Distribution theory:

We know from the previous section that there are functions in L^p-spaces which are differentiable in L^p-sense. That is, $\exists\, g \in L^p$ s.t. $\|\Delta_h f - g\|_p \to 0$ as $|h| \to 0$. However, there is a large class of functions which are neither differentiable nor their L^p-derivative exist. Though, there is a large sub-class of such functions whose derivative can be realized with the help of certain class of differentiable functions, known as "test functions".

For example, suppose f is diff. and g is a compactly supported differentiable function on $\mathbb{R}$. Then
$$\int_{-\infty}^{\infty} f'g = -fg\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} fg' = -\int_{-\infty}^{\infty} fg',$$
because g is compactly supported. Therefore, this gives way to realize the derivative of $f \in L^1_{loc}(\mathbb{R})$. For $g \in C_c^\infty(\mathbb{R})$, write
$$\Lambda_f(g) = \int_{\mathbb{R}} fg,$$
then the derivative of Λ_f can be defined by $\Lambda_f'(g) = -\int fg'$.

In fact, functional Λ_f is all time diff and its K^{th} derivative is given by

$$D^K \Lambda_f(g) = (-1)^K \int f D^K g \ , \text{ where } D = \frac{d}{dx}.$$

In order to discuss "distributions" in details, we need to devise a complete topology on $C_c^\infty(\mathbb{R}^n)$. Since the space $C_c^\infty(\mathbb{R}^n)$ cannot be made complete under sup norm, a complete top. on $C_c^\infty(\mathbb{R}^n)$ will be derived from a family of semi-norms (defined on compact subsets of $\mathbb{R}^n$). Thus, the space $C_c^\infty(\mathbb{R}^n)$ becomes a locally convex top. space.

Locally Convex topology:

Let $\{p_i : i \in I\}$ be a family of semi-norm on a top. vector space X. For finite set $F \subset I$, let

$$U_{F,\epsilon} = \bigcap_{i\in F} \{x \in X : p_i(x) < \epsilon\} = \bigcap_{i\in F} V_{i,\epsilon}.$$

Then each of $V_{i,\epsilon}$ is convex and balanced.

Let $\mathcal{B} = \{U_{F,\epsilon} : \epsilon > 0, F \subset I, \#(F) < \infty\}$.

Then the collection

$$\tau = \{ O \subset X : \forall x \in O, \exists U \in \mathcal{B} \text{ s.t. } x + U \subset O\}$$ is a topology on X.

Obviously, $\mathcal{T}$ contains ϕ & X and closed under arbitrary union. Now, let (3)

$$O = \bigcap_{i=1}^{k} O_i;\ O_i \in \mathcal{T}.$$

If $x \in O$, then $x \in O_i$ and $\exists\ U_{F_i, \epsilon_i} \in \mathcal{B}$ such that $x + U_{F_i, \epsilon_i} \subset O_i$. Write

$\epsilon = \min_{1 \le i \le k} \epsilon_i$ and $F = \bigcup_{i=1}^{k} F_i$. Then $\epsilon > 0$, and F is finite, and hence

$$x + U_{F, \epsilon} \subset \bigcap_{i=1}^{k} (x + U_{F_i, \epsilon_i}) \subset O.$$

The space $(X, \mathcal{T})$ is known as locally convex top. space.

Ex. Show that a locally convex top. v.s. X is Hausdorff iff $\{p_i : i \in I\}$ seperates pts in X.
(i.e $x \in X$, $x \neq 0$, implies $\exists\ i \in I$ s.t $p_i(x) > 0$).

Ex. Let X be a locally convex Hausdorff space whose topo. is induced by $\{p_i : i \in I\}$. Define

$$d(x, y) = \sum 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

Show that top. $\mathcal{T}_d$ coincides with $\mathcal{T}$.

Note that, in general setting, $U_{F,\epsilon}$ plays the role of $B_\epsilon(0)$ in $\mathbb{R}^n$, as $B_\epsilon(0)$, $\epsilon>0$ forms a local base at '0'. Hence, (4)

$$\mathcal{B} = \{ U_{F,\epsilon} : \epsilon > 0,\ F \subset I,\ \#(F) < \infty \}$$

is a local base at $0 \in X$.

