Norm linear spaces:

Norm linear space is eventually about mixing of linear structure of a vector space with some topological structure on the space.

Let $(X, +, \cdot)$ be a linear space over the field $F (= \mathbb{C}$ or $\mathbb{R})$.

Suppose X has a topological structure, say $(X, \mathcal{J})$ is a top. space too.

Now, the question is: how to mix top. structure with the linear structure. A linear space mainly concerned about two maps. For $x, y \in X$, $\alpha \in F$,

$$(x, y) \longmapsto x+y \quad (X \times X \longrightarrow X)$$

and $$(\alpha, x) \longmapsto \alpha x \quad (F \times X \longrightarrow X)$$

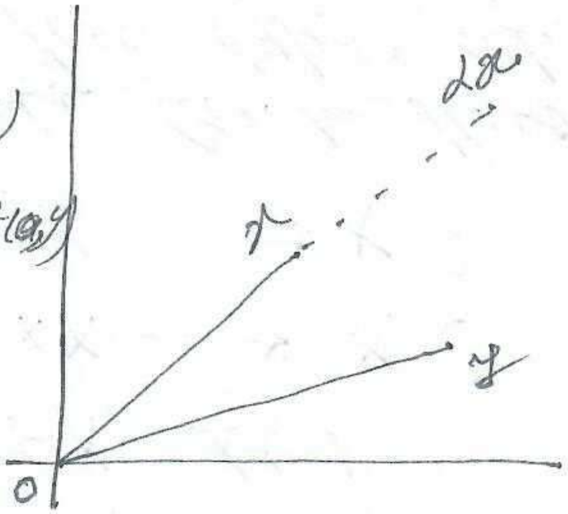
Therefore, a linear space can be thought of made by these two types of maps. But topology is all about continuity of functions on X. Thus, we can think of continuity of maps "$+$" & "$\cdot$" on product top. $X \times X$ and $F \times X$ respectively. In case both maps are continuous in their respective product topologies, we say X is a top. vector space

Now because of linearity and homogeneity of the space X, we can opt a sense of distance should satisfies the following set of rules.

i) $\text{dist}(0, \alpha x) = |\alpha| \, \text{dist}(0, x)$

ii) $\text{dist}(x, y) \leq \text{dist}(0, x) + \text{dist}(0, y)$

iii) when $\alpha = 0$, $\text{dist}(0, 0) = 0$.



Let $p = \text{dist} : X \times X \to [0, \infty)$ be defined by $p(x) = \text{dist}(0, x)$. Then

(i) $p(x) = 0$ for $x = 0$

(ii) $p(\alpha x) = |\alpha| \, p(x)$ (Absolute homogeneity)

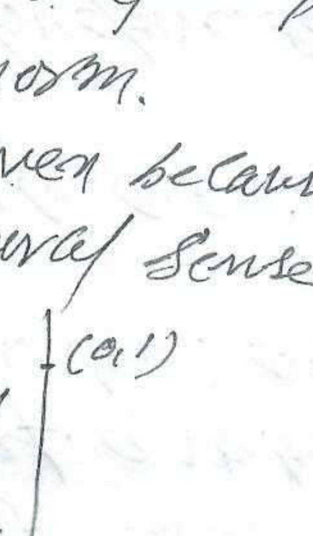
(iii) $p(x+y) \leq p(x) + p(y)$ (triangle inequality)

Here p is known as semi-norm.

The name semi-norm is given because it is little away from natural sense of distance. For example

$p : \mathbb{R}^2 \to [0, \infty)$, $p(x_1, x_2) = |x_1|$

is a semi-norm, and $p((0,1)) = 0$.



That is, the points on y-axis is at 0 distance.

away from the origin, does not look convincing as long as natural distance is concerned.

③

Let $\|\cdot\| : X \times X \longrightarrow F$ be map such that

(i) $\|x\| \geq 0, \forall x \in X$, and $\|x\| = 0$ iff $x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|, \forall (\alpha, x) \in F \times X$

(absolute homogeneity)

(iii) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$

(triangle inequality).

Then the map $\|\cdot\|$ is called a norm on X.

Note that $\|\cdot\|$ induces a metric on X by $d(x,y) = \|x-y\|$ that produces a top. on X. For $r>0$, $x \in X$,

$$B_r(x) = \{y \in X : \|x-y\| < r\}$$

is an open ball w.r.t the metric d.

Hence open sets can be defined accordingly.

Note that every metric on a linear space need not produce a norm.

For example, discrete metric on any linear space is not normable, because it fails to follow the absolute homogeneity.

For $x, y \in X$, define ④

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

If we define $\|x\| := d_0(0,x)$.
Then for $\alpha \in F$, $\|\alpha x\| \neq |\alpha| \|x\|$ $(x \neq 0)$ unless $\alpha = 1$.

Ex. Co-finite top. on $\mathbb{R}$ is not first countable and hence cannot be metrizable. Thus, co-finite top. on $\mathbb{R}$ does not ~~produce~~ a norm on $\mathbb{R}$.

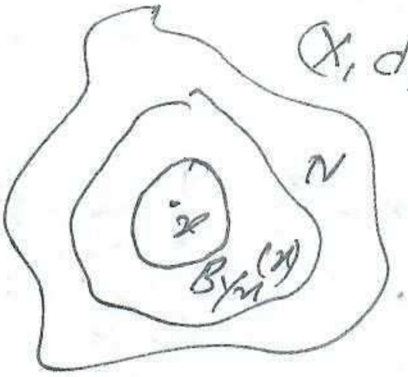
Ex. $\{f : \mathbb{R} \to \mathbb{R}\}$ is a linear space but not normable.

Defn: A top. space is 1st countable if each point $x \in X$ has a countable neighbourhood (nhd) basis.

Defn: Let $x \in X$. A nhd basis for x is a collection $\mathcal{B}_x$ of nhd of x such that for any nhd N of x, $\exists$ $B \in \mathcal{B}_x$ s.t. $B \subseteq N$.

Ex. Every metric space is 1st countable.
Let (X,d) be a metric space, and $x \in X$.
Then $\mathcal{B}_x = \{B_{1/n}(x) : n \in \mathbb{N}\}$ is a

Countable nhd. base for x.



Exercise: The quotient space $\mathbb{R}/\mathbb{N}$ is not 1st countable.

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Notice that the function $\|\cdot\|$ is uniformly w.r.t the metric induced by the norm.

For $x, y \in X$, we get

$$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|.$$

That is, $\|x\| - \|y\| \leq \|x-y\|$. By replacing x with y, we can write

$$\big|\|x\| - \|y\|\big| \leq \|x-y\|.$$

If $\epsilon > 0$, then for $\delta = \epsilon$, $\big|\|x\| - \|y\|\big| < \epsilon$ whenever $\|x-y\| < \delta$.

Hence $\|\cdot\|$ is uniformly continuous.

Ex. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function satisfying $f(\alpha x) = |\alpha| f(x)$, $\forall\, \alpha \in \mathbb{R}, x \in \mathbb{R}^n$.

Prove that

(i) $f(x+y) \leq f(x) + f(y)$, $\forall\, x, y \in \mathbb{R}^n$

(ii) $f(0) \geq 0$

(iii) $f(-x) \geq -f(x)$

(iv) $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$.

Further, what requires to make f a norm on $\mathbb{R}^n$?

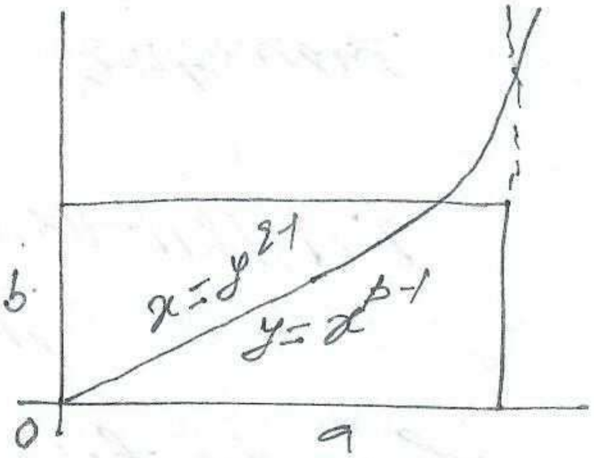
We need certain inequalities to deal with sequence spaces:

<u>Young's inequality</u>:

Let $1<p<\infty$ and $a,b\geq 0$. Then for $\frac{1}{p}+\frac{1}{q}=1$,
$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \quad \text{---} (*)$$

Let $y=x^{p-1}$, then $x=y^{q-1}$

$\left(\because \frac{1}{p}+\frac{1}{q}=1\right)$.



