Fourier Series:

Fourier Series basically deals with the problem of decomposing a given "nice" function into countably many symmetric functions, then relook at the superposition of those symmetric function to get the original function. In particular, reconstruct a given function out of countably many known information about the function.

Question: What are those symmetric nice functions?

We can see the existence of those elementary symmetric functions while discussing solution of wave equation and heat equation.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

and a variable separable solution:

$u(x,t) = \varphi(x)\psi(t)$. Then from (1),

$$\frac{\psi''(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda \text{ (say)}.$$

Hence,

$$\begin{cases} \psi''(t) - \lambda\psi(t) = 0 \\ \varphi''(t) - \lambda\varphi(t) = 0 \end{cases} \quad \text{—— (2)}$$

If $\lambda \geq 0$, then ψ will not oscillate w.r.t. time t. Hence, we only consider $\lambda < 0$ and write $\lambda = -m^2$, where $m \in \mathbb{Z}$. Here we consider countable many m as we promised earlier to determine the function only out of countable many known informations. (2)

Consider $\psi(t) = A\cos mt + B\sin mt$,

and $\varphi(x) = \tilde{A}\cos mx + \tilde{B}\sin mx$.

Suppose, the string is attached at $x=0$ and $x=\pi$. Then $\varphi(0) = \varphi(\pi) = 0$, that yields

$\tilde{A} = 0$ & $\tilde{B} \neq 0$.

If $m=0$, the solution is trivial. If $m \leq 1$, we may rewrite the coefficients and reduce this case to $m \geq 1$, because $\cos y$ and $\sin y$ are even and odd functions respectively.

Finally, we have

$$u_m(x,t) = (A_m\cos mt + B_m\sin mt)\sin mx.$$

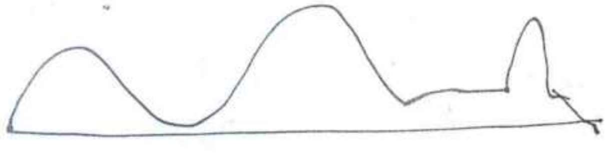
Since the wave equation (1) is linear, it follows that if u & v are two solutions of (1), then $\alpha u + \beta v$ is also a solution of (1). Thus, we can think of a general solution of (1) like

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx. \quad (3)$$

for the wave equation (1).

Now, suppose the initial position of the string (at $t=0$) is given by the graph of the function f on $[0,\pi]$ with $f(0) = f(\pi) = 0$.

Then $u(x,0) = f(x)$. Hence,



$$(3) \quad \sum_{n=1}^{\infty} A_n \sin nx = f(x).$$

Thus, given reasonable function f on $[0,\pi]$ with $f(0) = f(\pi) = 0$, we may find A_n so that $f(x) = \sum_{n=1}^{\infty} A_n \sin nx$?

If f is reasonable enough, we may think of evaluating

$$\int_0^{\pi} f(x) \sin mx \, dx = \int_0^{\pi} \left(\sum_{n=1}^{\infty} A_n \sin nx \right) \sin mx \, dx$$

$$= \sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= A_m \frac{\pi}{2}.$$

Hence, the nth sine coefficient of f is

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

We can extend this Fourier Sine Series on $[0,\pi]$ to $[-\pi,\pi]$ by assuming f is odd on $[-\pi,\pi]$.

Similarly, we can asked for an even function g on $[-\pi,\pi]$ to have expansion like (4)

$$g(x) = \sum_{m=0}^{\infty} A_m' \cos mx \quad ?$$

Since, an arbitrary function on $[-\pi,\pi]$ can be expressed as sum of odd and even functions, a reasonable function F on $[-\pi,\pi]$ can be thought of having expansion like

$$F(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=0}^{\infty} A_m' \cos mx \quad \text{——— (4)}$$

By using the Euler formula,

$$e^{ix} = \cos x + i \sin x,$$

we can re-write (4) as

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx} \quad ?$$

By analogy as to the earlier case, we can see that

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-inx} dx,$$

Since $\frac{1}{2\pi} \int e^{inx} e^{-imx} dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$

The number a_n is called n^{th} Fourier coefficient of F.

Question: Given any reasonable function F on $[-\pi,\pi]$ with Fourier coefficients defined as above, is it possible that

$$F(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx} \quad ? \qquad (5)$$

Joseph-Fourier (1768-1830) was the first who believed that an "arbitrary" function can be expressed as the series (5). However, his idea was implicit and later refined.

(5)

If we look at the wave equation little carefully, then we come to the fact that, it actually requires two initial conditions. Namely, initial position and initial velocity of the string. That is,

$$u(x,0) = f(x) \quad \& \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

From (3), we get

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{and} \quad g(x) = \sum_{m=1}^{\infty} m B_m \sin mx$$

Hence, convergence of series for g requires more decay on B_m.

Now, we consider the case of heat flow in an infinite plate. Namely,

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

When steady state reached, there is no ~~flow~~ exchange of heat in the plate implies

$\frac{\partial u}{\partial t} = 0$. That is, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —— (6)

A function satisfies (6) is known as harmonic function. Suppose the metallic plate is the unit disc $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and

$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$. (6)

By passing to the polar coordinates,

$x = r\cos\theta,\ y = r\sin\theta,\ r \geq 0,\ 0 \leq \theta \leq 2\pi,$

the steady-state heat equation reduces

to $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ —— (7)

Equation (7) together with initial condition $u(1,\theta) = f(\theta)$ is known as Dirichlet problem.

That is, we have given a temperature distribution f on the circle S^1 and waiting for temp. distribution inside the disc.

Further, $r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}$.

consider $u(r,\theta) = F(r)G(\theta)$. Then

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = \lambda \text{ (say)}.$$

That is, $\begin{cases} G''(\theta) + \lambda G(\theta) = 0 \quad \text{and} \\ r^2F''(r) + rF'(r) - \lambda F(r) = 0. \end{cases}$

Since G must be periodic, it follows that $\lambda \geq 0$. Let $\lambda = m^2$, $m \in \mathbb{Z}$. Then (7)

$$G(\theta) = \tilde{A}\cos m\theta + \tilde{B}\sin m\theta$$

$$\text{or } G(\theta) = A e^{im\theta} + B e^{-im\theta}$$

Case(i): If $m \neq 0$, then $F(r) = r^m$ & r^{-m}. Now, for $m > 0$, $r^{-m} \to \infty$ as $r \to 0$, so $F(r)G(\theta)$ is unbounded near "zero".

Case(ii): If $m = 0$, $F(r) = 1$ or $\log r$.

Hence, again solution is unbounded if $F(r) = \log r$. We reject these two cases, while solution is unbounded. Thus, we consider

$$u_m(r,\theta) = r^{|m|} e^{im\theta}, \quad m \in \mathbb{Z}.$$

Since, steady state heat equation ($\Delta u = 0 = \frac{\partial u}{\partial t}$) is linear, we can think of whether

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}$$

is a possible general solution for $\Delta u = 0$?