Defn. (i) A seqn $(x_i)_{i=1}^{\infty} \subset X$ is said to ~~converge~~ converges to $x \in X$ if $\forall\, U \in \mathcal{B}$, $\exists\, N = N_U \in \mathbb{N}$, such that $x - x_j \in U$, $\forall\, j \geq N$.

(ii) $(x_i)_{i=1}^{\infty} \subset X$ is called Cauchy seqn if $\forall\, U \in \mathcal{B}$, $\exists\, N \in \mathbb{N}$ s.t.

$$x_k - x_l \in U, \ \forall\, k, l > N.$$

(iii) X is called sequentially complete if every Cauchy seqn in X has limit in X.

Lemma: A sequence $(x_i)_{i=1}^{\infty} \subset X$ converges to $x \in X$ iff $\lim_{i \to \infty} p_\alpha(x_i - x) = 0$, $\forall\, \alpha \in I$.

Proof: Let $U_{j,\epsilon} = \{ x \in X : p_j(x) < \epsilon \}$. Then $\exists$ $N \in \mathbb{N}$ s.t. $p_j(x_i - x) < \epsilon$, $\forall\, j \geq N$. etc.

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1. Theorem: Let $\{p_i : i \in I\}$ be a separating family of semi-norms on a V.S. X, and write $V_{p_i,n} = \{x \in X : p(x) < \frac{1}{n}\}$.
Then $\mathcal{J} = \{V_{p_i,n} : i \in I, n \in \mathbb{N}\}$ forms a convex balanced local base for a topology $\mathcal{T}$ on X, which makes X into a locally convex space such that
(i) each p_i is continuous, and
(ii) a set $E \subset X$ is bounded iff $\forall i \in I$, $p_i(E)$ is bounded.

Proof: Let $x \in X$ and $x \neq 0$, then $\exists p_i$ s.t. $p_i(x) > 0$. Therefore, for some n, $n p_i(x) > 1$. implies $x \notin V(p_i; n)$, a nbd of 0. Hence $\{0\}$ is closed. Since $\mathcal{T}$ is translation-invariant, each $\{x\} \subset X$ is closed in $(X, \mathcal{T})$.

Addition is continuous: Let U be a nbd of $0 \in X$. Then $\bigcap_{i=1}^{m} V(p_i; n_i) \subset U$ (by def^n of $\mathcal{J}$)

Let $V = \bigcap_{i=1}^{m} V(p_i; 2n_i)$. Then $V + V \subset U$.

Consider $(x_1, x_2) \to x_1 + x_2$ and U be an open set containing $x_1 + x_2$. Then $U - (x_1 + x_2)$ is a nbd of 0. Hence, $\exists$ a nbd V of 0 s.t.

$$V+V \subset U-(x_1+x_2)$$

$$\Rightarrow (V+x_1)+(V+x_2) \subset U$$

$\Rightarrow$ addition is continuous.

For scalar multiplication. Let $x \in X$ and $\alpha \in \mathbb{C}$, U and V as above. Then $x \in sV$ for some $s>0$. Write $t = \frac{s}{1+|\alpha|s}$, and $y \in x + tV$ with $|\beta-\alpha| < \frac{1}{s}$. Then

$$\beta y - \alpha x = \beta(y-x) + (\beta-\alpha)x$$
$$\in |\beta| t V + |\beta-\alpha| s V$$
$$\subset V+V \subset U$$

Since $|\beta| t < (|\alpha| + \frac{1}{s})t = 1$, and V is balanced. Thus, $\beta(x+tV) \subset \alpha x + U$, this implies scalar multiplication is continuous.

(ii) Suppose E is a bounded subset of X. Since each $V(p_i, 1)$ is a nhd of 0, $\exists$ $k_i > 0$ s.t.

$$E \subset k_i V(p_i, 1) = V(p_i, k_i)$$

$$\Rightarrow p_i(x) < k_i, \ \forall i, \forall x \in E.$$

Conversely, suppose $p_i(x) < M_i$ $\forall x \in E$, $\forall i \in I$, then for each nhd U of 0,

$$U \supset \bigcap_{i=1}^{m} V(p_i, n_i)$$

$$\Rightarrow E \subset \bigcap_{i=1}^{m} V\left(p_i, \frac{1}{n_i}\right) = \bigcap_{i=1}^{m} M_i n_i V(p_i, n_i).$$

If any M_i, α_i, $i = 1, 2, \dots, m$, then

$$E \subset \alpha \bigcap_{i=1}^{m} V(p_i, \alpha_i) \subset \alpha V.$$

Hence E is bounded in (X, τ).