Now, from figure it is clear that
$$ab \leq \int_0^a x^{p-1}\,dx + \int_0^b y^{q-1}$$
$$= \frac{a^p}{p}+\frac{b^q}{q}.$$

Note that equality (in $(*)$) holds iff

$a^p = b^q$ (or $a = b^{q-1}$). For this

Consider $ab = \frac{a^p}{p}+\frac{b^q}{q}$, $\frac{1}{p}+\frac{1}{q}=1$.

Replace $a \to a^{\frac{1}{p}}$, $b \to b^{1/q}$ & $\frac{1}{p}=\alpha$

Then $a^{\alpha} b^{1-\alpha} = \alpha a + (1-\alpha) b$

or $t^{\alpha} - \alpha t - (1-\alpha) = 0$ iff $t = a/b$.

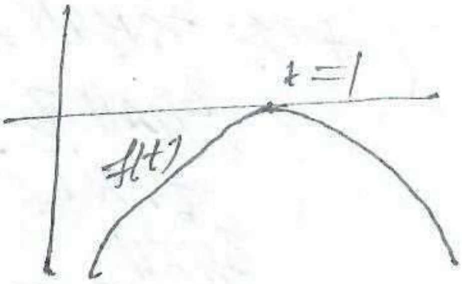
Let $f(t) = t^{\alpha} - \alpha t - (1-\alpha)$, $t \in (0,\infty)$ 7

Then $f(1) = 0$, $f'(t) = \alpha(t^{\alpha-1} - 1) = 0$ iff $t = 1$.

$\Rightarrow$ f attains its maxi at $t = 1$ and

$f(t) \leq f(1) = 0$.

Thus, $f(t) = 0$ iff $t = 1$



Ex. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Write $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. Then $(\mathbb{R}^n, \|\cdot\|_1)$ is a normed linear space. If $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$. Then by Cauchy-Schwarz inequality $(\mathbb{R}^n, \|\cdot\|_2)$ is normed linear space.

For $\|x\|_{\infty} = \sup_i |x_i|$, $(\mathbb{R}^n, \|\cdot\|_{\infty})$ is a n.l.s.

For $1 \leq p < \infty$, write $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, and $l_n^p := (\mathbb{R}^n, \|\cdot\|_p)$ will be a n.l.s.

<u>Space of Sequences</u>: Let $1 \leq p < \infty$. Let l^p be the space of all sequences that satisfies $\sum_{i=1}^{\infty} |x_i|^p < \infty$, $x = (x_1, x_2, \dots)$.

Then $(l^p, \|\cdot\|_p)$ or simply l^p will be a normed linear space.
To prove this, we need the following inequalities.

<u>Holder's inequality</u>:
Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x \in l^p$, and $y \in l^q$, it implies that $x \cdot y \in l^1$ and

$$\|x \cdot y\|_1 \le \|x\|_p \|y\|_p \quad \text{--- } (*)$$

(where $\frac{1}{\infty} = 0$, $x \cdot y = \sum_{i=1}^{\infty} x_i y_i$)

When $p = 1$, $q = \infty$, In this case $(*)$ is trivially holds.
Now, let $1 < p < \infty$. Then $1 < q < \infty$.
In the young's inequality, Substitute

$$a = a_j = \frac{|x_j|}{\|x\|_p} \ \& \ b = b_j = \frac{|y_j|}{\|y\|_q}.$$

Then

$$\sum_{j=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \le \sum_{j=1}^{n} \left(\frac{|x_j|^p}{p\,\|x\|_p{}^p} + \frac{|y_j|^q}{q\,\|y\|_q{}^q} \right)$$

$$\le \frac{1}{p} + \frac{1}{q} = 1. \ (\text{Notice that})$$

That is, $\sum_{j=1}^{n} |x_j y_j| \le \|x\|_p \|y\|_q$, $\forall n \in \mathbb{N}$
LHS is an increasing seq^n which bdd

above, hence $\|x \cdot y\|_1 \le \|x\|_p \|y\|_q$.

Note that for $p=\infty$, $\|x\|_p = \sup |x_i| < \infty$, then $(\ell^\infty, \|\cdot\|_\infty)$ is a n.l.s. (9)

Minkowski's inequality:

let $1 \le p \le \infty$. Then for $x, y \in \ell^p$, then

$$x+y \in \ell^p \text{ and } \|x+y\|_p \le \|x\|_p + \|y\|_p \quad \text{---} (*)$$

proof: For $p=1$, the proof is trivial.

let $1 < p < \infty$. Then

$$\|x+y\|_p = \left(\sum |x_j+y_j|^p\right)^{1/p}$$

$$\le \left(\sum (|x_j|+|y_j|)^p\right)^{1/p} \quad \text{---} (1)$$

$\because (|x_j|+|y_j|)^p = (|x_j|+|y_j|)^{p-1}|x_j| + (|x_j|+|y_j|)^{p-1}|y_j|$,

by Holder's inequality,

$$\sum (|x_j|+|y_j|)^{p-1}|x_j| \le \left(\sum (|x_j|+|y_j|)^{(p-1)q}\right)^{1/q} \left(\sum |x_j|^p\right)^{1/p}$$

Thus, $\sum (|x_j|+|y_j|)^p \le \left(\sum (|x_j|+|y_j|)^p\right)^{1/q} (\|x\|_p + \|y\|_p)$.

That is,

$$\left(\sum (|x_j|+|y_j|)^p\right)^{1-\frac{1}{q}} \le \|x\|_p + \|y\|_q$$

From (1), $\|x+y\|_p \le \left(\sum (|x_j|+|y_j|)^p\right)^{1/p} \le \|x\|_p + \|y\|_p$

Remarks (i) Note that equality in $\|x\cdot y\|_1 \le \|x\|_p \|y\|_q$ holds iff $\frac{|x_j|^p}{\|x\|_p^p} = \frac{|y_j|^q}{\|y\|_q^q}$.

(ii) Equality in $\|x+y\|_p \le \|x\|_p + \|y\|_p$ holds iff $x = \frac{\|x\|_p}{\|y\|_p} y$.

Now, if $x, y \in l^p$, then $x+y \in l^p$ can be seen directly without Minkowski's inequality. For $a, b > 0$,

$$(a+b)^p \le \{2\max\{a,b\}\}^p$$

i.e. $(a+b)^p \le 2^p (a^p + b^p)$

$$\Rightarrow \sum_{j=1}^{n} |x_j + y_j|^p \le 2^p \left(\sum_{j=1}^{n} |x_j|^p + \sum_{j=1}^{n} |y_j|^p\right)$$

$$\le 2^p \left(\|x\|_p^p + \|y\|_p^p\right) < \infty.$$

$$\Rightarrow \|x+y\|_p \le 2\left(\|x\|_p^p + \|y\|_p^p\right)^{1/p} < \infty.$$

Thus, l^p is closed under $\|\cdot\|_p$. Hence $(l^p, \|\cdot\|_p)$ is a n.l.s.

Ex. Since we know that any convergent seq^n is bounded, it follows that space c of all seq^{n}s under the norm

$\|x\|_\infty = \sup |x_i|$ is a normed linear space. Further, the space c_0 of all seq^ns converges to 0 is also a n.l.s. (12)

i.e. $x = (x_1, x_2, \ldots)$

$\lim_{n\to\infty} |x_n| = 0$.

That is, $(c_0, \|\cdot\|_\infty)$ is a linear subspace of $(c, \|\cdot\|_\infty)$.