Here, for a reasonable function f on $[0, 2\pi]$

$$u(1,\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta).$$

Question: Given a reasonable function f on $[0, 2\pi]$ with $f(0) = f(2\pi)$, can we find the coefficients a_m so that

$$f(\theta) = \sum a_m e^{im\theta} ?$$

Function on Circle:

Let $S^1 = \{ e^{i\theta} : 0 \le \theta < 2\pi \}$.
Consider $\varphi : \mathbb{R} \to S^1$, by $\varphi(x) = e^{ix}$. Then φ is a group homomorphism, with $\operatorname{Ker}\varphi = 2\pi\mathbb{Z}$.
Hence, $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. If $f : S^1 \to \mathbb{C}$, then can identify f on $\mathbb{R}$ by $\tilde{f} : \mathbb{R} \to \mathbb{C}$ such that $\tilde{f}(x) = \tilde{f}(\tilde{x} + 2\pi n) = f(\tilde{x})$.

That is, function on S^1 can be identified with 2π-periodic function on $\mathbb{R}$ which allow us understand notion of continuity, differentiability etc. for functions on S^1. Further, Lebesgue measure on S^1 can also be identified by mean of, f is integrable on S^1 if the corresponding ~~function~~ 2π-periodic function which again we denote by f is Lebesgue integrable on $[0, 2\pi)$, and we write

$$\int_{S^1} f(t)\,dt := \int_0^{2\pi} f(x)\,dx.$$

Now onwards, we identify S^1 as $[0, 2\pi)$ and the Lebesgue measure dt on S^1 as the restriction of Lebesgue measure on $\mathbb{R}$ to $[0, 2\pi)$.

Therefore, dt on S^1 is translation invariant. That is, for $t_0 \in S^1$, (9)

$$\int_{S^1} f(t-t_0)\,dt = \int_{S^1} f(t)\,dt,$$

Since the corresponding function f on $\mathbb{R}$ is 2π-periodic.

An expression of the form $P_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}$, where $|a_n| + |a_{-n}| \neq 0$, is known as trigonometric polynomial of degree N.

Likewise, $S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}$

is known as trigonometric series.

For $n \in \mathbb{Z}$, and $f \in L^1(S^1)$, the nth Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt.$$

The Fourier Series of $f \in L^1(S^1)$ is the expression of type

$$S(f) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

Hence, the nth partial sum of the F.S. $S_N(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$

is a trigonometric poly of degree n.

Lemma: Let $f, g \in L^1(S^1)$, then (10)

(i) $\widehat{f+g}(n) = \hat{f}(n) + \hat{g}(n)$,

(ii) $(\alpha f)\hat{}(n) = \alpha \hat{f}(n)$, $\forall \alpha \in \mathbb{C}$,

(iii) $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

(iv) If $\tau_{t_0} f(t) = f(t-t_0)$, $t_0 \in S^1$, then

$$(\tau_{t_0} f)\hat{}(n) = e^{-int_0} \hat{f}(n)$$

(v) $|\hat{f}(n)| \leq \frac{1}{2\pi} \int |f(t)| dt = \|f\|_1$.

Cor: If $f_j \in L^1(S^1)$ & $\|f_j - f\|_1 \to 0$, then
$\hat{f_j}(n) \to \hat{f}(n)$ absolutely (& uniformly).

Theorem: Let $f : [0, 2\pi] \longrightarrow \mathbb{C}$ ($\text{or } \mathbb{R}$). Then f is absolutely continuous iff f' exists a.e. and $f(x) = f(0) + \int_0^x f'(t) dt$.
(For a proof see Carothers p. 374.)

Theorem: Let $f \in L^1(S^1)$ and $\hat{f}(0) = 0$. Define

$$F(t) = \int_0^t f(s) ds.$$

Then F is continuous 2π-periodic function and $\hat{F}(n) = \frac{1}{in} \hat{f}(n)$, if $n \neq 0$.

Proof: For $t_k \to t_0$, $F(t_k) - F(t_0) = \int_0^{2\pi} \chi_{[t_0, t_k)}^{(s)} f(s) ds$

Since $\chi_{[t_0,t_k)}(s) f(s) \to 0$ pointwise a.e. and $f \in L^1(S^1)$, by DCT, it follows that $F(t_k) - F(t_0) \to 0$ as $k \to \infty$. Hence, F is cont. on S^1. (11)

Notice that
$$\sum_{k=1}^{l} |F(t_k) - F(t_{k-1})| \le \sum_{k=1}^{l} \int_0^{2\pi} \chi_{[t_k, t_{k-1})}(s) |f(s)| ds.$$

Hence, R.H.S. tends to "0" when $l \to \infty$. This implies that F is absolutely continuous. Thus, F is differentiable a.e. Also
$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(s) ds = \hat{f}(0) = 0.$$

Now, integrating by part, we get
$$\hat{F}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} F(t) dt = -\frac{1}{2\pi} \int_0^{2\pi} F'(t) \frac{e^{-int}}{-in} dt$$
$$= \frac{1}{in} \hat{f}(n).$$

Ex. Let $f(\theta) = \theta$, $-\pi \le \theta < \pi$. Then
$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta} d\theta = \frac{(-1)^{n+1}}{in}, \text{ if } n \ne 0,$$
$\hat{f}(0) = 0$. Thus,
$$f(\theta) \sim \sum \frac{(-1)^{n+1}}{in} e^{in\theta} = 2 \sum (-1)^{n+1} \frac{\sin n\theta}{n}.$$

It is easy to see that series in R.H.S. is pointwise convergent, but showing it converges to $f(\theta)$ is not easy and we see later!

Ex. $f(\theta) = \frac{(\pi-\theta)^2}{4}$, $0 \le \theta \le 2\pi$ (12)

$f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}$.

The Fourier series is uniformly convergent, but it converges to $f(\theta)$ is not easy.

Theorem: For $f, g \in L^1(S^1)$; write

$$h(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t-s)\, g(s)\, ds.$$

Then $h \in L^1(S^1)$ and $\|h\|_1 \le \|f\|_1 \|g\|_1$.

Moreover, $\hat{h}(n) = \hat{f}(n)\, \hat{g}(n)$.

Proof: $\int |h(t)|\, dt \le \frac{1}{2\pi}\int \left(\frac{1}{2\pi}\int |f(t-s)||g(s)|\, ds\right) dt$

$= \frac{1}{2\pi}\int \left(\frac{1}{2\pi}\int |f(t-s)|\, dt\right) |g(s)|\, ds$ (by Fubini's thm)

$= \frac{1}{2\pi}\int \|f\|_1 |g(s)|\, ds = \|f\|_1 \|g\|_1$.

Further, $\hat{h}(n) = \frac{1}{2\pi}\int h(t)\, e^{-int}\, dt$

$= \frac{1}{4\pi^2}\int\int f(t-s)\, e^{-in(t-s)}\, dt\; g(s)\, e^{-ins}\, ds$

$= \frac{1}{2\pi}\int \hat{f}(n)\, g(s)\, e^{-ins}\, ds$

$= \hat{f}(n)\, \hat{g}(n)$.