Topology of the spaces $C^\infty(\Omega)$ and $\mathcal{D}_K$:

We define a topo. on $C^\infty(\Omega)$ which makes $C^\infty(\Omega)$ a Frechet space with Heine-Borel property such that the space

$$\mathcal{D}_K = \{\phi \in C^\infty(\Omega),\ \text{supp}\,\phi \subset K\},$$

where K is a cpt set in Ω, is a closed subspace of $C^\infty(\Omega)$.

Define a seqⁿ of compact sets in Ω such that $K_i \subset K_{i+1}^{\circ}$ by

$$K_i = \{x \in \Omega : d(x, \partial\Omega) \geq \tfrac{1}{i}\} \cap \overline{B_i},$$

where $B_i = \{x \in \mathbb{R}^n : |x| < i\}$.

For $f \in C^\infty(\Omega)$, define

$$p_N(f) = \sup\{|D^\alpha f(x)| : x \in K_N,\ |\alpha| \leq N\}$$

then $\{p_N\}_{N=1}^{\infty}$ is a separating family of semi-norms that make $C^\infty(\Omega)$ a metrizable locally convex top. space (exercise). (By the previous thm)

For $x \in \Omega$, define $\delta_x(f) = f(x)$, then each δ_x is a cont. linear functional in the top. induced by $\{p_N\}_{N=1}^{\infty}$.

That is, $P_N(f_i) \to 0 \Rightarrow |f_i(x)| \le P_N(f_i) \to 0$. (8)

It is easy to see that

$$\mathcal{D}_K = \bigcap_{x \in \Omega \setminus K} \operatorname{Ker} \delta_x .$$

Hence $\mathcal{D}_K$ is a closed subspace of $C^\infty(\Omega)$.

Notice that the collection

$$V_N = \{ f \in C^\infty(\Omega) : P_N(f) < \tfrac{1}{N} \}, \quad N = 1, 2, \ldots$$

forms a convex balanced local base at $0 \in \mathcal{E}^\infty(\Omega)$.

If $\{f_i\}$ is a Cauchy seq^n in $C^\infty(\Omega)$, then for each V_N, $\exists\ l_N \in \mathbb{N}$ s.t.

$$f_i - f_j \in V_N, \quad \forall\ i, j > l_N$$

$$\Rightarrow P_N(f_i, f_j) < \tfrac{1}{N}, \text{ and hence}$$

$$|D^\alpha f_i(x) - D^\alpha f_j(x)| < \tfrac{1}{N}, \quad \forall x \in K_N$$

That is, $D^\alpha f_i \to g_\alpha$ on each compact set K_N in Ω. In particular, $f_i(x) \to g_0(x)$.

Thus, $g_0 \in C^\infty(\Omega)$ and $g_\alpha = D^\alpha g_0$. This implies that $f_i \to g_0$ in the topo. of $\mathcal{E}^\infty(\Omega)$. Hence, $C^\infty(\Omega)$ is a Frechet space and same is true for $\mathcal{D}_K$.

Suppose $E \subset \mathcal{E}^\infty(\Omega)$ is closed and bounded. Then by the previous theorem A, $\exists\ 0 < M_N < \infty$ such that $P_N(f) < M_N$, $N = 1, 2, \ldots$, $\forall f \in E$.

Thus, $|D^\alpha f| < M_N$ on K_N, $|\alpha| \leq N$. Hence $\{D^\beta f : f \in E\}$ is an equicontinuous family on K_{N-1}, if $|\beta| \leq N-1$. (9)

By MVT $|f(x) - f(y)| \leq \|D^1 f\|_\infty |x-y|$ — (1)

Replacing $f \to D^\beta f$ in (1), we get

$$|D^\beta f(x) - D^\beta f(y)| \leq \|D^{\beta+1} f\|_\infty \|x-y\|$$
$$\leq \|f\|_N \|x-y\|.$$

By Arzela-Ascoli theorem, every seq^n in E has convergent subsequence. Hence E is compact in $C^\infty(\Omega)$. Thus, $C^\infty(\Omega)$ has Heine-Borel property.

