Assignment 5

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $L^\infty(X,S,\mu)$ contains an almost non-zero function for every measure space (X,S,μ).
   (b) If $f:(X,S,\mu)\to\mathbb{R}$ is bounded almost everywhere, then f is measurable.
   (c) If for $1\le p<\infty$, $L^\infty(X,S,\mu)\subset L^p(X,S,\mu)$, then μ is a finite measure.
   (d) Let $\mathcal{T}(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha\in\mathbb{N}$. Then $\mathcal{T}(\mathbb{R})$ is dense $L^2(\mathbb{R})$.
   (e) There exists $f\in L^\infty(X,S,\mu)$ such that $\mu(\{x\in X:|f(x)|=\|f\|_\infty\})=0$.
   (f) Let (X,S,μ) be a σ- finite measure space with $\mu(\{x\})=0$ for all $x\in X$. It is possible that $(\mu\times\mu)(\{(x,y)\in X\times X:x=y\})>0$.
2. Show that the space of all essentially bounded simple functions is dense in $L^\infty(X,S,\mu)$.
3. Let $1\le p<\infty$. For $f\in L^p(X,S,\mu)$ and $\alpha>0$ show that $\mu\{x\in X:|f(x)|\ge\alpha\}\le\left(\frac{\|f\|_p}{\alpha}\right)^p$. Further, for $1<p<\infty$ show that the series $\sum\limits_{n=1}^{\infty}\mu\{x\in X:|f(x)|\ge n\}$ is convergent.
4. Let $1\le p<\infty$ and $f,f_n\in L^p(X,S,\mu)$ be such that $\|f_n-f\|_p\to 0$. For each $\epsilon>0$, show that $\mu\{x\in X:|f_n(x)-f(x)|>\epsilon\}\to 0$.
5. Suppose that $f_n\in L^p(X,S,\mu)$, for $1\le p<\infty$, with $\|f_n\|_p\le 1$ and $f_n\to f$ point-wise a.e. Show that $f\in L^p(X,S,\mu)$ and $\|f\|_p\le 1$.
6. Let (X,S,μ) be a measure space and $0<p<1$. Then for $f,g\in L^+\cap L^p(X,S,\mu)$ show that $\|f+g\|_p\ge\|f\|_p+\|g\|_p$.
7. Let $\{E_n\}$ be sequence of disjoint measurable sets. Show that $\sum_{n=1}^\infty \alpha_i\chi_{E_i}\in L^p(X,S,\mu)$ if and only if $\sum_{n=1}^\infty|\alpha_i|^p\mu(E_i)<\infty$.
8. Let f and g be disjointly supported functions in $L^p(X,S,\mu)$. Prove that $\|f+g\|_p^p=\|f\|_p^p+\|g\|_p^p$.
9. Let $1\le p<\infty$ $f\in L^p(\mathbb{R},M,m)$. Then show that $\|f(x+h)-f(x)\|_p\to 0$ as $|h|\to 0$.
10. Let (X,S,μ) be a finite measure space. Let $1\le p<q\le\infty$, where $p^{-1}+q^{-1}=1$. For $f\in L^q(X,S,\mu)$, show that $\|f\|_p\le(\mu(X))^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_q$. Further, deduce that $L^q(X,S,\mu)$ is a proper dense subspace of $L^p(X,S,\mu)$.
11. Suppose $f\in L^\infty(X,S,\mu)$ is supported on a set of finite measure. Then show that f is in $L^p(X,S,\mu)$ for all $p\ge 1$ and $\lim\limits_{p\to\infty}\|f\|_p=\|f\|_\infty$.
12. For $1<p<\infty$, prove that $L^1(\mathbb{R},M,m)\cap L^p(\mathbb{R},M,m)$ is a proper dense subspace of $L^p(\mathbb{R},M,m)$.
13. Let $1\le p,\ q\le\infty$ and $p^{-1}+q^{-1}=r^{-1}$. If $f\in L^p(X,S,\mu)$ and $g\in L^q(X,S,\mu)$, then prove that $fg\in L^1(X,S,\mu)$ and $\|fg\|_r\le\|f\|_p\|g\|_q$. (A generalized Holder's inequality.)
14. Let $1\le p<q<r\le\infty$. Then prove that $L^q(X,S,\mu)\subset L^p(X,S,\mu)+L^r(X,S,\mu)$.
15. Let $1\le p<q<r\le\infty$. Show that $L^p(X,S,\mu)\cap L^r(X,S,\mu)\subset L^q(X,S,\mu)$ and $\|f\|_q\le\|f\|_p^\lambda\|f\|_r^{1-\lambda}$, where $\lambda\in(0,1)$ is given by $q^{-1}=\lambda p^{-1}+(1-\lambda)r^{-1}$.
16. Let $1\le p<\infty$ and $p^{-1}+q^{-1}=1$. For $f\in L^p(X,S,\mu)$, prove that

$$\|f\|_p=\sup\left\{\left|\int_X fg\,d\mu\right|:g\in L^q(X,S,\mu)\text{ and }\|g\|_q=1\right\}.$$

17. Let (X, S, μ) be a σ-finite measure space. Then show that $\|f\|_\infty = \sup\limits_{\|g\|_1=1} \left| \int\limits_X fg d\mu \right|$.