Ex. Show that the following strict inclusions hold.

$$\ell^1 \subsetneq \ell^2 \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$$

Ex. For $x = (x_1, \ldots x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$), show that

$$\|x\|_\infty \le \|x\|_1 \le \sqrt{n}\, \|x\|_2 \le n \|x\|_\infty.$$

(Hint: $x = (x_n) \in \ell^1$, then $x \in \ell^\infty$, and

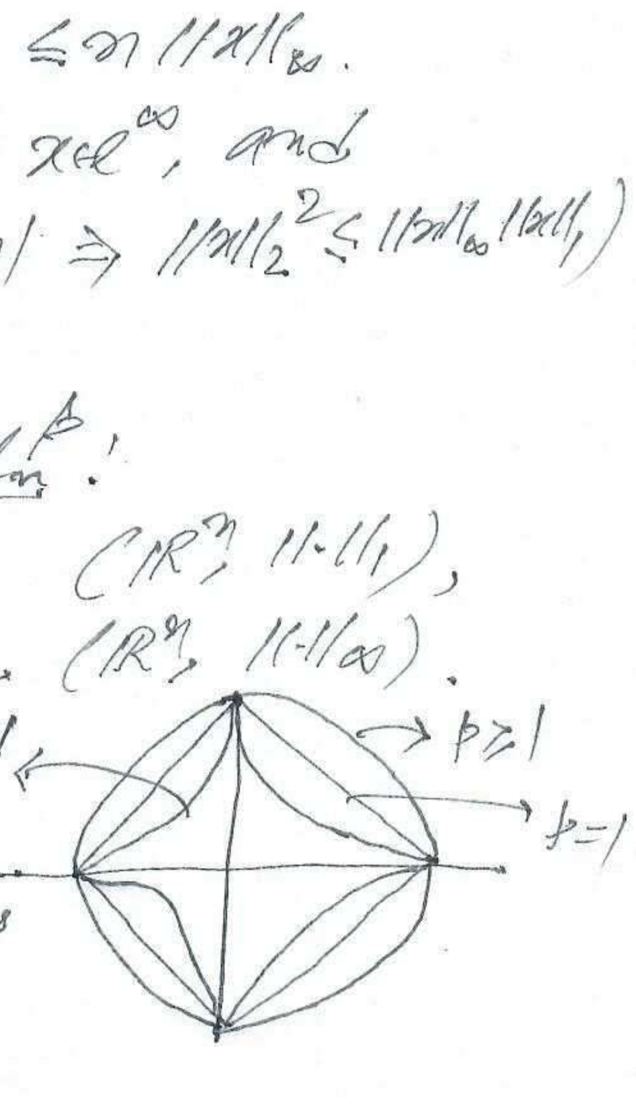
$\sum |x_n|^2 \le \sum \|x\|_\infty |x_n| \Rightarrow \|x\|_2^2 \le \|x\|_\infty \|x\|_1$)

$\Rightarrow \ell^2 \subset \ell^1$)

<u>Geometry of Sphere in ℓ^p_n</u>:

For $0 \le p \le \infty$, Consider $(\mathbb{R}^n, \|\cdot\|_1)$, $(\mathbb{R}^n, \|\cdot\|_2)$, $(\mathbb{R}^n, \|\cdot\|_p)$, $(\mathbb{R}^n, \|\cdot\|_\infty)$.

Ex. Trace the following figure for different values of p.



<u>Space of finite seq^{n}s c_{00}:</u> (12)

Let c_{00} be the space of all sequences having finitely many non-zero terms. That is,

$$c_{00} = \{ x = (x_1, \ldots x_n, 0, 0, \ldots) : x_i \in F \}.$$

Then x is a bounded seq^n and

$$\|x\|_\infty := \max_{1 \le i \le n} |x_i| < \infty$$ defines a norm on c_{00}.

Notice that the space of all seq^{n}s c_{00} is dense in all ℓ^p; $1 \le p < \infty$, which we see later. However, closure of c_{00} is c_0 which is a closed proper subspace of ℓ^∞.

For $x^n = (1, \frac{1}{2}, \ldots \frac{1}{n}, 0, \ldots) \in c_{00}$,

and $x = (1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots)$

$$\|x - x^n\|_\infty = \sup_{k \ge n} \frac{1}{k+1} = \frac{1}{n+1} \to 0.$$

But $x \notin c_{00}$, hence c_{00} is not in ℓ^∞. In addition, c_{00} is not even open in ℓ^∞.

For this, let $\epsilon > 0$ be arbitrarily small.
Then for $B_\epsilon(0) \in l^\infty$, $(\epsilon/2, \epsilon/2, \ldots) \in B_\epsilon(0)$,
but $(\epsilon/2, \epsilon/2, \ldots) \notin C_{00}$. Hence
$B_\epsilon(0) \not\subseteq C_{00}$, for any $\epsilon > 0$.

On the other hand if $B_\epsilon(0) \subset l^\infty$, implies
$B_\epsilon(0) \subset l^\infty$. Then for any $x \in l^\infty$,
$\frac{\epsilon x}{\epsilon \|x\|_\infty} \in B_\epsilon(0) \subset C_{00} \Rightarrow x \in C_{00}$,
because C_{00} is a linear space.
That means, $l^\infty \subset C_{00}$, which is absurd.

Notice that $C_{00} \subsetneq l^p$, $1 \le p \le \infty$. But C_{00} is neither closed nor open in l^p.

Consider $x_n = \left(\frac{\epsilon^p}{2^{n+1}}\right)^{\frac{1}{p}}$, $1 \le p < \infty$,
and write $x = (x_1, x_2, \ldots)$. Then
$x \in B_\epsilon(0) \subset l^p$, but $x \notin C_{00}$.

Now, write $X^n = (x_1, \ldots x_n, 0, 0, \ldots) \in C_{00}$.
Then $\|X^n - x\|_p^p = \sum_{k=n+1}^{\infty} \frac{\epsilon^p}{2^{k+1}} \to 0$.

But $x \notin C_{00}$.

Def^n: A set $A \subset X$ (n.l.s) is said to be dense if $\forall\, x \in X$, $\exists\, x_n \in A$ such that $x_n \to x$. (14)

Note that for $x = (x_1, x_2, \cdots) \in \ell^p$, $1 \le p < \infty$, and $X_n = (x_1, x_2, \cdots x_n, 0, \cdots) \in C_{00}$.

$$\|x - X_n\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \longrightarrow 0 \ (\because x \in \ell^p).$$

Hence $X_n \xrightarrow{\ell^p} x$. Thus, C_{00} is dense in ℓ^p, $1 \le p < \infty$. That is $\overline{C_{00}} = \ell^p$.

Further, $\overline{C_{00}} = C_0$. For this, let

$x = (x_1, x_2, \cdots x_n, \cdots) \in C_0$. Then

$\lim x_n = 0$. For $\epsilon > 0$, $\exists\, n_0 \in N$

such that $|x_n| < \epsilon/2 \ \forall\, n \geq n_0$. —(1)

Now, write $X_n = (x_1, x_2, \cdots x_n, 0, 0, \cdots)$

then $X_n \in C_{00}$, and

$$\|x - X_n\|_\infty = \sup_{n \geq n_0} |x_{n+1}| \le \epsilon/2 \ \forall\, n \geq n_0.$$

$\Rightarrow X_n \xrightarrow{\|\cdot\|_\infty} x$.

Remark: $\overline{C_{00}} = C_0 \subsetneq \ell^\infty$. That is, C_0 is not dense in ℓ^∞.

Space of functions:

Let $C[a,b] = \{ f : [a,b] \to \mathbb{R} \ (\text{or } \mathbb{C}) \text{ conti} \}$. (15)

Define $\|f\|_u = \sup_{a \le t \le b} |f(t)|$. Then

$(C[a,b], \|\cdot\|_u)$ is a normed linear space.

We know that any conti function on a compact set in a metric space is bounded. Hence for $K \subset X$, a metric space. $C(K)$ is a normed linear space with $f \in C(K)$, $\|f\|_u = \sup_{x \in K} |f(x)|$.

Note that $\|\cdot\|_\infty$ also used for $\|\cdot\|_u$.

Let $R[a,b]$ be the space of all Riemann integrable functions on $[a,b]$. Define

$$\|f\|_1 := \int_a^b |f(x)|\, dx.$$

Then $\left|\int_a^b (f+g)\right| \le \int_a^b |f+g| \le \int_a^b |f| + \int_a^b |g|$.

Hence $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$.

Also $\|\alpha f\|_1 = |\alpha| \|f\|_1$, but $\|f\|_1 = 0$, need not implies $f \equiv 0$. However, the function is zero almost everywhere.

If $B[a,b]$ is the space of all bounded function on $[a,b]$. Then $(B[a,b], \|\cdot\|_\infty)$ is a n.l.s. with $\|f\|_\infty = \sup_{a\le t\le b} |f(t)|$. 16

Note that $C[a,b] \subsetneq R[a,b] \subsetneq B[a,b]$.

L^1-space:

Let $(\mathbb{R}, M, m)$ be Lebesgue measure space. Let $L^1(\mathbb{R}, M, m)$ be the space of all L.m'ble function on $\mathbb{R}$ s.t. $\int_{\mathbb{R}} |f| dm < \infty$.

Then L^1 is a n.l.s. by identifying $[0] = \{g \in L^1 : g = 0 \text{ a.e. } m\}$.

Note that $\int_{\mathbb{R}} |f| = 0 \Rightarrow f = 0$ a.e.

for this, let $E = \{x \in \mathbb{R} : |f(x)| > 0\}$.

then $E = \cup E_n$, where $E_n = \{x : |f(x)| \ge \frac{1}{n}\}$.

Now, $m(E_n) = n \int_{E_n} \frac{1}{n} dm \le \int_{E_n} |f| dm \le \int_{\mathbb{R}} |f| dm = 0$.