Since $d(f,0) = \sum 2^{-n} \frac{P_N(f)}{1+P_N(f)} < 2$, the top. on $C^\infty(\Omega)$ is not normable.

Now, for each fixed $K \subset \Omega$, $\mathcal{D}_K$ is a Frechet space and $\mathcal{D}(\Omega) = C_c^\infty(\Omega) = \bigcup_{K \subset \Omega} \mathcal{D}_K$ is known as space of test functions.

For $\varphi \in \mathcal{D}(\Omega)$, define

$$\|\varphi\|_N = \sup\{|D^\alpha \varphi(x)| : x \in \Omega, |\alpha| \leq N\},$$

for $N = 0, 1, 2, \ldots$

Note that restriction of these norms to $\mathcal{D}_K$ gives the same top. as do the semi-norms $\{P_N\}_{N=1}^\infty$. For this, let $K \subset \Omega$, compact.

then $\exists\ N_0 \in \mathbb{N}$ s.t. $K \subset K_N,\ \forall N \geq N_0$ and for these $N \geq N_0$,

$$\|\varphi\|_N = P_N(\varphi),\ \forall\ \varphi \in \mathcal{D}_K$$

Since $\|\varphi\|_N \leq \|\varphi\|_{N+1} \cdots$ and

$$P_N(\varphi) \leq P_{N+1}(\varphi) \cdots,$$

the top. given by either sequence $\{P_N\}_{N=N_0}^{\infty}$ or $\{\|\cdot\|_N\}_{N=N_0}^{\infty}$ will be same. thus, two top. on $\mathcal{D}_K$ coincides. Therefore,

$$V_N = \{\varphi \in \mathcal{D}_K : \|\varphi\|_N < \tfrac{1}{N}\}$$ form a local base for $\mathcal{D}_K$.

notice that $\{\|\cdot\|_N\}_{N=0}^{\infty}$ can be used to define a locally convex metricable top. on $\mathcal{D}(\Omega)$, but this topology is not. complete. for $\varphi \in \mathcal{D}(\Omega)$, supp $\varphi \subset [0,1]$, $\varphi > 0$ on $(0,1)$

$$\varphi_m(x) = \varphi(x-1) + \tfrac{1}{2}\varphi(x-2) + \cdots + \tfrac{1}{m}\varphi(x-m)$$

is a Cauchy seq. in this top. but $\lim \varphi_m$ is not compactly supported. this happens because $\{P_N\}_{N=0}^{\infty}$ is not enough to prevent Cauchy seq. leaking towards the boundary of Ω. So that we can add more semi-norm to the family $\{P_N\}_{N=0}^{\infty}$ that makes more functions on $\mathcal{D}(\Omega)$ to be continuous. Now, we define another top. τ on $\mathcal{D}(\Omega)$ in which Cauchy sequences do converge, however τ is not metrizable.

(i) let $\beta = \{ W \in \mathcal{D}(\Omega) :$ W-convex, balanced sets with $\mathcal{D}_K \cap W \in \tau_K, \forall$ cpt $K \subset \Omega \}$.

(ii) $\tau = \{$unions of the form $\phi + W$, $\phi \in \mathcal{D}(\Omega)$, and $W \in \beta \}$.

Note that the top. τ is different than the top. generated by P_n's, as the top. τ includes more semi-norms. For example let $\varphi \in \mathcal{D}(\Omega)$ and $\{x_i\} \subset \Omega$ be seq^n having no limit pt, then for any $c_i > 0$, only

$$p(\varphi) = \sup c_i |\varphi(x_i)| < \infty \quad (\because \text{ finitely}$$

many i for each φ), is a semi-norm on $\mathcal{D}(\Omega)$ and p restricted to each $\mathcal{D}_K$ is continuous. In fact, $W = \{ \varphi \in \mathcal{D}(\Omega) : p(\varphi) < c \}$ is convex balanced and belongs to β is a τ-nbd of $0 \in \mathcal{D}(\Omega)$.

This forces every τ-bdd set (or c.c. ? in $\mathcal{D}(\Omega)$) to be concentrated on a common compact set $K \subset \Omega$. This we see in the next theorem. That is, a seq^n $\varphi_i \in \mathcal{D}(\Omega)$ converges to 0 if supp $\varphi_i \subset K$, $\forall i = 1, 2, \ldots$

Theorem: (a) τ is topology on $\mathcal{D}(\Omega)$ and β is a local base for τ.