Ex. Does $\exists$ $f, g \in L^1(S^1)$ such that

$$f * g(s) = 1 ?$$

Ex. Let $f \in L^1(S^1)$ and $\varphi(t) = e^{int}$, then

$$\varphi * f(t) = \frac{1}{2\pi} \int f(s) e^{+in(t-s)} ds$$
$$= e^{int} \hat{f}(n).$$

Hence, if $P_N(t) = \sum_{n=-N}^{N} a_n e^{int}$, then

$$P_N * f(t) = \sum_{n=-N}^{N} a_n \hat{f}(n) e^{int}.$$

i.e Convolution of a trigonometric poly. with any function is a trigonometric poly.

Now, Consider the Fourier Series of $f \in L^1(S^1)$ as

$$f(t) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

Let $D_N(t) = \sum_{n=-N}^{N} e^{int}$ and $S_N(f)(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{int}$.

Then $S_N(f)(t) = D_N * f(t)$. The function D_N is known as Dirichlet Kernel.

Further, $D_N(t) = \dfrac{\sin((N+\frac{1}{2})t)}{\sin t/2}$, if $t \neq 0$

and $D_N(0) = 2N+1$.

(Hint: put $w = e^{it}$, then $D_N(t)$ is the sum of two geometric series etc).

Hence the earlier question of Convergence of Fourier Series can be rephrased as: whether the partial sum seq^n $S_N(f)$ converges to f pointwise. That is, when $\lim_{N\to\infty} D_N * f(t) = f(t)$? (4)

Recall back the heat equation (steady state),

$$\Delta u = 0, \quad u(r,\theta) = \sum a_m r^{|m|} e^{im\theta}$$

Let $P_r(\theta) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{im\theta}, \quad 0 \le r < 1, \; \theta \in [-\pi, \pi)$

Then the series in RHS converges absolutely and uniformly. Hence,

$\hat{P}_r(n) = r^{|n|}$ and we have

$$P_r * f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \, r^{|n|} e^{int}$$

The function $P_r(\theta)$ is known as Poisson Kernel and can be represented as

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

(Hint: Series for $P_r(\theta)$ is sum of two geometric series etc.).

Thus, we can asked when

$$\lim_{r\to 1} P_r * f(\theta) = f(\theta) ?$$

The function $P_r * f$ is called the Abel mean of Fourier Series $S(f)$.

Now, the question is, does there exist a family of "good Kernels" (weight functions or averaging functions) for the Fourier series that leads the series to the given function?

that is, if $f \in L^1(S^1)$, can we find a seq^n $K_n \in L^1(S^1)$ s.t. $f * K_n \to f$?

Def^n: A sequence of functions $\{K_n\}_{n=1}^{\infty}$ is "good Kernels" if

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1; \ \forall n \geq 1.$

(ii) $\exists\ M > 0$ such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(t)|\,dt \leq M, \ \forall n \geq 1$

(iii) for each $\delta > 0$, $\int_{\delta < |t| \leq \pi} |K_n(t)|\,dt \to 0$ as $n \to \infty$.

Theorem: Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of good Kernels in $[-\pi,\pi]$ and $f \in R[-\pi,\pi]$ (Riemann integrable)

Then $(f * K_n)(x) \longrightarrow f(x)$, if x is point of continuity of f and the above limit is uniform if f is continuous on $[-\pi,\pi]$.

Proof: Since f is cont at x, for $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x-y) - f(x)| < \epsilon$, if $|y| < \delta$.

$\Rightarrow f*K_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]\,dy$

(by prop. (i) of K_n). (16)

$$\Rightarrow |f*K_n(x) - f(x)| \le \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

$$+ \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

$$\le \frac{\epsilon}{2\pi}\int_{|y|<\delta} |K_n(y)|\,dy + \frac{2B}{2\pi}\int_{\delta<|y|\le\pi} |K_n(y)|\,dy,$$

where $|f(x)| \le B$, $\forall x \in [-\pi,\pi]$.

$\Rightarrow |f*K_n(x) - f(x)| < C\epsilon$, for large n.

If f is cont on $[-\pi,\pi]$, then we can find one $\delta>0$ that serves for each x. Hence,

$f*K_n \to f$ unif in this case.

Cor: If $\{K_n\}_{n=1}^{\infty}$ is a seqn of good kernels in $L^1(S^1)$ and $f \in L^1(S^1)$. Then

$$f*K_n \to f \text{ in } L^1(S^1).$$

Proof: Since $\overline{C([-\pi,\pi])} = L^1([-\pi,\pi])$. For $f \in L^1$ and $\epsilon>0$, $\exists$ g conti. such that

$$|f(x) - g(x)| < \epsilon, \ \forall x \in [-\pi,\pi].$$

That is $\|f-g\|_1 < 2\pi\epsilon$. —— (1)

From the above result

$g * K_n(x) \longrightarrow g$ unif. (17)

i.e. $|g*K_n(x) - g(x)| < \epsilon$, for large n, $\forall x$.

$\Rightarrow \|g*K_n - g\|_1 < 2\pi\epsilon$ — (2)

This implies,

$$\|f*K_n - f\|_1 \le \|(f-g)*K_n\|_1 + \|g*K_n - g\|_1 + \|f-g\|_1$$

$$\le \|f-g\|_1 \|K_n\|_1 + 4\pi\epsilon$$

$$\le \epsilon . 2 + 4\pi\epsilon, \text{ for large } n.$$

Remark: Dirichlet Kernel is not a good Kernel for Fourier Series.

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin t/2}, \ \forall t \neq 0,$$

Since $|\sin x| < |x|$, it follows that

$$\int_{-\pi}^{\pi} |D_n(t)|\,dt \geq \frac{2}{\pi}\int_0^{\pi} |\sin(n+\tfrac{1}{2})t| \frac{dt}{t}$$

$$= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi} |\sin t| \frac{dt}{t}$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin t|\,dt$$

$$= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \to \infty \text{ as } n \to \infty.$$

That is, Dirichlet Kernel D_n fails to satisfy prop (ii) of a good kernel. In fact, it is also clear from the above calculation that

$$\int_{\delta \le |t| \le \pi} |D_n(t)|\, dt \not\to 0 \text{ as } n \to \infty.$$ (18)

However, $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\, dt = 1$. Thus, if we write $F_n(t) = \dfrac{D_0(t) + D_1(t) + \cdots + D_{n-1}(t)}{n}$,

where $D_k(t) = \sum_{l=-k}^{k} e^{ilt}$, then we will show that $\{F_n\}_{n=1}^{\infty}$ is a family of good kernels. This is known a Fejer kernels, and $F_n * f$ is known as Cesaro partial sum of the Fourier series for f.

In general, for a seq^n $\{a_n\}_{n=1}^{\infty}$ of complex numbers, let $S_n = a_1 + \cdots + a_n$. Then the series $\sum a_n$ is said to be Cesaro-summable if $\sigma_n = \dfrac{S_1 + \cdots + S_n}{n}$ is convergent.

Ex. $1 - 1 + 1 - 1 + \cdots \infty = \sum_{n=0}^{\infty} (-1)^n$, then

$S_n \in \{0, 1\}$ (Hint: n = even v. n = odd)

and hence $\sigma_n \to \frac{1}{2}$.