Hence, $m(E) \le \sum m(E_n) = 0$.

Thus, $\int |f| = 0 \Rightarrow f = 0$ a.e.

We know that $R[a,b] \subset L^1([a,b], \mathcal{M}, m)$,

and $\int_a^b |f(x)|\,dx = \int_{[a,b]} |f|\,dm$. (17)

Thus, if $\int_a^b |f(x)|\,dx = 0$, then $f = 0$ a.e. m.

Hence $(R[a,b], \|\cdot\|_1)$ is normed linear space if we identify $[0] = \{g : g = 0 \text{ a.e.}\}$. That is, zero function, we mean almost zero function.

For $1 \le p < \infty$, we define $L^p(\mathbb{R}, \mathcal{M}, m)$ as the space of all m'ble functions on $\mathbb{R}$ s.t. $\int_{\mathbb{R}} |f|^p\,dm < \infty$. Write

$$\|f\|_p = \left(\int_{\mathbb{R}} |f|^p\,dm\right)^{1/p}.$$

Then following inequalities hold.

<u>Holder's inequality:</u>

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $f \in L^p(\mathbb{R})$ & $g \in L^q(\mathbb{R})$, $fg \in L^1(\mathbb{R})$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$ ---- $(*)$

Remark: Equality in $(*)$ holds iff

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}.$$

Proof: We know that $ab \le \frac{a^p}{p} + \frac{b^q}{q}$. Let
$a = \frac{|f|}{\|f\|_p}$, $b = \frac{|g|}{\|g\|_q}$. Then

$$\int_{\mathbb{R}} \frac{|fg|}{\|f\|_p \|g\|_q} \le \int_{\mathbb{R}} \frac{|f|^p}{p\|f\|_p^p} + \int_{\mathbb{R}} \frac{|g|^q}{q\|g\|_q^q}$$

$$\Rightarrow \|fg\|_1 \le \|f\|_p \|g\|_q.$$

Minkowski inequality:

Let $1 \le p < \infty$ & $f, g \in L^p(\mathbb{R})$. Then
$f+g \in L^p(\mathbb{R})$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.
Equality holds iff $f = g$ a.e.

Proof: For $p=1$, $f, g \in L^1$, we have

$$\left|\int (f+g)\right| \le \int |f| + \int |g|.$$

$$\text{ie } \|f+g\|_1 \le \|f\|_1 + \|g\|.$$

Now consider $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Then $p = q(p-1)$ and hence

$$\int |f+g|^{(p-1)q} = \int |f+g|^p < \infty.$$

$$\Rightarrow |f+g|^{p-1} \in L^q(\mathbb{R}) \text{ and } |f| \in L^p(\mathbb{R}).$$

By Holder's inequality, we get

$$\|f+g\|_p^p \le \int |f+g|^{p-1}(|f|+|g|)$$ (19)

$$= \int |f+g|^{p-1}|g| + \int |f+g|^{p-1}|g|$$

$$\le \|f\|_p \, \|(f+g)^{p-1}\|_q + \|g\|_q \, \|(f+g)^{p-1}\|_q \quad \text{— (1)}$$

But

$$\|(f+g)^{p-1}\|_q^q = \int |f+g|^{(p-1)q} = \int |f+g|^p$$

$$= \|f+g\|_p^p \quad \text{——— (2)}$$

From (1) and (2), we get

$$\|f+g\|_p^{p(1-\frac{1}{q})} \le \|f\|_p + \|g\|_p$$

$$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_q \quad \text{—— (*)}$$

<u>Exercise</u>: Show that equality in (*) holds iff $f = \alpha g$ for some $\alpha > 0$.

For $1 \le p < \infty$, if we define

$\|f\|_p = \left(\int |f|^p\right)^{1/p} < \infty$, then the $(L^p(\mathbb{R}), \|\cdot\|_p)$ is a normed linear space. Because $\|f\|_p = 0$ iff $f = 0$ a.e., $\|\alpha f\|_p = |\alpha| \|f\|_p$ and by Minkowski inequality, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Notice that, in general, $L^1(\mathbb{R}) \subsetneq L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$. (20)

For this, let $f(x) = \frac{1}{\sqrt{x}} \chi_{[0,1]}(x)$. Then $f \in L^1(\mathbb{R})$, but $f \notin L^2(\mathbb{R})$.

On the other hand, $g(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$, $g \in L^2(\mathbb{R})$ but $g \notin L^1(\mathbb{R})$.

$$\int_{\mathbb{R}} |g| \, dm = 2 \int_{[0,\infty)} \frac{1}{1+x} \, dm = \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dx}{1+x} \geq \sum \frac{1}{n+1} = \infty.$$

(By Beppo-Levi theorem)

Ex. Let $f(x) = \frac{1}{\sqrt{x}} \chi_{[0,1]}(x)$, and write $f_n(x) = f(x-n)$. Define

$g = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. Then $g \in L^1(\mathbb{R})$, but $g \notin L^2(\mathbb{R})$. For this, consider

$$\int g \, dm = \sum \frac{1}{2^n} \int f_n \, dm = \sum \frac{1}{2^n} \int_{[n, n+1]} \frac{1}{\sqrt{x-n}} \, dm$$

$$= \sum \frac{1}{2^n} \int_{[0,1]} \frac{1}{\sqrt{x}} \, dm = \sum \frac{1}{2^n} \cdot 2 = 4.$$

Now, $\int_{\mathbb{R}} g^2 dm = \sum \frac{1}{2^{2n}} \int_{\mathbb{R}} |f_n|^2 dm = \sum \frac{1}{2^{2n}} \int_0^1 \frac{1}{x} dm$

$\Rightarrow \int_{\mathbb{R}} g^2 dm = \infty.$

(Hint: use the fact that if $E_1 \cap E_2 = \phi$, then χ_{E_1} & χ_{E_2} are L.I.)

Banach Spaces:

A normed linear space $(X, \|\cdot\|)$ is said to be complete if every Cauchy seq^n in X has limit in X.

That is, if $x_n \in X$ is C.S. then $\exists\, x \in X$ such that $x_n \to x$.

A complete normed linear space is known as Banach space.

Ex. $(\mathbb{R}, |\cdot|)$ is a Banach space.

Let $x_n \in \mathbb{R}$ be a C.S., Then x_n is bounded in $\mathbb{R}$. By Bolzano-Weierstrass theorem, $\exists$ a subseq^n $x_{n_k} \to x \in \mathbb{R}$.

Hence, $x_n \to x$.

Note that this follows the following:

Ex. Show that every C.C. in a metric space having conv. subsequence is convergent. (22)

(Hint: $d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x)$)

Ex. $(\mathbb{R}^n, \|\cdot\|_p)$ is complete for any p; $1 \le p \le \infty$.

Let $1 \le p < \infty$, and write $x^k = (x_1^k, \dots, x_n^k)$ be a C.C. in $(\mathbb{R}^n, \|\cdot\|_p)$. Then

$$\|x^k - x^l\|_p \le \left(\sum |x_j^k - x_j^l|^p\right)^{1/p} < \epsilon, \ \forall\ l, k \ge N.$$

$$\Rightarrow |x_j^l - x_j^k| < \epsilon, \ \forall\ l, k \ge N.$$

$\Rightarrow (x_j^l)_{l=1}^{\infty}$ is a C.C. in $\mathbb{R}$, and hence conv. Say $x_j^l \to x_j \in \mathbb{R}$, $j = 1, 2, \dots, n$.

Write $x = (x_1, x_2, \dots, x_n)$. Then

$$\|x^k - x\|_p < n^{\frac{1}{p}} \epsilon, \ \forall\ k \ge N.$$

$$\Rightarrow x^k \to x \in \mathbb{R}^n$$

For $p = \infty$, $\|x^k - x^l\|_\infty < \epsilon$, $\forall\ l, k \ge N$

$$\Rightarrow |x_j^k - x_j^l| < \epsilon \ \text{etc.}$$

(Similar argument as above work).

Ex. Let $1 \le p \le \infty$. Show that $(\ell^p, \|\cdot\|_p)$ is Complete.

For $1 \le p < \infty$, write $x^k = (x_1^k, \dots, x_n^k, \dots)$.
Suppose x^k be a C.S. in $(\ell^p, \|\cdot\|_p)$.
Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$\|x^l - x^k\|_p < \epsilon, \ \forall \ k, l \ge N.$$

$$\Rightarrow \sum_{j=1}^{n} |x_j^k - x_j^l|^p \le \sum_{j=1}^{\infty} |x_j^k - x_j^l|^p < \epsilon^p \quad \text{—(1)}$$

for each fixed n. But then it reduce to $(\mathbb{R}^n, \|\cdot\|_p)$ for each fixed $n \in \mathbb{N}$, which is complete.