(b) τ makes $\mathcal{D}(\Omega)$ into a locally convex top. v. space.

Proof: To prove (a), it is enough to prove that for $V_1, V_2 \in \tau$ and $\phi \in V_1 \cap V_2$, $\exists$ $W \in \beta$ such that $\phi + W \subset V_1 \cap V_2$.

By definition $\exists$ $\phi_i + W_i \in \tau$ such that

$\phi \in \phi_i + W_i \subset V_i$, $i = 1, 2$.

Choose $K \subset \Omega$ cpt so that $\phi_1, \phi_2, \phi \in \mathcal{D}_K$.

Since $\mathcal{D}_K \cap W_i$ is open in $\mathcal{D}_K$ and $\phi - \phi_i \in \mathcal{D}_K \cap W_i$, it follows that $\phi - \phi_i \in (1 - \delta_i) W_i$ for $\delta_i > 0$.

(this is like, if $x \in B_\epsilon(x) \subset W$, then $x \in (1-\delta) B_{\epsilon/2}(x) \subset (1-\delta)W$)

By the convexity of W_i, we get

$\phi - \phi_i + \delta_i W_i \subset (1 - \delta_i) W_i + \delta_i W_i = W_i$

So $\phi + \delta_i W_i \subset \phi_i + W_i \subset V_i$; $i = 1, 2$.

Hence $\phi + (\delta_1 W_1) \cap (\delta_2 W_2) \subset V_1 \cap V_2$. This proves (a).

(b) Let $\phi_1, \phi_2 \in \mathcal{D}(\Omega)$ be distinct and write

$W = \{ \phi \in \mathcal{D}(\Omega) : \|\phi\|_0 < \|\phi_1 - \phi_2\|_0 \}$.

Then $W \in \mathcal{B}$, and $\phi_2 \notin \phi_1 + W$. Since ϕ_2 is arbitrary, it implies that $\{\phi_1\}$ is closed set relative to τ. (13)

Notice that for every pair of $\psi_1, \psi_2 \in \mathcal{D}(\Omega)$,

$$(\psi_1 + \tfrac{1}{2}W) + (\psi_2 + \tfrac{1}{2}W) = (\psi_1 + \psi_2) + W,$$

hence addition is continuous in $(\mathcal{D}(\Omega), \tau)$.

Pick $\alpha_0 \in \mathbb{C}$, and $\phi_0 \in \mathcal{D}(\Omega)$. Then $\phi_0 \in \frac{1}{2}sW$ for some $s > 0$. Let $|\alpha - \alpha_0| < \frac{1}{s}$ and $t = \frac{s}{2(1+|\alpha_0|s)}$.

Then for $\phi \in \phi_0 + tW$.

$$\begin{aligned} \alpha\phi - \alpha_0\phi_0 &= \alpha(\phi - \phi_0) + (\alpha - \alpha_0)\phi_0 \\ &\in |\alpha| t W + \tfrac{1}{2}W \\ &\in \tfrac{1}{2}W + \tfrac{1}{2}W = W, \end{aligned}$$

Since $|\alpha| t < (|\alpha| + \frac{1}{s})t = \frac{1}{2}$. Thus,

$$\alpha(\phi_0 + tW) \subset \alpha_0\phi_0 + |\alpha| t W \subset \alpha_0\phi_0 + W.$$

Hence scalar multiplication is continuous.

Now onward by $\mathcal{D}(\Omega)$ we mean $(\mathcal{D}(\Omega), \tau)$.

<u>Theorem</u>:

(a) A convex balanced subset $V \in \mathcal{D}(\Omega)$ is open iff $V \in \mathcal{B}$.

(b) Topology τ_K of $\mathcal{D}_K \subset \mathcal{D}(\Omega)$ coincides with the top. on $\mathcal{D}_K$ that inherited from $\mathcal{D}(\Omega)$. (V.V.I)

(c) If E is a bounded subset of $\mathcal{D}(\Omega)$, then $E \subset \mathcal{D}_K$ for some compact $K \subset \Omega$ and $\exists\ 0 \le M_N < \infty$ s.t.

$$\|\phi\|_N \le M_N \quad \forall\ \phi \in E,\ N = 0, 1, 2, \ldots$$

(d) $\mathcal{D}(\Omega)$ has the Heine-Borel property.