Let $\sigma_n(f)(x) = \dfrac{S_0(f)(x) + \cdots + S_{n-1}(f)(x)}{n}$.

Since $S_n(f) = f * D_n$, it follows that

$$\sigma_n(f) = f * F_n, \text{ where}$$

$$F_n = \frac{D_0 + D_1 + \cdots + D_{n-1}}{n}.$$

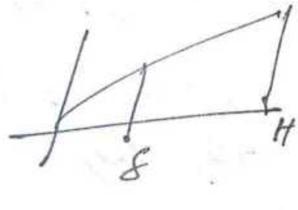
Exercise: Show that

(i) $F_n(x) = \frac{1}{n}\,\frac{\sin^2\left(\frac{nx}{2}\right)}{\sin^2(x/2)}$, if $x \neq 0$.

(ii) $F_n(0) = 1$ (since F_n continuous at $x=0$).

(iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t)\,dt = 1$.

Notice that ~~if~~ for $\delta > 0$, $\exists\, C_\delta > 0$ such that
$\sin^2\left(\frac{x}{2}\right) > C_\delta$, if $\delta \le |x| \le \pi$.



Hence, $F_n(x) \le \frac{1}{n C_\delta}$, $\forall |x| > \delta$.

Therefore, $\int_{\delta \le |x| \le \pi} F_n(x)\,dx \le \frac{(\pi-\delta)}{C_\delta}\,\frac{1}{n} \to 0$ as $n \to \infty$.

Hence, $\{F_n\}_{n=1}^{\infty}$ is a family of good kernels.

Thus, if $f \in R[-\pi,\pi]$, then the Fourier series is Cesaro summable to f at the pt of continuity of f and uniformly Cesaro summable if f is continuous.

Remark: If $f \in R[-\pi,\pi]$ and $\hat{f}(n) = 0$, $\forall n \in \mathbb{Z}$, then $f = 0$ ~~on $[-\pi,\pi]$~~ at all points of continuity of f.

Since $S_n(f)(t) = \sum_{k=-n}^{n} \hat{f}(k)\, e^{ikt} = 0$,

$f * F_n(t) = 0 \Rightarrow f(t) = 0$, if f cont at

Uniqueness theorem:

If $f \in L^1(S^1)$ is such that $\hat{f}(n) = 0,\ \forall n \in \mathbb{Z}$, (20)

then $f = 0$ on S^1 a.e.

Proof: For $f \in L^1(S)$ and $\epsilon > 0$, $\exists\ g \in C(S^1)$

such that $\|f - g\|_1 < \epsilon$.

Now, $\|f\|_1 \leq \|f * F_n - f\|_1$

$$\leq \|f * F_n - g * F_n\|_1 + \|g * F_n - g\|_1 + \|g - f\|_1$$

$$\leq \|f - g\|_1 \cdot 1 + \|g * F_n - g\|_1 + \|g - f\|_1.$$

Since g is cont, for $\epsilon > 0$, $\|g * F_n - g\|_1 < \epsilon$

for $n \geq N_0$. Hence,

$$\|f\|_1 < 3\epsilon, \quad \forall \epsilon > 0.$$

Thus, $\|f\|_1 = 0 \Leftrightarrow f = 0$ a.e.

Remark: Continuous function on S^1 can be unif approximated by trigonometric ~~function~~ polynomials. That is, if $f \in C[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then $\sigma_n(f) = f * F_n$ is a trigonometric poly. and we know that

$$f * F_n \to f \text{ unif.}$$

That is, $\{f * F_n : n \in \mathbb{N}\}$ is dense in $\{f \in C[-\pi, \pi] \cdot \text{s.t. } f(\pi) = f(-\pi)\}$.

We also notice that if $f \in L^1(S^1)$, then for $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$\|f * F_n - f\|_1 < \epsilon, \quad \forall n \geq n_0.$$

Hence, trigonometric polys are dense in $L^1(S^1)$.

Riemann-Lebesgue lemma:

If $f \in L^1(S^1)$, then $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

Proof: For $\epsilon > 0$, $\exists$ a trigonometric poly P s.t. $\|f - P\|_1 < \epsilon$ ($\because P = f * F_N$ etc)

Let $|n| > \deg P$. Then

$|\hat{f}(n)| = |\hat{f}(n) - \hat{P}(n)| \leq \|f - P\|_1 < \epsilon$, if $|n| > \deg P$. That is $|\hat{f}(n)| < \epsilon$, for large n.

Hence, $\lim \hat{f}(n) = 0$.

Abel mean summability:

A series $\sum_{n=0}^{\infty} a_n$ is said to be Abel summable to s if the series

$$A(r) = \sum a_n r^n \text{ is convergent}$$

for each $0 \leq r < 1$, and $\lim_{r\to 1} A(r) = s$.

Ex. Every conv series is Abel summable.

Consider $1 - 2 + 3 - 4 + 5 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1)$.

Then $A(r) = \sum_{n=0}^{\infty} (-1)^n (n+1) r^n = \frac{1}{(1+r)^2} \to \frac{1}{4}$.

Show that the above series is not Cesaro summable.

Now, consider the Fourier series of $f \in R[-\pi,\pi]$ as

$$f(t) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{int}.$$

Let $A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n)\, e^{in\theta}$. Then

$$A_r f(\theta) = (f * P_r)(\theta), \text{ where}$$

$$(*) \text{ — } \quad P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Then $P_r(\theta)$ is a good kernel in the following sense.

(i) $\frac{1}{2\pi}\int P_r(\theta)\, d\theta = 1$

(ii) $\lim_{r\to 1} \int_{\delta \le |\theta| \le \pi} P_r(\theta)\, d\theta = 0, \ \forall \delta > 0.$

Proof: (i) easily follows from (*), since the series converges uniformly for each $0 \le r < 1$.

To prove (ii), let $\frac{1}{2} < r < 1$. Then

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta).$$

For $0 < \delta < |\theta| \le \pi$, $1 - 2r\cos\theta + r^2 \ge c_\delta$. Hence,

$$P_r(\theta) < \frac{1-r^2}{c_\delta}, \quad \forall \delta > 0$$

$$\Rightarrow \frac{1}{2\pi}\int_{\delta < |\theta| < \pi} P_r(\theta)\, d\theta \le \frac{1-r^2}{c_\delta} \to 0 \text{ as } r \to 1.$$

Theorem: Let $f \in R[-\pi,\pi]$. Then

(i) $A_r f(\theta) = P_r * f(\theta) \to f(\theta)$, if θ is pt of continuity of f

(ii) $A_r f \to f$ uniformly if f is continuous.

Proof: Proof of this result is same as to the Fejer Kernel when we consider continuous parameter $r \in [0,1)$.

Cor: Since $\overline{C(S^1)} = L^1(S^1)$, it follows that

$$\|P_r * f - f\|_1 \to 0 \text{ as } r \to 1 \text{ for } f \in L^1(S^1).$$

Theorem: Let $U(r,\theta) = f * P_r(\theta)$. Then

(i) U is twice differentiable on the unit disc $D = \{re^{i\theta} : 0 \le r < 1, -\pi \le \theta < \pi\}$.