From (1), (x_j^k) is a C.S. for $j = 1, 2, \dots, n$.
Hence $x_j^k \to x_j \in \mathbb{R}$, $j = 1, 2, \dots, n$.
Now, letting $k \to \infty$ in (1), we get

(2) $\sum |x_j - x_j^l|^p \le \epsilon^p$, $\forall \ l \ge N$.

Note that LHS of (2) is an $\uparrow$ seqn in n, which is bounded above.
Thus, letting $n \to \infty$, $\sum_{j=1}^{\infty} |x_j - x_j^l|^p \le \epsilon^p$.

i.e. $\|x - x^l\|_p \leq \epsilon$, $\forall\, l \geq N$

Now, $\|x\|_p \leq \|x - x^N\|_p + \|x^N\|_p$ (24)

$$\leq \epsilon + \|x^N\|_p < \infty.$$

Hence, $x \in l^p$.

For $p = \infty$, $\sup\limits_{1 \leq j \leq m} |x_j^l - x_j^k| \leq \sup\limits_{j \in N} |x_j^l - x_j^k| < \epsilon$

Since $(\mathbb{R}^m, \|\cdot\|_\infty)$ is complete, a similar argument as the above would give the result.

Ex. $(c_0, \|\cdot\|_\infty)$ is a complete normed linear space.

Since every closed subspace of a complete metric space is complete, it is enough to show that $(c_0, \|\cdot\|_\infty)$ is a ~~$(l^\infty, \|\cdot\|_\infty)$~~ closed subspace of $(l^\infty, \|\cdot\|_\infty)$.

Let $x^k = (x_1^k, \cdots x_n^k, \cdots) \in c_0$ and $x^k \to x$. That is, $\sup\limits_{j \in N} |x_j^k - x_j| < \epsilon/2$, for all $k \geq N$.

Claim $x \in C_0$. We have

$$|x_j^N - x_j| \le \sup_{j \in \mathbb{N}} |x_j^N - x_j| < \epsilon/2 \quad , \forall j \ge 1 \qquad \text{—(1)}$$

Since $x_j^N \in C_0 \Rightarrow \lim_{j \to \infty} x_j^N = 0$.

letting $j \to \infty$ in (1), it follows that

$$\lim_{j \to \infty} |x_j| \le \epsilon/2 \quad \forall j \ge 1.$$

That is $x_j \to 0$. Thus, $x \in C_0$.

Ex. The space $(C[a,b], \|\cdot\|_\infty)$ is a Complete n.l.s.

let $f_n \in C[a,b]$ be a C.S. Then for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$(*) \quad \|f_m - f_n\| < \epsilon/2 \ , \forall m, n \ge N.$$

$$\Rightarrow |f_n(t) - f_m(t)| < \epsilon/2 \ , \forall m, n \ge N.$$

For each fixed $t \in [a,b]$, $f_n(t)$ is a C.S. in $\mathbb{R}$. Therefore $f_n(t) \to f(t) \in \mathbb{R}$. f is well-defined, because limit is unique.

Notice that N is independent of t.

letting $m \to \infty$ in $(*)$ (26)

$$\Rightarrow |f(t) - f_m(t)| \le \epsilon/2 < \epsilon, \ \forall m \ge N.$$

and $\forall t \in [a,b]$.

Now, $|f(t) - f(s)| \le |f(t) - f_N(t)|$

$$+ |f_N(t) - f_N(s)| + |f_N(s) - f(t)|$$

$\Rightarrow |f(t) - f(s)| < \epsilon$ if $|t-s| < \delta$.

($\because f_N$ is uniformly conti on $[a,b]$)

However, $(C[0,1], \|\cdot\|_1)$ is not complete.

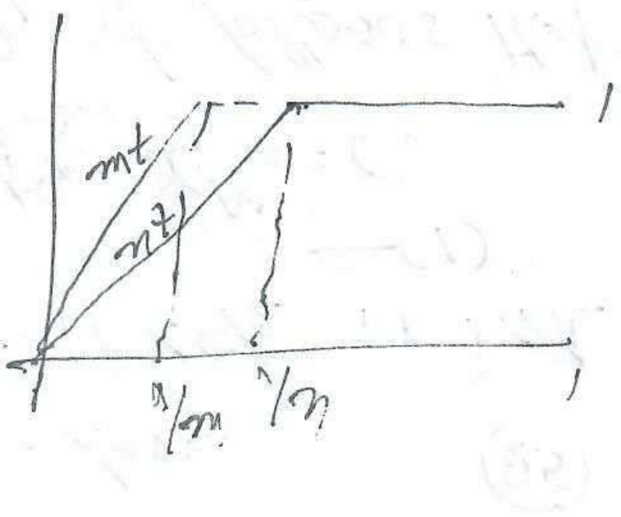
Consider $f_n(t) = \begin{cases} nt & 0 \le t < 1/n \\ 1 & 1/n \le t \le 1. \end{cases}$

Then $f_n \in C[0,1]$. Let $1/m < \frac{1}{n}$ $(n < m)$

$\|f_n - f_m\|_1 = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right)$

$\|f_n - f_m\|_1 \to 0$ as $m, n \to \infty$.

$\Rightarrow f_n$ is C.C.



in $(C[0,1], \|\cdot\|_1)$. But,

$$f(t) = \lim f_n(t) = \begin{cases} 0 & t=0 \\ 1 & 0<t\le 1 \end{cases}$$

which is not continuous on $[0,1]$.

Note: $t=0$, $f_n(0)=0 \Rightarrow f(0)=0$

$0<t_0<1 \Rightarrow t_0 > \frac{1}{n_0}$ for some $n_0 \Rightarrow n_0 t_0 > 1$.

$\Rightarrow t_0 > \frac{1}{n_0} > \frac{1}{n_0+1} \Rightarrow t_0 > \frac{1}{n}$, $n \geq n_0$.

$\Rightarrow f_n(t_0) = 1 \Rightarrow \lim f_n(t_0) = 1$.

Proposition: Let $1 \le p < q < \infty$. Then $L^q([0,1]) \subset L^p([0,1])$.

Proof:

$$\int_{[0,1]} |f|^p = \int_{\{x:|f(x)|<1\}} |f|^p + \int_{\{x:|f(x)|\geq 1\}} |f|^p$$

$$\le m\{x:|f(x)|<1\} + \int_{\{x:|f(x)|\geq 1\}} |f|^q$$

$$\le 1 + \int_{\{x:|f(x)|\geq 1\}} |f|^q < \infty.$$

Further, let $r = q/p$. Then $r>1$.

Write $\frac{1}{r} + \frac{1}{s} = 1$. Now, $|f|^p = |f|^{q/r}$

$\Rightarrow |f|^p \in L^r([0,1])$.

$$\int |f|^p = \int |f|^{\frac{q}{r}} \cdot 1 \le \| |f|^q \|_r \, \|1\|_s$$

$$\Rightarrow \|f\|_p \le \left(\int |f|^q\right)^{\frac{1}{rp}} \cdot 1 = \|f\|_q .$$

Theorem: For $1 \le p < \infty$, the space $L^p(\mathbb{R})$ is complete. Moreover, if $f_n \to f$ in $L^p(\mathbb{R})$, then $\exists$ a subsequence f_{n_k} of f which converges pointwise a.e. m.

Proof: Let $\{f_n\}$ be a Cauchy seq^n in $L^p(\mathbb{R})$.

Then $\|f_{n_{j+1}} - f_{n_j}\|_p < \frac{1}{2^j}$, $\forall j \geq 1$ (Ex)

Write $f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j})$ — (1)

and $g = |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|$ — (2)

Then $S_k(g) = |f_{n_1}| + \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}| \uparrow g$ p.w.

By Minkowski inequality

$$\|S_k(g)\|_p \le \|f_{n_1}\|_p + \sum_{j=1}^{k} \frac{1}{2^j} \le \|f_{n_1}\|_p + 1 < \infty .$$

i.e $\int_{\mathbb{R}} S_K(g)^p \uparrow$ & bounded above.

Hence by Monotone Conv. thm,

$\int_{\mathbb{R}} g^p = \lim_{K\to\infty} \int S_K(g)^p < \infty$. Thus,

$g \in L^p(\mathbb{R})$. From (1) & (2) we get

$|f| \le g \in L^p(\mathbb{R})$.

This implies, f is finite a.e. on $\mathbb{R}$.

Hence, $S_K(f) \xrightarrow[a.e]{p.w} f \Rightarrow f_{n_K} \xrightarrow[a.e]{p.w} f$.