(e) $\{\phi_i\}$ is a Cauchy seqn in $\mathcal{D}(\Omega)$, then $\{\phi_i\} \subset \mathcal{D}_K$ for some $K \subset \Omega$, K cpt.

(f) $\phi_i \to 0$ in $\mathcal{D}(\Omega)$, then $\exists$ a cpt set $K \subset \Omega$ s.t. supp $\phi_i \subset K$, $\forall i$ & $\partial^\alpha \phi_i \to 0$ unif, $\forall \alpha$.

(g) In $\mathcal{D}(\Omega)$, every Cauchy seqn is convergent.

Proof: (a) Suppose $V \in \tau$. Claim $V \in \beta$. Consider $\phi \in \mathcal{D}_K \cap V$. By previous theorem, $\exists$ $W \in \beta$ s.t. $\phi + W \subset V$.

$$\Rightarrow \phi + (\mathcal{D}_K \cap W) \subset \mathcal{D}_K \cap V$$

Since $\mathcal{D}_K \cap W$ is open in $\mathcal{D}_K$, implies $\mathcal{D}_K \cap V$ is open in $\mathcal{D}_K$, for each $V \in \tau$. —(*)

Conversely, if $V \in \beta$, then $V \in \tau$, since $\beta \in \tau$.

(b) Let $V \in \tau$, then $\mathcal{D}_K \cap V \in \tau_K$ (by (a)(*)).

that is, $\tau \cap \mathcal{D}_K \subset \tau_K$, $\forall K \subset \Omega$.

Conversely, suppose $E \in \tau_K$, for some $K \subset \Omega$.

Claim. $E = \mathcal{D}_K \cap V$, for some $V \in \mathcal{T}$. (15)

Let $\phi \in E$, then $\exists N$ and $\delta > 0$ s.t

$$\{\psi \in \mathcal{D}_K : \|\psi - \phi\|_N < \delta\} \subset E$$

& $\{\psi \in \mathcal{D}_K : \|\psi\|_N < \delta\} \subset E - \phi$.

Let $W_\phi = \{\psi \in \mathcal{D}_K : \|\psi\|_N < \delta\}$. Then

$W_\phi \cap \mathcal{D}_K \in \mathcal{T}_K$ (an open ball in $\mathcal{D}_K$)

$\Rightarrow W_\phi \in \mathcal{B}$, and

$\mathcal{D}_K \cap (\phi + W_\phi) = \phi + W_\phi \cap \mathcal{D}_K \subset \phi + E - \phi = E$.

Let $V = \bigcup_{\phi \in E} (\phi + W_\phi)$. Then

$E = \bigcup_{\phi \in E} ((\phi + W_\phi) \cap \mathcal{D}_K)$

$=$ Union of all balls around $\phi \in E$.

$= V \cap \mathcal{D}_K$.

(c) Let E be a bounded set in $\mathcal{D}(\Omega)$. Suppose $E \not\subset \mathcal{D}_K$, for any K. Then $\exists\ \phi_m \in E$ and a seq'n $x_m \in \Omega$ having no limit pt such that $\phi_m(x_m) \neq 0$, $m = 1, 2, \ldots$

Let $W = \{\phi \in \mathcal{D}(\Omega) : |\phi(x_m)| < \frac{1}{m} \phi_m(x_m), \ m = 1, 2, \ldots\}$.

Since each K contains only finitely many x_m,

$W \cap \mathcal{D}_K = \{\phi \in \mathcal{D}_K : |\phi(x_m)| < \frac{1}{m} \phi_m(x_m)\}$

is open in $\mathcal{D}_K$. For this, let $\phi \in W \cap \mathcal{D}_K$.

Then $|\varphi(x_m)| < \frac{1}{m}|\varphi_m(x_m)|$; $m = 1, 2, \ldots, l$.

$p(\varphi) = \sup_{1 \le m \le l} |\varphi(x_m)| < c_l$, where $c_l = \max_{1 \le m \le l} \frac{1}{m}|\varphi_m(x_m)|$.

Since p is cont, it follows that $W \cap D_K$ is open in D_K. Thus, $W \in \beta$. Since $\varphi_m \notin mW$ for any m, it follows that E is not bounded.

