ by $\nu(E) = \int_E f d\mu$, whenever $E \in S$. Show that $\nu(X)$ is finite if $f \in L^1(X, S, \mu)$. Does the converse true?

16. For $f \in L^+ \cap L^1(\mathbb{R}, M, m)$, define $g(x) = \sum\limits_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Show that $g \in L^1(\mathbb{R}, M, m)$ and $\int\limits_{\mathbb{R}} g dm = \int\limits_{\mathbb{R}} f dm$.

17. Construct a function $f \in L^1(\mathbb{R}, M, m)$ such that $\lim n^2 m\{x \in \mathbb{R} : |f(x)| \geq n\} = \infty$.

18. Let $f \in L(X, S, \mu)$. Suppose there exists an increasing sequence $E_n \in S$ such that $\cup_{n=1}^{\infty} E_n = X$ and $\lim \int_{E_n} |f| d\mu < \infty$. Show that $f \in L^1(X, S, \mu)$.

19. Suppose $f_n, f : (X, S, \mu) \to [0, \infty]$ are measurable functions such that f_n converges to f point-wise and $f_n \leq f$. Show that $\int_X f d\mu = \lim \int_X f_n d\mu$.

20. Let $f_n : (X, S, \mu) \to \bar{\mathbb{R}}$ be sequence of measurable functions that f_n increases to f point-wise. If $f, f_n \in L^1(X, S, \mu)$, then show that $\overline{\lim} \int_X f_n d\mu \leq \int_X f d\mu$.

21. Let $f_n : X \to [0, \infty]$ be a sequence of measurable functions and $f_n \to f$ point wise. Suppose there exists $M > 0$ such that $\sup\limits_{n\geq 1} \int_X f_n \leq M$. Show that $f \in L^1(X, S, \mu)$.

22. Let $f \in L^1(X, S, \mu)$. Then show that for each $\epsilon > 0$ there exists $\delta > 0$ and set $E \in S$ such that $\int\limits_E |f|\, d\mu < \epsilon$, whenever $\mu(E) < \delta$.

23. Let $f \in L^1(X, S, \mu)$ be arbitrary and let $E_n = \{x \in X : |f(x)| \geq n\}$. If $0 < p \leq 1$, then show that $\lim\limits_{n\to\infty} n^p\, \mu(E_n) = 0$.

24. Let $f \in L^1(\mathbb{R}, M, m)$ be such that $\int\limits_I f = 0$, for any open interval $I \subset \mathbb{R}$, then show that $f = 0$.

25. Let $\mu(\mathbb{R}) < \infty$ and $f_n \in L^1(\mathbb{R}, M, \mu)$ be such that $f_n \to f$ uniformly. Show that $f \in L^1(X, S, \mu)$ and $\int\limits_X f = \lim \int\limits_X f_n$.

26. Let $f_n : X \to [0, \infty]$ be a decreasing sequence of measurable functions and $f_n \to f$ point wise. If $f_1 \in L^1(X, S, \mu)$. Then show that $\int\limits_X f = \lim \int\limits_X f_n$.

27. Let $f_n, g : X \to \overline{\mathbb{R}}$ be measurable functions such that $f_n \leq g$, $\forall\, n \in \mathbb{N}$ and $g \in L^1(X, S, \mu)$. Show that $\limsup \int\limits_X f_n \leq \int\limits_X \limsup f_n$.

28. Let $f_n : X \to [0, \infty]$ be a sequence of measurable functions and $f_n \to f$ point wise such that $\int_X f = \lim \int_X f_n < \infty$. Show that $\int_E f = \lim \int_E f_n$, for any $E \in S$.

29. Let $f, g, f_n, g_n \in L^1(X, S, \mu)$ be such that $|f_n| \leq g_n,\ f_n \to f$ and $g_n \to g$ point wise. Show that $\int_X g = \lim \int_X g_n$ implies $\int_X f = \lim \int_X f_n$.

30. Let $f_n, f \in L^1(X, S, \mu)$ be such that $f_n \to f$ point wise. Prove that $\lim \int_X |f_n - f| = 0$ if and only if $\int_X |f| = \lim \int_X |f_n|$.

31. $f : X \to [0, \infty]$ be a measurable function. Show that f is integrable on (X, S, μ) if and only if $\sum_{n=-\infty}^{\infty} 2^n \mu\{x \in X : 2^n \leq f(x) \leq 2^{n+1}\} < \infty$.

32. Let $\mu(X) < \infty$ and $f : X \to [0, \infty]$ be a measurable function. Show that $f \in L^1(X, S, \mu)$ if and only if $\sum_{n=0}^{\infty} \mu\{x \in X : f(x) \geq n\} < \infty$.