(ii) If θ is point of continuity of f, then

$$U(r,\theta) \to f(\theta) \text{ as } r \to 1,$$

and this limit is uniform if f is cont on $[-\pi,\pi]$.

(iii) If f is cont on $[-\pi,\pi]$, then $U(r,\theta)$ is unique solution of $\Delta U = 0$ with $\lim_{r\to 1} U(r,\theta) = f(\theta)$.

Proof: $U(r,\theta) = \sum r^{|n|} \hat{f}(n) e^{in\theta}$

Since the series and its derivative (w.r.t. r & θ) both are uniformly convergent, term-by-term differentiation is allowed.

In fact, $G(r,\theta)$ is C^{∞} - function on D.

Since $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$, it is easy to verify $\Delta u = 0$, if $u = P_r * f$.

A proof for (ii) is followed by the previous result.

(iii) Let $v(r,\theta)$ be another solution of $\Delta u = 0$ with $\lim_{r\to 1} v(r,\theta) = f(\theta)$. Then

$$v(r,\theta) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \quad (\because \Delta v = 0),$$

where $a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} v(r,\theta)\, d\theta$.

Since v is two time differentiable,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2}(r,\theta)\, e^{-in\theta} d\theta = -n^2 a_n(r).$$

Hence, from $\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

it follows that

$$a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0.$$

This gives $a_n(r) = A_n r^n + B_n r^{-n}$ if $n \neq 0$.

Since v is bounded on D, letting $r \to 0$, implies $B_n = 0$. That is,

$$v(r,\theta) = \sum A_n r^n e^{in\theta} \xrightarrow{\text{unif}} f(\theta)$$

$$\Rightarrow A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

For $n=0$, $a_0(r) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$.
Thus for each $0 \le r < 1$, Fourier series of v is same as series for u. By uniqueness, it follows that $u = v$.

Ex. If $\{J_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ are two family of good kernel for $L^1(S^1)$, then $\{J_n * K_n\}_{n=1}^{\infty}$ is a good kernel for $L^1(S^1)$.

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} J_n * K_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int J_n(t-s)K_n(s)\,ds\,dt$

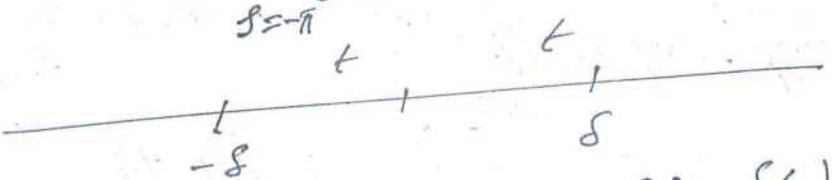
$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} J_n(t-s)\,dt\right)K_n(s)\,ds$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot K_n(s)\,ds$ ($\because L^1(S^1)$ is translation invariant)

$= 1.$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |J_n * K_n(t)|\,dt \le \frac{1}{2\pi}\int_{-\pi}^{\pi} M\,|K_n(s)|\,ds \le MN < \infty.$

(iii) Let $\delta > 0$, then
$$\int_{\delta<|t|\le\pi} |K_n * J_n(t)|\,dt \le \int_{s=-\pi}^{\pi}\left(\int_{\delta<|t|\le\pi} |K_n(t-s)|\,dt\right)|J_n(s)|\,ds$$



Let $|s| < \delta/2$, then $r = t-s \in (-\delta/2, \delta/2)^c$. Then

(**) $\int_{|s|<\delta/2}\left(\int_{\delta/2<|r|<\pi} |K_n(r)|\,dr\right)|J_n(s)|\,ds \to 0$ as $n\to\infty$,

since $\int_{\delta/2<|s-t|<\pi} |K_n(t-s)|\,dt \to 0$ as $n \to \infty$, (Exercise) while $|s| < \delta/2$. (Use the fact that $\tau_x f \to f$ is cont on $L^1(S^1)$).

That is, if $\int_{\delta < |t| \le \pi} |K_n(t)|\, dt \to 0, \ \forall \delta > 0,$ (26)

then $\left| \int_{\delta < |t| < \pi} (\tau_s K_n(t) - K_n(t))\, dt \right| \le \int_{\delta \le |t| \le \pi} |\tau_s K_n(t) - K_n(t)|\, dt < \epsilon$

For $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $\int |K_n(t)| dt < \epsilon, \ \forall n \ge n_0$. for small $|s| < \delta'$.

However, $\int_{|s| > \delta/2} \int_{|t| > \delta} |K_n(t-s)|\, |K_n(s)|\, ds\, dt$

$\le \int_{|s| > \delta/2} M\, |K_n(s)|\, ds \to 0$ as $n \to \infty$.

<u>Lemma</u>: Let $f : [-\pi, \pi] \to \mathbb{C}$ be such that

$|f(x) - f(y)| \le M |x - y|, \ \forall x, y \in [-\pi, \pi]$

for some $M > 0$. Then $S_n(f) \to f$ uniformly.

Note that $|x - y| = \min\{ |x-y|, |x - y \pm 2\pi| \}$.

= distance between x & y modulo 2π.

Proof: $S_n(f)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x))\, D_n(t)\, dt$.

Since $D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{\sin t/2}, \quad t \ne 0,$

$|S_n(f)(x) - f(x)| \le \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \frac{\cos t/2}{\sin t/2} \sin nt\, dt \right|$

$+ \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) \cos nt\, dt \right|$

Let $g(t) = \left(\frac{f(x-t) - f(x)}{t/2}\right)\left(\frac{t/2}{\sin t/2}\right)\cos t/2$, if $t \neq 0$.

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Then $|g(t)| \leq 2M \left|\frac{t/2}{\sin t/2}\right|$, if $t \neq 0$.

Since $\lim_{t \to 0} \frac{t/2}{\sin t/2} = 1$, it follows that g is a bounded function on $[-\pi, \pi]$ and continuous on $[-\pi,\pi] \setminus \{0\}$. Hence, $g \in \mathcal{R}[-\pi,\pi]$.

Let $h(t) = f(x-t) - f(x)$. Then

$$|S_n(f)(x) - f(x)| \leq \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} g(t)\sin nt\, dt\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} h(t)\cos nt\, dt\right|$$

$$= \frac{1}{2}\left|\hat{g}(n) - \hat{g}(-n)\right| + \frac{1}{2}\left|\hat{h}(n) + \hat{h}(-n)\right|.$$

$\longrightarrow 0$ (By R-L lemma), whenever $x \in [-\pi,\pi]$.

Cor: If $f \in \mathcal{R}[-\pi,\pi]$ and f is differentiable at x_0, then $S_n(f)(x_0) \longrightarrow f(x_0)$.

(Hint: Define $F(t) = \begin{cases} \frac{f(x_0-t) - f(x_0)}{t}, & \text{if } t \neq 0 \\ -f'(x_0) & \text{if } t = 0. \end{cases}$

Cor: If $f \in C^1[-\pi,\pi]$, then $S_n(f) \longrightarrow f$ uniformly.