Thus every bounded set $E \subset D(\Omega)$ must lies in some D_K. By (b), E is bounded in D_K. This implies

$$\sup\{\|\varphi\|_N : \varphi \in E\} \le M_N < \infty, \; N = 0, 1, 2 \ldots$$

(d) It follows from (c), since D_K has the HBP. If E is a closed and bounded set in $D(\Omega)$, then E is closed and bounded in D_K, hence compact. Thus, E is compact in $D(\Omega)$.

(e) If $\{\varphi_i\}$ is a C.S. in $D(\Omega)$, then it is bounded and hence $\varphi_i \in D_K$, for some K. By (b) $\{\varphi_i\}$ is C.S. relative to D_K.

(f) It is just restatement of (e).

Finally (g) follows from (b), (e) and completeness of D_K ($\because D_K$ is a Frechet space).

Theorem: Let Λ be a linear map from $\mathcal{D}(\Omega)$ to a locally convex space Y. Then FAE.

(i) Λ is continuous. (ii) Λ is bounded.

(iii) If $\varphi_i \to 0$ in $\mathcal{D}(\Omega)$, then $\Lambda\varphi_i \to 0$ in Y.

(iv) If $\forall K \subset \Omega$, the restriction $\Lambda : \mathcal{D}_K \to Y$ is continuous.

Proof: (i) $\Rightarrow$ (ii) is known.

(ii) $\Rightarrow$ (iii): Suppose Λ is bounded and $\varphi_i \to 0$ in $\mathcal{D}(\Omega)$. Then $\varphi_i \to 0$ in some $\mathcal{D}_K$, and hence $\Lambda|_{\mathcal{D}_K}$ is bounded. ~~Hence,~~ $\Lambda : \mathcal{D}_K \to Y$ is continuous, that $\Lambda\varphi_i \to 0$ in Y.

(iii) $\Rightarrow$ (iv): Suppose $\{\varphi_i\} \subset \mathcal{D}_K$ and $\varphi_i \to 0$ in $\mathcal{D}_K$. Then by (b) of the previous theorem, $\varphi_i \to 0$ in $\mathcal{D}(\Omega)$. By (iii) $\Lambda\varphi_i \to 0$ in Y.

(iv) $\Rightarrow$ (i): Let U be a convex balanced nbd of $0 \in Y$, and write $V = \Lambda^{-1}(U)$. Then V is a convex and balanced set in $\mathcal{D}(\Omega)$. By (a) of the previous theorem, $V \in \tau$ If $\mathcal{D}_K \cap V \in \tau_K$ for each $K \subset \Omega$. By (iv) $\mathcal{D}_K \cap V \in \tau_K$, hence $V \in \tau$. Hence Λ is continuous.

Defn: A linear functional Λ on $\mathcal{D}(\Omega)$ which is continuous in the top. τ of $\mathcal{D}(\Omega)$ is called distribution. (18)

The space of all distribution is denoted by $\mathcal{D}'(\Omega)$.

Theorem: let Λ be a linear functional on $(\mathcal{D}(\Omega), \tau)$. Then FAE.

(i) $\Lambda \in \mathcal{D}'(\Omega)$

(ii) for each compact set $K \subset \Omega$, $\exists$ $N \in \mathbb{N}$ and $C > 0$ s.t.

(*) —— $|\Lambda \varphi| \leq C \|\varphi\|_N$, $\forall$ $\varphi \in \mathcal{D}_K$.

This result is nothing but equivalence of (i) and (iv) in the previous theorem.

Note that if N in the (*) is independent of choice of K, then the minimum of such N's is called order of distribution Λ. If no such N exists, then we say Λ has ∞ order.

Remark: Since each $\mathcal{D}_K$ is closed, it is obvious that $\mathcal{D}_K$ has no interior in $\mathcal{D}(\Omega)$. Since $\exists$ a countable seqn of cpt sets in Ω s.t. $\Omega = \bigcup_{i=1}^{\infty} K_i$, $K_i \subset K_{i+1}^{\circ}$, we get

$$\mathcal{D}(\Omega) = \bigcup_{i=1}^{\infty} \mathcal{D}_{K_i}.$$

Since C.C. in $\mathcal{D}(\Omega)$ does converge in $\mathcal{D}(\Omega)$, by Baire Category theorem, $(\mathcal{D}(\Omega), \tau)$ cannot be metrizable.