18. Let $1 \leq p < \infty$ and $f \in L^+(X, S, \mu) \cap L^p(X, S, \mu)$. Define $f_n(x) = \min\{n, f(x)\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty} \int\limits_X |f_n - f|^p d\mu = 0$.

19. Define a linear functional on $L^1(\mathbb{R}, M, m)$ by $T(f) = \int\limits_{\mathbb{R}} \frac{f(x)}{1+|x|} dm(x)$. Show that T is bounded and $\|T\| = 1$.

20. For $f \in L^1(\mathbb{R}, M, m,)$ define $F(x) = \int\limits_{[0,x]} f(t) dm(t)$. Show that $F \in L^1([0,1], M, m)$ and it satisfies $\|F\|_1 \leq \|f\|_1$.

21. Let $\mathcal{A}$ be the monotone class generated by all closed sets in $\mathbb{R}$. If E and F are closed subsets $\mathbb{R}$, then show that $E + F$ belongs to $\mathcal{A}$.

22. Let $\mathcal{B}(\mathbb{R}^2)$ be the σ-algebra generated by Borel subsets of $\mathbb{R}^2$ (i.e. σ-algebra generated by open subsets of $\mathbb{R}^2$). Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

23. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Show that $G_f = \{(x, y) \in X \times \mathbb{R},\ y = f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$. If $(X, S, \mu) = (\mathbb{R}, M, m)$, then show that $m \times m(G_f) = 0$.

24. Let(X, S, μ) be a σ-finite measure space. Let $f : (X, S, \mu) \to [0, \infty]$ be measurable. Show that $A_f = \{(x, y) \in X \times [0, \infty],\ y \leq f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times m(A_f) = \int_X f(x) d\mu(x)$.

25. Let $f : (X, S, \mu) \to [0, 1]$ be a measurable function on the finite measure space (X, S, μ). Show that $A_f = \{(x, y) \in X \times [0, 1] : f(x) \leq y\}$ is $S \otimes \mathcal{B}(\mathbb{R})$- measurable and $(\mu \times m)(A_f) = \mu(X) - \int\limits_X f(x) d\mu(x)$.

26. Let $f(x, y) = e^{-xy} \sin x$ and $D = [0, \infty) \times [1, \infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ and $\int_0^\infty \int_1^\infty f(x, y) dy dx = \int_1^\infty \int_0^\infty f(x, y) dx dy$.

27. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$ and $D = [0, 1] \times [1, \infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, M \otimes M, m \times m)$.

28. Let $f \in L^1(X, S, \mu)$ and $g \in L^1(Y, T, \nu)$. Define $\varphi(x, y) = f(x)g(y)$. Show that φ is measurable and $\varphi \in L^1(X \times Y, S \otimes T, \mu \times \nu)$.

29. Let $f \in L^1(0, a)$ and define $g(x) = \int_x^a \frac{f(t)}{t} dt$. Then show that $g \in L^1(0, a)$ and compute $\int_0^a g(x) dx$.

30. Let $X = Y = [0, 1],\ S = T = \mathcal{B}[0, 1]$ and $\mu = \nu = m$. Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Compute $\int_0^1 \int_0^1 f(x, y) dy dx$ and $\int_0^1 \int_0^1 f(x, y) dx dy$. Whether $f \in L^1(m \times m)$?

31. Let (X, S, μ) be a finite measure space and $f : X \to [1, \infty]$ be a measurable function. Compute $\mu \times m\{(x, y) \in X \times \mathbb{R} :\ y < f(x)\}$.

32. Let $E, F \in M(\mathbb{R})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \chi_E(x)\chi_F(x - y)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable and $\int\limits_{\mathbb{R}^2} f\, d(m \times m) = m(E)m(F)$.

33. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 :\ y \geq x^2 \text{ and } y \leq 1\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Find $m \times m(\mathbb{D})$.

34. Let $P(x, y)$ be a polynomial on $\mathbb{R}^2$. Show that the set $S = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 1\}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m\,(S)$.

35. Let $f : (\mathbb{R}^2, M \otimes M, m \times m) \to \overline{\mathbb{R}}$ be a measurable function. If either of f^+ or f^- belongs to $L^1(\mathbb{R}^2, M \otimes M, m \times m)$, then show that $\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f \, dm \, dm = \int\limits_{\mathbb{R}^2} f \, d(m \times m)$.

36. Let $f \in L^1(\mathbb{R}, M, m)$. If $\varphi(x, y) = \dfrac{f(x + y)}{1 + y^2}$, then show that φ is $M \otimes M$- measurable and $\varphi \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$.