Assignment 5

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $L^\infty(X, S, \mu)$ contains an almost non-zero function for every measure space (X, S, μ).
   (b) If $f : (X, S, \mu) \to \mathbb{R}$ is bounded almost everywhere, then f is measurable.
   (c) If for $1 \leq p < \infty$, $L^\infty(X, S, \mu) \subset L^p(X, S, \mu)$, then μ is a finite measure.
   (d) Let $\mathcal{S}(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha \in \mathbb{N}$. Then $\mathcal{S}(\mathbb{R})$ is dense $L^2(\mathbb{R})$.
2. Show that the space of all essentially bounded simple functions is dense in $L^\infty(X, S, \mu)$.
3. Let (X, S, μ) be a measure space and $1 \leq p < \infty$. For $f \in L^p(X, S, \mu)$ and $\alpha > 0$ show that $\mu\{x \in X : |f(x)| \geq \alpha\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$.
4. Suppose that $f_n \in L^p(X, S, \mu)$, for $1 \leq p < \infty$, with $\|f_n\|_p \leq 1$ and $f_n \to f$ point-wise a.e. Show that $f \in L^p(X, S, \mu)$ and $\|f\|_p \leq 1$.
5. Let (X, S, μ) be a measure space and $0 < p < 1$. Then for $f, g \in L^+ \cap L^p(X, S, \mu)$ show that $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.
6. Let $\{E_n\}$ be sequence of disjoint measurable sets. Show that $\sum_{n=1}^\infty \alpha_i \chi_{E_i} \in L^p(X, S, \mu)$ if and only if $\sum_{n=1}^\infty |\alpha_i|^p \mu(E_i) < \infty$.
7. Let f and g be disjointly supported functions in $L^p(X, S, \mu)$. Prove that $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$.
8. Let $1 \leq p < \infty$ $f \in L^p(\mathbb{R}, M, m)$. Then show that $\|f(x+h) - f(x)\|_p \to 0$ as $|h| \to 0$.
9. Let (X, S, μ) be a finite measure space. Let $1 \leq p < q \leq \infty$, where $p^{-1} + q^{-1} = 1$. For $f \in L^q(X, S, \mu)$, show that $\|f\|_p \leq (\mu(X))^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_q$. Further, deduce that $L^q(X, S, \mu)$ is a proper dense subspace of $L^p(X, S, \mu)$.
10. Suppose $f \in L^\infty(X, S, \mu)$ is supported on a set of finite measure. Then show that f is in $L^p(X, S, \mu)$ for all $p \geq 1$ and $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$.
11. For $1 < p < \infty$, prove that $L^1(\mathbb{R}, M, m) \cap L^p(\mathbb{R}, M, m)$ is a proper dense subspace of $L^p(\mathbb{R}, M, m)$.
12. Let $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} = r^{-1}$. If $f \in L^p(X, S, \mu)$ and $g \in L^q(X, S, \mu)$, then prove that $fg \in L^1(X, S, \mu)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$. (A generalized Holder's inequality.)
13. Let $1 \leq p < q < r \leq \infty$. Then prove that $L^q(X, S, \mu) \subset L^p(X, S, \mu) + L^r(X, S, \mu)$.
14. Let $1 \leq p < q < r \leq \infty$. Show that $L^p(X, S, \mu) \cap L^r(X, S, \mu) \subset L^q(X, S, \mu)$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda \in (0, 1)$ is given by $q^{-1} = \lambda p^{-1} + (1 - \lambda) r^{-1}$.
15. Let $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. For $f \in L^p(X, S, \mu)$, prove that
$$\|f\|_p = \sup\left\{\left|\int_X fg d\mu\right| : g \in L^q(X, S, \mu) \text{ and } \|g\|_q = 1\right\}.$$
16. Let (X, S, μ) be a σ-finite measure space. Then show that $\|f\|_\infty = \sup\limits_{\|g\|_1 = 1} \left|\int_X fg d\mu\right|$.
17. Let $1 \leq p < \infty$ and $f \in L^+(X, S, \mu) \cap L^p(X, S, \mu)$. Define $f_n(x) = \min\{n, f(x)\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty} \int_X |f_n - f|^p d\mu = 0$.

18. Let $\mathcal{B}(\mathbb{R}^2)$ be the σ-algebra generated by Borel subsets of $\mathbb{R}^2$ (i.e. σ-algebra generated by open subsets of $\mathbb{R}^2$). Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

19. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Show that $G_f = \{(x, y) \in X \times \mathbb{R},\ y = f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$. If $(X, S, \mu) = (\mathbb{R}, M, m)$, then show that $m \times m(G_f) = 0$.

20. Let(X, S, μ) be a σ-finite measure space. Let $f : (X, S, \mu) \to [0, \infty]$ be measurable. Show that $A_f = \{(x, y) \in X \times [0, \infty],\ y \leq f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times m(A_f) = \int_X f(x) d\mu(x)$.

21. Let $f(x, y) = e^{-xy} \sin x$ and $D = [0, \infty) \times [1, \infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ and $\int_0^\infty \int_1^\infty f(x, y) dy dx = \int_1^\infty \int_0^\infty f(x, y) dx dy$.

22. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$ and $D = [0, 1] \times [1, \infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, M \otimes M, m \times m)$.

23. Let $f \in L^1(X, S, \mu)$ and $g \in L^1(Y, T, \nu)$. Define $\varphi(x, y) = f(x)g(y)$. Show that φ is measurable and $\varphi \in L^1(X \times Y, S \otimes T, \mu \times \nu)$.

24. Let $f \in L^1(0, a)$ and define $g(x) = \int_x^a \frac{f(t)}{t} dt$. Then show that $g \in L^1(0, a)$ and compute $\int_0^a g(x) dx$.

25. Let $X = Y = [0, 1],\ S = T = \mathcal{B}[0, 1]$ and $\mu = \nu = m$. Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Compute $\int_0^1 \int_0^1 f(x, y) dy dx$ and $\int_0^1 \int_0^1 f(x, y) dx dy$. Whether $f \in L^1(m \times m)$?

26. Let (X, S, μ) be a finite measure space and $f : X \to [1, \infty]$ be a measurable function. Compute $\mu \times m\left\{(x, y) \in X \times \mathbb{R} :\ y < f(x)\right\}$.

27. Let $E, F \in M(\mathbb{R})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \chi_E(x)\chi_F(x - y)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable and $\int\limits_{\mathbb{R}^2} f\, d(m \times m) = m(E)m(F)$.

28. Let $\mathbb{D} = \left\{(x, y) \in \mathbb{R}^2 :\ y \geq x^2 \text{ and } y \leq 1\right\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Find $m \times m(\mathbb{D})$.

29. Let $P(x, y)$ be a polynomial on $\mathbb{R}^2$. Show that the set $S = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 1\}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m\left(S\right)$.

30. Let $f : (\mathbb{R}^2, M \otimes M, m \times m) \to \overline{\mathbb{R}}$ be a measurable function. If either of f^+ or f^- belongs to $L^1\left(\mathbb{R}^2, M \otimes M, m \times m\right)$, then show that $\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f\, dm\, dm = \int\limits_{\mathbb{R}^2} f\, d(m \times m)$.

31. Let $f \in L^1(\mathbb{R}, M, m)$. If $\varphi(x, y) = \dfrac{f(x + y)}{1 + y^2}$, then show that φ is $M \otimes M$- measurable and $\varphi \in L^1\left(\mathbb{R}^2, M \otimes M, m \times m\right)$.