(there $S_K(f) = f_{n_1} + \sum_{j=1}^{K} (f_{n_{j+1}} - f_{n_j})$).

Note that $|f_{n_K} - f|^p \xrightarrow[a.e]{p.w} 0$ & we have

$|f_{n_K} - f|^p \le 2^p \left(|f_{n_K}|^p + |f|^p\right)$

$\le 2^p \left(S_K(g)^p + g^p\right) \le 2^{p+1} g^p \in L^1(\mathbb{R})$.

By Dominated Conv thm,

$\lim \int |f_{n_K} - f|^p = 0$

i.e $\lim \| f_{n_K} - f \|_p = 0$, and f_n is a C.C. in $L^p(\mathbb{R})$, it follows that $f_n \to f$ in $L^p(\mathbb{R})$.

Functions Vanishing at ∞:

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be vanishing at ∞ if $\lim_{|x| \to \infty} f(x) = 0$. If f is continuous, then f is bounded, while $\lim_{|x| \to \infty} f(x) = 0$. In fact, for $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x)| < \epsilon\ \forall x,\ |x| \geq \frac{1}{\delta}$. Hence for fixed ϵ, f is bounded on $|x| \geq \frac{1}{\delta}$, and by continuity, f is bounded in $|x| \leq \frac{1}{\delta}$. Thus, f is bdd.

Let $C_0(\mathbb{R}) = \{ f : \mathbb{R} \xrightarrow{\text{cont}} \mathbb{R} \ \&\ \lim_{|x| \to \infty} f(x) = 0 \}$.

Then $(C_0(\mathbb{R}), \|\cdot\|_u)$ is a complete n.l.s. where $\|f\|_u = \sup_{x \in \mathbb{R}} |f(x)| < \infty$.

If f_n is a c.s. in $(C_0(\mathbb{R}), \|\cdot\|_u)$, then for $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$\sup_{x \in \mathbb{R}} |f_n(x) - f_m(x)| < \epsilon, \ \forall n, m \geq n_0.$$

$$\Rightarrow |f_n(x) - f_m(x)| < \epsilon,\ n, m \geq n_0,$$

for $x \in \mathbb{R}$. Let $\lim_{n \to \infty} f_n(x) = f(x)$, since

$f_n(x)$ is a c.c in $\mathbb{R}$. Then

$$|f(x) - f_n(x)| \le \epsilon, \quad \forall\, n \ge n_0, \forall\, x \in \mathbb{R}$$

Given $f_n \in C_0(\mathbb{R})$, letting $|x| \to \infty$, we get

$$\lim_{|x|\to\infty} |f(x)| \le \epsilon, \quad \forall\, \epsilon > 0.$$

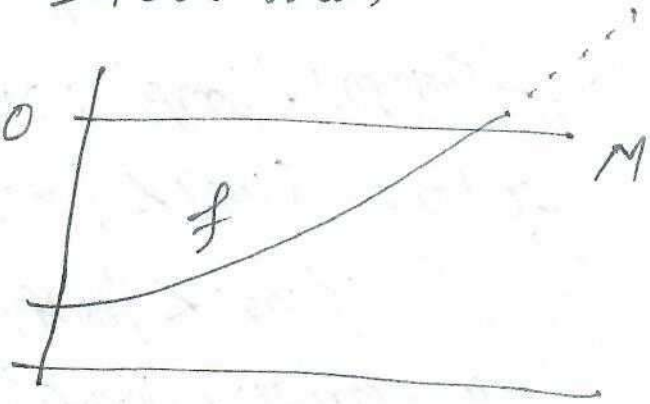
i.e. $\lim_{|x|\to\infty} |f(x)| = 0.$

$L^\infty(\mathbb{R})$ - Space:

A measurable function f on $\mathbb{R}$ is said to be essentially bounded on $\mathbb{R}$ w.r.t. m if $\exists\, M \ge 0$ such that

$$m\{x \in \mathbb{R} : |f(x)| > M\} = 0$$

(i.e. $|f(x)| \le M$ a.e. x)



Notice that if $|f(x)| > M+1$, then $|f(x)| > M$.
Hence $m\{x \in \mathbb{R} : |f(x)| > M+1\} = 0$.
Thus, we need to minimize M for f.
Denote
$$\|f\|_\infty := \inf\{M : |f(x)| \le M \text{ a.e. } x\}$$
$$= \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)|.$$

If no such M exists for f, then we

write $\|f\|_\infty = \infty$, by the convension that $\inf \Phi = \infty$. (32)

By def^n of $\|f\|_\infty$, for each $n \in \mathbb{N}$, $\exists M_n > 0$ such that

$$\|f\|_\infty + \frac{1}{n} > M_n.$$

Then $\{x \in \mathbb{R} : |f(x)| > \|f\|_\infty\}$

$$= \bigcup \{x \in \mathbb{R} : |f(x)| > \|f\|_\infty + \tfrac{1}{n}\}$$

$$\subset \bigcup \{x \in \mathbb{R} : |f(x)| > M_n\}.$$

Since $m\{x \in \mathbb{R} : |f(x)| > M_n\} = 0$, it follows that $m\{x \in \mathbb{R} : |f(x)| > \|f\|_\infty\} = 0$.

Hence, $|f(x)| \le \|f\|_\infty$ a.e. x.

It is clear that $\|f\|_\infty \le \sup_{x \in \mathbb{R}} |f(x)|$, however, both of them need not be same.

ex $f = \chi_Q$, Q set of rationals.

Then $\|f\|_\infty = 0 < \sup_{x \in \mathbb{R}} |f(x)| = 1$.

Now, consider $f(x) = \frac{1}{\sqrt{x}}$, $x > 0$. Then $f \notin L^\infty(\mathbb{R})$. Since $\frac{1}{\sqrt{x}} \le M \Rightarrow 0 < \frac{1}{M^2} < x$, which is absurd.

In general, if E is an arbitrary L-msble set of $\mathbb{R}$, $L^\infty(E) \not\subseteq L^p(E)$, for $1 \le p < \infty$.

(33)

However, if $m(E) < \infty$, then
$$L^\infty(E) \subset L^p(E), \quad 1 \le p < \infty.$$
Let $f \in L^\infty(E)$, then
$$\int_E |f|^p \le m(E)\, \|f\|_\infty^p, \text{ since}$$
$$|f(x)| \le \|f\|_\infty \text{ a.e.}$$
Thus, $\|f\|_p \le (m(E))^{1/p} \|f\|_\infty$.

Notice that $\|f\|_\infty = 0$ iff $|f(x)| \le 0$ a.e., i.e $f = 0$ a.e.

Also, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Hence $L^\infty(E)$ is a normed linear space.

Remark: For $f \in L^\infty(\mathbb{R})$, if $0 < \alpha < \|f\|_\infty$, then $m\{x \in \mathbb{R} : |f(x)| > \alpha\} > 0$.

Theorem: $L^\infty(\mathbb{R})$ is a complete n.l.s.

Proof: Let $\{f_n\}$ be a C.S. in $L^\infty(\mathbb{R})$.

Then for $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$\|f_n - f_m\|_\infty < \epsilon, \ \forall\ n, m \geq N.$$

But then,

$$|f_n(x) - f_m(x)| < \epsilon, \ \forall\ \text{a.e. } x, \ \forall\ m, n \geq N.$$

This implies,

$$|f_n(x) - f_m(x)| < \epsilon, \ \forall\ x \in E_N^c, \ \forall\ m, n \geq N$$

where $E_N = \bigcup_{m,n \geq N} E_{m,n}$, and

$$E_{m,n} = \{x \in \mathbb{R} : |f_n(x) - f_m(x)| \geq \epsilon\}.$$

But $m(E_N) = 0$. Thus, for each $x \in E_N^c$, $\{f_n(x)\}$ is a C.C. in $\mathbb{R}$.

Let $f(x) = \lim f_n(x)$, $x \in E_N^c$

Then $|f_m(x) - f(x)| \leq \epsilon, \ \forall\ m \geq N$

$$\Rightarrow \|f_m - f\|_\infty \leq \epsilon, \ \forall\ m \geq N.$$

$$\|f\|_\infty \leq \|f_N - f\|_\infty + \|f_N\|_\infty < \epsilon + \|f_N\|_\infty < \infty.$$

Hence, $f \in L^\infty$, and $f_m \to f$ in L^∞.

Ex. Let $E \in \mathcal{M}$ (σ-alg. of all L-msble sets of $\mathbb{R}$). Then $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$, if $m(E) < \infty$.