(Hint: Use MVT).

Notice that if f is piece-wise C^1-function, then $S_n(f) \longrightarrow f$ uniformly too.

Question: Does every continuous function f on S^1 has Fourier series which converges to f at each point of S^1?

To discuss this, we need the following lemma. (28)

Lemma: Let $f \in R[-\pi,\pi]$ and f is bounded on $[-\pi,\pi]$ by M. Then $\exists$ a sequence f_n of continuous functions on $[-\pi,\pi]$ such that

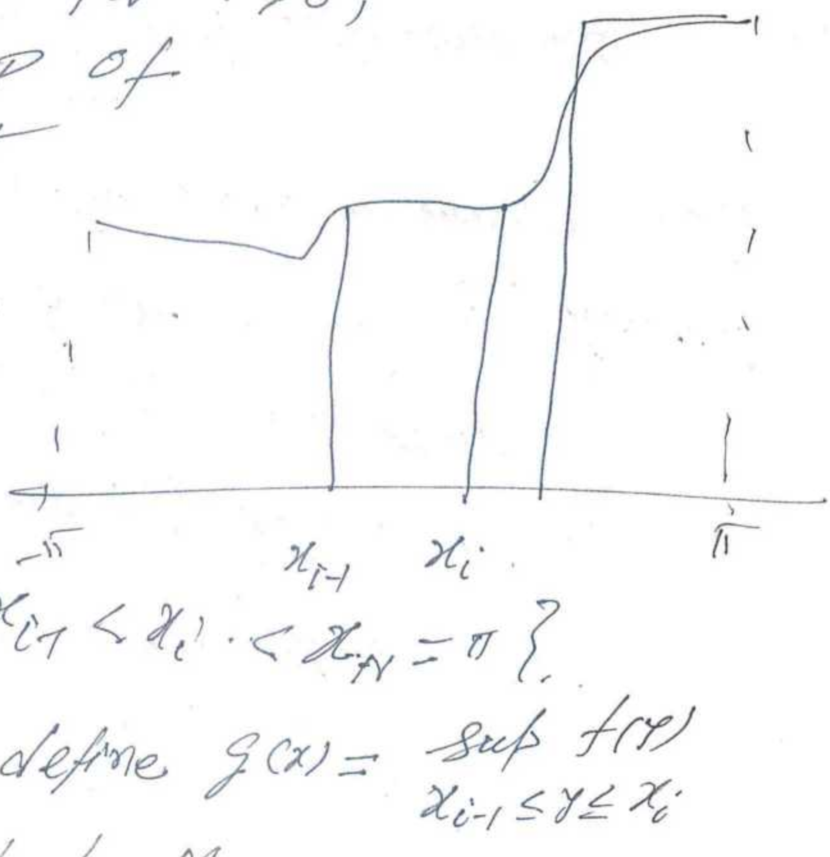
(i) $|f_n(x)| \leq M, \ \forall n \in \mathbb{N}, \forall x \in [-\pi,\pi]$.

(ii) $\int_{-\pi}^{\pi} |f(x) - f_n(x)|\, dx \to 0$ as $n \to \infty$.

Proof: First we consider f as a real valued function. For $\epsilon > 0$, $\exists$ a partition P of $[-\pi,\pi]$ such that

$U(P,f) - L(P,f) < \epsilon$, $\longrightarrow (1)$

where



$P = \{-\pi = x_0 < x_1 < \cdots < x_{i-1} < x_i \cdots < x_N = \pi\}$.

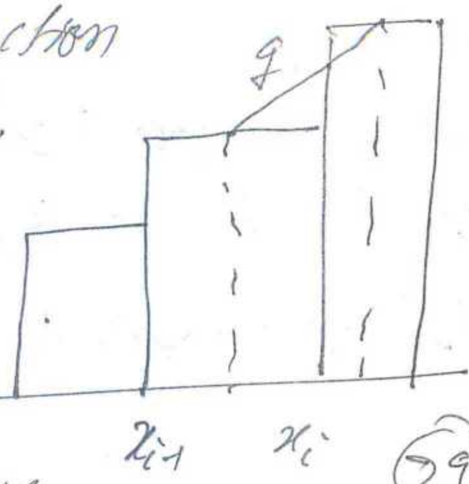
For $x \in [x_{i-1}, x_i)$, define $g(x) = \sup_{x_{i-1} \leq y \leq x_i} f(y)$

Then g is bounded by M.

$$\int_{-\pi}^{\pi} |g(x) - f(x)|\, dx = \int_{-\pi}^{\pi} (g(x) - f(x))\, dx < \epsilon \ (\text{by } (1)).$$

Let $\delta > 0$, and $x \in (x_i - \delta, x_i + \delta)$, define

$\tilde{g}(x)$ to be the linear function joining $g(x-\delta)$ and $g(x+\delta)$, and $\tilde{g} = 0$ near $-\pi$ & π. Then $\tilde{g}$ is a continuous periodic function which differ with g on N many intervals each of length less than 2δ surrounding the partitioning points. Hence



$$\int_{-\pi}^{\pi} |g(x) - \tilde{g}(x)|\,dx \le (2M) N (2\delta).$$

For δ sufficiently small, $\int_{-\pi}^{\pi} |g(x) - \tilde{g}(x)|\,dx < \epsilon$.

$$\Rightarrow \int_{-\pi}^{\pi} |f(x) - \tilde{g}(x)|\,dx < 2\epsilon.$$

For $2\epsilon = \frac{1}{n}$, $\tilde{g} = f_n$. Thus,

$$\int_{-\pi}^{\pi} |f(x) - f_n(x)|\,dx \longrightarrow 0 \text{ as } n \to \infty.$$

<u>Remark</u>: If $f \in R[-\pi, \pi]$ has only finitely many points of discontinuity, then $\tilde{g}_n(x) \to f(x)$ point-wise.

Now, let $X = C(S^1)$, and define $\Lambda_n : X \longrightarrow X$ by

$$\Lambda_n(f) = S_n(f)(0).$$

Then $\{\Lambda_n\}$ is a seq^n of linear functionals on X and

$$|\Lambda_n(f)| \le \|D_n\|_1 \|f\|_\infty$$

& $\|\Lambda_n\| \le \|D_n\|_1$.

We claim that $\|\Lambda_n\| = \|D_n\|_1$.

i.e $\|\Lambda_n\| = \int_{-\pi}^{\pi} |D_n(t)|\, dt$. Then for each
To this, let $g(t) = \text{sign}\, D_n(t)$. Then for each fixed n, g has only finitely many points of discontinuity. Hence, $\exists$ $g_m \in C[-\pi,\pi]$ s.t $|g_m(t)| \le 1$ and $g_m(t) \to g(t)$ as $m \to \infty$ for each t in $[-\pi,\pi]$. (By previous lemma).