We know that

$$\|f\|_p \leq \|f\|_\infty \, m(E)^{1/p}.$$

Therefore,

$$\overline{\lim} \|f\|_p \leq \|f\|_\infty \, \overline{\lim} \left(m(E)\right)^{1/p}$$
$$= \|f\|_\infty \quad \text{—— (1)}$$

Now, for $\epsilon > 0$, $\exists \ \delta > 0$ s.t.

$m\{x \in E : |f(x)| > \|f\|_\infty - \epsilon\} > \delta$, by def^n of $\|\cdot\|_\infty$. Let

$$G = \{x \in E : \|f\|_\infty - \epsilon < |f(x)|\}.$$

Then

$$\int_E |f|^p \, dm \geq \int_G |f|^p \, dm \geq (\|f\|_\infty - \epsilon)^p m(G).$$

$$\Rightarrow \|f\|_p \geq (\|f\|_\infty - \epsilon) \left(m(G)\right)^{1/p}.$$

Since $m(G) > 0$, it follows that

$$\underline{\lim} \|f\|_p \geq (\|f\|_\infty - \epsilon) \, \underline{\lim} \left(m(G)\right)^{1/p}$$

i.e. $\underline{\lim} \|f\|_p \geq (\|f\|_\infty - \epsilon) \cdot 1, \quad \forall \epsilon > 0.$ —— (2)

From (1) & (2)

$$\underline{\lim} \|f\|_p \geq \|f\|_\infty \geq \overline{\lim} \|f\|_p.$$

Characterization of Banach spaces:

We know that every conv. series on $\mathbb{R}$ need not be absolutely conv. For instance, $\sum \frac{(-1)^n}{n}$ is not abs. conv.

However, every abs. conv. series is convergent. Suppose $\sum |x_n| < \infty$, $x_n \in \mathbb{R}$. Then

$$0 \le x_n + |x_n| \le 2|x_n|$$

$$\therefore \left|\sum_{k=1}^{n} x_k\right| \le \sum_{k=1}^{n} |x_k| \le \sum_{k=1}^{\infty} |x_k| < \infty.$$

Note that $S_n = \sum_{k=1}^{n} (x_k + |x_k|)$ $\uparrow$ & bounded above. Hence S_n is convergent. Thus,

$$\sum_{k=1}^{\infty} x_k = \sum (x_k + |x_k|) - \sum |x_k| < \infty.$$

Ex. Let $x^k = (0, 0, \ldots, \frac{1}{k^2}, 0, \ldots 0)$

$$\in (c_{00}, \|\cdot\|_\infty).$$

Then $\sum x^k$ is abs. conv. since

$$\sum \|x^k\|_\infty = \sum \frac{1}{k^2} < \infty$$

But $S_n = \sum_{k=1}^{n} x^k = (1, \frac{1}{2^2}, \ldots, \frac{1}{n^2}, 0, \ldots)$

$$\to (1, \frac{1}{2^2}, \ldots \frac{1}{n^2}, \frac{1}{(n+1)^2}, \ldots) \notin c_{00}.$$

Hence $\sum x^k$ is not convergent. But note that $\sum x^k$ converges to a pt in c_0, which is completion of c_{00}. (37)
($\because \overline{c_{00}} = c_0$).

<u>Theorem</u>: A n.l.s. $(X, \|\cdot\|)$ is a Banach space iff every abs. conv series in $(X, \|\cdot\|)$ is convergent.

Proof: Suppose X is complete, let $(x_n) \in X$ be such that
$$\sum_{n=1}^{\infty} \|x_n\| < \infty \quad \text{——} (1)$$
(abs. summable).

Write $y_n = \sum_{k=1}^{n} x_k$. If $m > n$, then
$$\|y_m - y_n\| = \|\sum_{k=n+1}^{m} x_k\| \leq \sum_{k=n+1}^{m} \|x_k\| \to 0,$$
as $m, n \to \infty$ (by (1)).

Therefore, $\{y_n\}$ is a C.S in $(X, \|\cdot\|)$ and hence conv. say to y. Hence
$$\sum_{k=1}^{\infty} x_k = y \in X.$$

Conversely, suppose every abs. conv. series in X is convergent. We claim X is complete. (38)

Let (y_n) be a C.S. in X. We need to show that (y_n) has a conv. subseq^n.

Since (y_n) is a C.S. seq^n, for $\epsilon = \frac{1}{2} > 0$, $\exists$ $n_1 \in \mathbb{N}$ s.t. $\forall$ $m \geq n_1$,

$$\|y_{n_1} - y_m\| < \tfrac{1}{2} \quad \text{——— (1)}$$

For $\epsilon = 1/4$, $\exists$ $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $\|y_{n_2} - y_m\| < \frac{1}{4}$ — (2)
$\forall$ $m \geq n_2$.

By continuing this process, we get
$n_1 < n_2 < \cdots < n_k < \cdots$ such that
for $m \geq n_k$, $\|y_{n_k} - y_m\| < \frac{1}{2^k}$.

In particular, $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k}$ (3)

Write $x_k = y_{n_{k+1}} - y_{n_k}$. Then

$$\sum_{k=1}^{n} \|x_k\| \leq \sum_{k=1}^{n} \frac{1}{2^k} < \sum \frac{1}{2^k} < \infty, \; \forall \, n \geq 1.$$

$\Rightarrow \sum_{k=1}^{\infty} \|x_k\| < \infty$. By hypothesis,

$\sum x_k$ is conv. That is, $\sum_{k=1}^{n} x_k \to x \in X$. 39

$\Rightarrow y_{n_{k+1}} - y_{n_1} \to x$.

That is, $y_{n_{k+1}} \longrightarrow y_{n_1} + x \in X$. Thus, $\{y_n\}$ is a conv. seqn in X.

Defn: Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a linear space X. We say that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ if $\exists$ $\alpha, \beta > 0$ such that

$$\alpha \|x\|_2 \le \|x\|_1 \le \beta \|x\|_2, \ \forall x \in X.$$

Ex. Show that all norms on a finite dim. n.l.s. are equivalent.

Let $X = \text{span}\{e_1, e_2, \cdots, e_n\}$, and $\|\cdot\|_2$ be the Euclidean norm on X and $\|\cdot\|$ be an arbitrary norm on X.

For $x = x_1 e_1 + \cdots + x_n e_n \in X$, define $T: X \longrightarrow \mathbb{R}$ by

$$T(x_1, x_2, \cdots x_n) = \| x_1 e_1 + \cdots + x_n e_n \|.$$

Then $\big| \| T(x_1 e_1 + \cdots + x_n e_n) \| - \| T(y_1 e_1 + \cdots + y_n e_n) \| \big|$
$\le \| (x_1 - y_1) e_1 + \cdots + (x_n - y_n) e_n \|$, implies that

T is conti on $(X, \|\cdot\|)$. Further, $T(x_1, \ldots, x_n) = 0$ iff $x_i = 0 \ \forall\, i = 1, 2, \ldots, n$.
Notice that

$$\|x_1 e_1 + \cdots + x_n e_n\| \leq |x_1| \|e_1\| + \cdots + |x_n| \|e_n\|$$
$$\leq \sqrt{x_1^2 + \cdots + x_n^2} \cdot \sqrt{\|e_1\|^2 + \cdots + \|e_n\|^2}$$
$$= \|x\|_2 \cdot (\text{constant})$$

Hence T is continuous in $(X, \|\cdot\|_2)$. Since $S_2 = \{x \in X : \|x\|_2 = 1\}$ is compact in $(X, \|\cdot\|_2)$, T attains its bounds, say m & $M > 0$ on T. That is,

$$m \leq \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \leq M, \quad -(1)$$

where $(x_1, \ldots, x_n) \in S_2$. Thus, if $0 \neq x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ (or $\mathbb{R}^n$), then $\frac{x}{\|x\|_2} \in S_2$ and hence

$$m \|x\|_2 \leq \|x\| \leq M \|x\|_2.$$

Note that $m = \inf\limits_{x \in S_2} T(x) = T(x_0)$, for some $x_0 \in S_2$. Hence, $m > 0$.

<u>Remark</u>: If $\|\cdot\|_1$ & $\|\cdot\|_2$ are two equivalent norms on X, then both generate the same topology on X.

Hint: If $\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1$, then the above fact is being followed by

(i) $B^2_{r\alpha}(0) \subseteq B^1_r(0)$ (41)

(ii) $B^1_{r/\beta}(0) \subseteq B^2_r(0)$,

because every open is union of open balls.

However, in infinite dim n.l.s. two norm need not be equivalent. For instance, $\|\cdot\|_\infty$ & $\|\cdot\|_1$ are not equivalent on $C[0,1]$. We know that

$\|f\|_1 = \int |f| \leq \|f\|_\infty$, but $\nexists\ \beta > 0$

s.t. $\|f\|_\infty \leq \beta \|f\|_1$, $\forall f \in C[0,1]$.

let $f_n(t) = t^n$, then $1 \leq \beta \frac{1}{n+1} \to 0$.