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Therefore, $\lim_{m\to\infty} \Lambda_n(g_m) = \lim_{m\to\infty} \int_{-\pi}^{\pi} g_m(-t) D_n(t)\, dt$

$$= \int_{-\pi}^{\pi} g(-t) D_n(t)\, dt \quad (\text{By DCT})$$

$$= \int_{-\pi}^{\pi} |D_n(t)|\, dt = \|D_n\|_1.$$

Thus, $\|\Lambda_n\| = \|D_n\|_1 \to \infty$ as $n \to \infty$
that is, $\{\Lambda_n\}_{n=1}^{\infty}$ is not uniformly bounded seq^n in $B(X, \mathbb{C})$, hence by uniform boundedness principle (UBP), $\exists$ $f \in C[-\pi,\pi]$ s.t

$\Lambda_n(f) = S_n(f)(0)$ is not bounded.

Therefore, F.S. of f at '0' does not converge to $f(0)$.

Notice that by translation, we can show that for each $x \in [-\pi,\pi]$, $\exists$ a function $f \in C[-\pi,\pi]$ whose Fourier series does not converge to $f(x)$ of x.

In fact, for each $x \in [-\pi,\pi]$, we can create a dense class of continuous functions say E_x s.t.

$S_n(f)(x) \to \infty$. (See Rudin, Real & Complex).

Convergence of Fourier Series in $L^2(S^1)$:

We have seen that F.S. of $f \in C(S^1)$ need not converge to f uniformly. Similarly, we can also see that F.S. of $f \in L^1(S^1)$ need not converge to f in L^1-norm. (For this define $\Lambda_n(f) = S_n(f)$, $f \in L^1(S^1)$ and use $\|F_n\|_1 = 1$). However, because of the self duality of the space $L^2(S^1)$, for $f \in L^2(S^1)$ we shall see that $S_n(f) \to f$ in L^2-norm.

For $f, g \in R[0,2\pi]$, define an inner product by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta$ and

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|^2 d\theta$$

Let $e_n(\theta) = e^{in\theta}$. Then $\{e_n : n \in \mathbb{Z}\}$ forms an ONS in $L^2(S^1)$, because

$$\langle e_n, e_m\rangle = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

Let $\langle f, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt = a_n$. Then

$S_N(f) = \sum_{|n|\le N} a_n e_n$. Note that

$$f - \sum_{|n|\le N} a_n e_n \perp e_n, \quad \forall\, |n| \le N.$$

Hence $\left(f - \sum_{|n|\le N} a_n e_n\right) \perp \sum_{|n|\le N} b_n e_n$, whenever $b_n \in \mathbb{C}$.

By Pythagorean theorem, from

$$f = f - \sum_{|n|\le N} a_n e_n + \sum_{|n|\le N} a_n e_n,$$

it follows that

$$\|f\|_2^2 = \|f - \sum_{|n|\le N} a_n e_n\|_2^2 + \sum_{|n|\le N} |a_n|^2$$

or $\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n|\le N} |a_n|^2$ —(1)

Since $f \in L^2(S^1)$, we get $\sum_{|n|\le N} |a_n|^2 \le \|f\|_2^2 < \infty$, for each $N \in \mathbb{N}$. (Bessel inequality).

Best approximation lemma:

Let $f \in L^2[0, 2\pi]$ and $a_n = \hat{f}(n)$. Then

$$\|f - S_N(f)\|_2 \le \|f - \sum_{|n|\le N} c_n e_n\|,$$ for any

seq^n $(c_n) \subset \mathbb{C}$. Moreover, equality holds if $c_n = a_n$, $\forall$ $|n| \le N$.

Proof: $f - \sum_{|n|\le N} c_n e_n = f - S_N(f) + \sum_{|n|\le N} (a_n - c_n) e_n$

Let $a_n - c_n = b_n$. Then by orthogonality,

$$\|f - \sum_{|n|\le N} c_n e_n\|_2^2 = \|f - S_N(f)\|_2^2 + \|\sum b_n e_n\|_2^2 \quad —(1)$$

$\Rightarrow \|f - S_N(f)\|_2 \le \|f - \sum c_n e_n\|_2$.

But equality holds iff $\|\sum b_n e_n\|_2^2 = 0$ iff $b_n = 0$.

That is, Fourier approximation is best among any other approximation of the form $\sum_{|n|\le N} c_n e_n$.

<u>Mean Square Convergence:</u>

If $f\in R[-\pi,\pi]$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-S_N(f)(x)|^2 dx \to 0$ as $N\to\infty$ (or $\|f-S_N(f)\|_2\to 0$). 33

Proof: First, we suppose f is continuous. Then for $\epsilon>0$, $\exists$ a trigonometric poly P s.t.

$$|f(x)-P(x)|<\epsilon \;, \forall\; x\in[-\pi,\pi].$$

Let deg $P=k$. Then $\langle P, e_n\rangle \ne 0$, if $|n|=k$, and by best approxi. lemma,

$$\|f-S_N(f)\|_2^2 < \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-P(x)|^2 \le \epsilon, \forall N\ge k.$$

Now, if $f\in R[-\pi,\pi]$, then for $\epsilon>0$, $\exists$ $g\in C[-\pi,\pi]$ s.t. $\sup|g(x)|\le \sup|f(x)|\le M$ and

$$\int|f(x)-g(x)|\,dx<\epsilon^2$$

Hence, $\|f-g\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-g(x)|\,|f(x)-g(x)|\,dx$

$$\le \frac{2M}{2\pi}\epsilon^2 \longrightarrow (2)$$

Since, $\|g-S_N(g)\|_2<\epsilon \longrightarrow (3)$

$\forall N\ge K$. From (2) and (3), we get

$$\|f-S_N(f)\|_2 \le \|f-g\|_2+\|g-S_N(g)\|_2+\|S_N(g-f)\|_2$$
$$\le \sqrt{\tfrac{2M}{2\pi}}\,\epsilon+\epsilon+\sum_{|n|\le N}|(f-g)^\wedge(n)|^2$$

$\|f - S_N(f)\|_2 \le \sqrt{\frac{m}{\pi}}\,\epsilon + \epsilon + \|f-g\|_2 < \sqrt{\frac{m}{\pi}}\,\epsilon + 2\epsilon, \ \forall N > k.$

Cor: If $f \in L^2(S^1)$, then $\|f - S_N(f)\|_2 \to 0$.

Since $\overline{R[-\pi,\pi]} = L^2[-\pi,\pi]$.

Further, $\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n|\le N} |a_n|^2$

implies $\|f\|_2^2 = \lim_{N\to\infty} \sum_{|n|\le N} |a_n|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$.

(Parseval identity).

Hence, the set $\{e_n : n \in \mathbb{Z}\}$ is a complete ONS. For this, let $f \in L^2(S^1)$ and $\langle f, e_n\rangle = 0$, $\forall n \in \mathbb{N}$, then $f = 0$ a.e. by uniqueness of F.S, since $L^2(S^1) \subset L^1(S^1)$.

Now, for $f, g \in L^2(S^1)$,

$\langle f, g\rangle = \langle \lim_{N\to\infty} \sum_{|n|\le N} \langle f, e_n\rangle e_n, g\rangle$

$= \lim_{N\to\infty} \sum \langle f, e_n\rangle \langle e_n, g\rangle = \sum \langle f, e_n\rangle \overline{\langle g, e_n\rangle}$

i.e. $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\,\overline{\hat{g}(n)}$.

Result: Let $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. Then $\exists$ unique $f \in L^2(S^1)$ such that $\hat{f}(n) = a_n$.

Proof: Consider $\sum a_n e_n(t) = \sum a_n e^{int}$.

then $\sum |a_n e^{int}|^2 dt = \sum |a_n|^2 < \infty$.

That is, $\sum a_n e^{int}$ is absolutely summable in $L^2(S^1)$. Set $f = \sum a_n e^{int}$. Then (35)
$f \in L^2(S^1)$, and $\langle f, e_n \rangle = a_n = \hat{f}(n)$.
Since F.S. of an L^2-function is unique, it follows that f must be unique.

Now, we end the topic Fourier Series by the following optimal result about the convergence of Fourier Series.

<u>Theorem</u>: Let $f \in R[-\pi, \pi]$ and $\hat{f}(n) = O(\frac{1}{n})$. Then $S_n(f;t) \to f(t)$, if t is a point of continuity of f, and limit is uniform, if f is cont. on $[-\pi, \pi]$.

<u>Proof</u>: We know that

$$\sigma_n(f;t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j) e^{ijt}$$

$$= S_n(f;t) - \sum_{|j| \le n} \frac{|j|}{n+1} \hat{f}(j) e^{ijt}$$

Since, $\sigma_n(f;t) \to f(t)$, at the point of cont. of f, we need to show that, the residual in the RHS is negligible.

For $0 < n < m$, define

$$\sigma_{m,n}(f;t) = \frac{S_{m+1}(f;t) + \cdots + S_n(f;t)}{n-m}$$ (36)

$$(1) \qquad = \frac{(n+1)\sigma_{n+1}(f;t) - (m+1)\sigma_{m+1}(f;t)}{n-m}$$

That is,

$$\sigma_{m,n} = S_m + \sum_{m<|j|\le n} \frac{n+1-|j|}{n-m} \hat{f}(j)\, e_j,$$

where $e_j(t) = e^{ijt}$.

For each fixed $k \in \mathbb{N}$, from (1),

$$\sigma_{kn,(k+1)n}(f;t) = \frac{\{(k+1)n+1\}\sigma_{(k+1)n+1}(f;t) - (kn+1)\sigma_{kn+1}(f;t)}{n}$$

$$\to (k+1)f(t) - kf(t) \quad \text{as } n \to \infty$$

$$= f(t).$$

Further, if $nk \le m < (k+1)n$, then

$$\left|\sigma_{kn,(k+1)n}(f;t) - S_m(f;t)\right| \le \sum_{kn<|j|\le(k+1)n} |\hat{f}(j)|$$

$$\le 2\sum_{j=nk+1}^{(k+1)n} \frac{A}{j} \le \frac{2nA}{kn} = \frac{2A}{k}.$$

Now, for fixed k_0, choose $n_0 \ge k_0$ s.t.

$$\forall\, n \ge n_0, \quad \left|\sigma_{k_0 n,(k_0+1)n}(f;t) - f(t)\right| < \epsilon/2. \qquad (3)$$

For $\epsilon > 0$, select k_0 so large that $\frac{2A}{k_0} < \epsilon/2$.

Then for $m > k_0 n_0$, ~~and~~ for some $n \ge n_0$,

$k_0 n_0 \le k_0 n \le m < (k_0+1)n$,

$$\left|\sigma_{k_0 n,(k_0+1)n}(f;t) - S_m(f;t)\right| < \frac{2A}{k_0} < \epsilon/2 \quad -(4)$$

From (3) and (4), for $m \ge k_0 n_0 = N_0$ (say),

$$|S_m(f;t) - f(t)| < \epsilon.$$

<u>Isoperimetric problem:</u>

Let γ be a simple closed curve in $\mathbb{R}^2$ of length l and it encloses the area A. Then $A \le \frac{l^2}{4\pi}$

Equality holds iff γ is a circle

Proof: By using dilation, we can assume that $l = 2\pi$. Then $A \le \pi$.

Let $\gamma : [0, 2\pi] \xrightarrow{C^1} \mathbb{R}^2$ be given by

$\gamma(t) = (x(t), y(t))$, such that

$$(x'(t))^2 + (y'(t))^2 = 1$$

(i.e. γ was traced by a particle with constant speed).

Then $\frac{1}{2\pi}\int_0^{2\pi} \left((x'(t))^2 + (y'(t))^2\right)dt = 1 \quad -(1)$

Since γ is closed, $x(t)$ & $y(t)$ are 2π-periodic. Hence

$x(t) \sim \sum a_n e^{int}$ and $y(t) \sim \sum b_n e^{int}$.

As γ is given smooth, γ can be consider to be conti diff (ie $\gamma \in C^1[0, 2\pi]$), and

$$x'(t) \sim \sum a_n in e^{int}, \quad y'(t) \sim \sum b_n in e^{int}.$$

By the Parseval identity, (1) gives (38)

$$\sum_{n=-\infty}^{\infty} |n|^2 \left(|a_n|^2 + |b_n|^2\right) = 1 \quad (2)$$

Since $x(t)$ and $y(t)$ are real-valued, we have $a_n = \overline{a_{-n}}$ and $b_n = \overline{b_{-n}}$.

Now, by bilinear form of the Parseval identity,

$$A = \frac{1}{2} \left| \int_0^{2\pi} \left(x(t) y'(t) - x'(t) y(t)\right) dt \right|$$

$$= \pi \left| \sum_{n=-\infty}^{\infty} n \left(a_n \overline{b_n} - b_n \overline{a_n}\right) \right| \quad -(3)$$

Here, $|a_n \overline{b_n} - b_n \overline{a_n}| \leq 2 |a_n| |b_n| \leq |a_n|^2 + |b_n|^2$

Since, $|n| \leq n^2$. From (3), we get

$$A \leq \pi \sum |n|^2 \left(|a_n|^2 + |b_n|^2\right) = \pi \quad (\text{by } (2)).$$

When $A = \pi$, it follows that

$$x(t) = a_{-1} e^{-it} + a_0 + a_1 e^{it}$$

$$\& \ y(t) = b_{-1} e^{-it} + b_0 + b_1 e^{it} \quad (\text{from } (3)).$$

From (2), $2\left(|a_1|^2 + |b_1|^2\right) = 1$, $(\because a_{-1} = \overline{a_1}, \ b_{-1} = \overline{b_1})$.

i.e. $a_1 = \frac{1}{2} e^{i\alpha}$, $b_1 = \frac{1}{2} e^{i\beta}$.
The fact that $1 = 2 | a_1 \bar{b}_1 - \bar{a}_1 b_1 |$, we get
$$|\sin(\alpha - \beta)| = 1 \Rightarrow \alpha - \beta = k\pi/2$$
$$\Rightarrow x(t) = a_0 + \cos(\alpha + t), \quad y(t) = b_0 \pm \sin(\alpha + t).$$