Ex. Show that $\|\cdot\|_1$ & $\|\cdot\|_2$ are not equivalent on $L^2([0,1])$.
(Hint: $L^2([0,1]) \subseteq L^1([0,1])$)

Ex. Show that $\|\cdot\|_1$ & $\|\cdot\|_2$ are not equivalent on l^1.

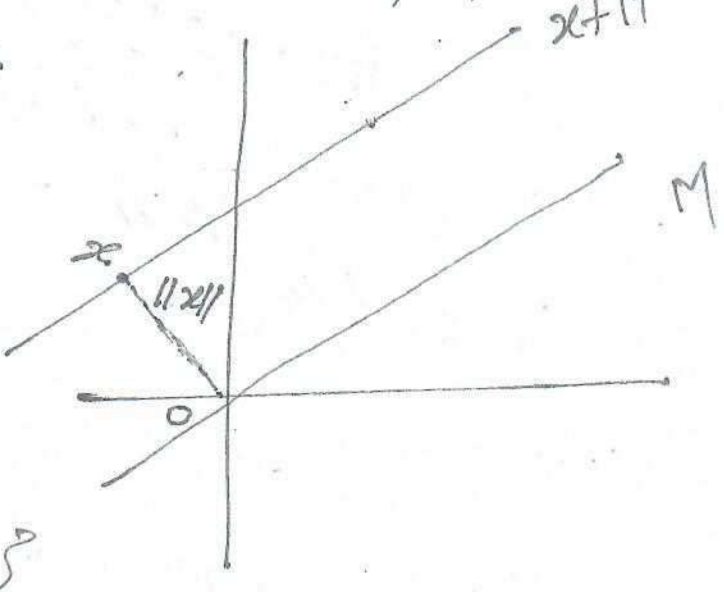
Quotient space:

Let X be a normed linear space, and M be a closed subspace of X.

For $x, y \in X$, define

$x \sim y$ iff $y - x \in M$.

Then $\sim$ is an equivalence ~~class~~ relation.



$$\tilde{x} = \{y \in X : x \sim y\}$$
$$= \{y \in X : y - x \in M\}$$
$$= \{y \in X : y \in x + M\}$$
$$= x + M.$$

Let $X/M = \{x + M : x \in X\} = \{\tilde{x} : x \in X\}$.

Then X/M is a linear space with $\tilde{0} = M$ as zero vector.

Define $\|x + M\| := \text{dist}(x + M, M)$. Then

$\|\tilde{x}\| = \|x + M\| = \text{dist}(x, M) = \inf\limits_{m \in M} \|x + m\|$.

Note that

(i) $\|\tilde{x}\| \geq 0$

(ii) $\|\tilde{x}\| = 0$ iff $\tilde{x} = \tilde{0}$.

If $\|\check{x}\| = 0$, then $\inf_{m\in M} \|x + m\| = 0$,

implies, $x \in \overline{M} = M \Rightarrow \check{x} = M = \check{0}$ ......(*)

If $\check{x} = \check{0}$, then $\|\check{x}\| = \inf_{m\in M} \|0 + m\| = 0$.

(iii) If $\check{x}, \check{y} \in X/M$, then

$$\begin{aligned} \|\check{x} + \check{y}\| &= \|x + y + M\| \\ &= \inf \|x + y + m\| \\ &= \inf_{m\in M} \|x + y + 2m\| \\ &\leq \|\check{x}\| + \|\check{y}\|. \end{aligned}$$

Note that closeness of M is required, otherwise norm on X does not induce a norm on X/M.

If M is <u>not</u> closed then $\exists\, m_k \in M$ s.t. $m_k \to x \notin M$. But then

$\|\check{x}\| = \inf_{m\in M} \|x + m\| \leq \|x - m_k\| \to 0$,

although $\check{x} \neq \check{0}$.

Ex. Let $X = C[0,1]$, and define

$\|f\|_1 = \int_0^1 |f(t)|\,dt$, for $f \in X$.

Then $M = \{f \in X : f(0) = 0\}$ is not closed in $(X, \|\cdot\|_1)$. For this, let

(44)

(*) $f_n(t) = \begin{cases} nt & 0 \le t \le 1/n \\ 1 & 1/n < t \le 1. \end{cases}$

Then $f_n \in M$, but $\|f_n - 1\|_1 \to 0$, however $1 \notin M$. Thus, M is not closed.

In fact, $\|\tilde{f}\| = 0, \ \forall f \in X$.

Here, $\|\tilde{f}\| = \inf\{\|f+g\|_1 : g \in M\}$
$= \inf\{\|h\|_1 : h - f \in M\}$
$= \inf\{\|h\|_1 : (h-f)(0) = 0\}$

Let $h_n(t) = f(0)(1 - f_n(t))$, where f_n is given by (*). Then $h_n(0) = f(0)$.

Thus, $\|\tilde{f}\| \le \|h_n\|_1 = \frac{|f(0)|}{2n} \to 0$.

However, M is a closed subspace of $(X, \|\cdot\|_\infty)$ and hence X/M is a n.l.s. w.r.t the norm induced by $\|\cdot\|_\infty$. In fact, X/M is linearly isomorphic to $\mathbb{C}$.

Define $\varphi : X/M \to \mathbb{C}$ by

$\varphi(\tilde{f}) = f(0)$. Then φ is well defined. If $g \in \tilde{f}$, then $g \sim f$ iff $g - f \in M$ iff $g(0) = f(0) = \varphi(\tilde{g})$.

Obviously, φ is linear. (45)

(i) φ is 1-1: $\varphi(\tilde{f}) = 0 \Rightarrow f(0) = 0$

$\Rightarrow f \sim 0 \Rightarrow \tilde{f} = \tilde{0}$.

(ii) φ is onto: Note that

$X = \overline{\text{Span}}\{1, x, x^2, \dots\}$.

So, for $\alpha \in \mathbb{C}$, let $\varphi(\tilde{f}) = \alpha$. Then for $f(x) = \alpha$, $f(0) = \alpha$.

Question: Does φ continuous?

Notice that

$$\|\tilde{f}\| = \inf\{\|h\|_\infty : h(0) = f(0)\}$$

But $\|h\|_\infty \geq |h(0)| = |f(0)|$

$\Rightarrow \|\tilde{f}\| \geq |f(0)|$.

Also, for $h_0(x) = f(0)$, $\|\tilde{f}\| \leq \|h_0\|_\infty$

i.e. $\|\tilde{f}\| \leq |f(0)|$.

Hence, $\|\tilde{f}\| = |f(0)|$.

Now, let $\tilde{f}_n \to \tilde{0}$, then $0 = \lim |f_n(0)|$.
But then, $\varphi(\tilde{f}_n) = f_n(0) \to 0$. (46)
Thus, φ is continuous linear map.
(Note that conti of a linear map is equivalent to its conti at $\tilde{0}$ ").
In fact, $\varphi^{-1}(f(0)) = \tilde{f}$, and

$$f_n(0) \to 0 \Rightarrow \|\tilde{f}_n\| = |f_n(0)| \to 0.$$

Hence φ^{-1} is cont. Thus, φ is a linear top. homeomorphism.

Ex. For $x = (x_1, x_2, \ldots) \in \ell^\infty(\mathbb{N})$, and $\tilde{x} \in \ell^\infty(\mathbb{N})/c_0(\mathbb{N})$. Show that

$$\|\tilde{x}\| = \limsup_{n\to\infty} |x_n|.$$

Ex. For $x \in c$ (the space of all conv seqns), show that for $\tilde{x} \in c(\mathbb{N})/c_0(\mathbb{N})$,

$$\|\tilde{x}\| = \lim |x_n|.$$

Further, deduce that $c(\mathbb{N})/c_0(\mathbb{N}) \cong \mathbb{C}$.
(Hint: $\varphi(\tilde{x}) = \lim x_n$ etc.)

<u>Theorem:</u> Let M be a closed subspace of a Banach space X. Then X/M is a Banach space. 47

Pf: Suppose $\{\tilde{x}_n\}$ be a seq^n in X/M such that $\sum_{n=1}^{\infty} \|\tilde{x}_n\| < \infty$.

Since $\|\tilde{x}_n\| = \inf_{m \in M} \|x_n + m\|$, for $\epsilon = \frac{1}{2^n} > 0$, $\exists\, m_n \in M$ s.t.

$$\|x_n + m_n\| \le \|\tilde{x}_n\| + \frac{1}{2^n}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n + m_n\| \le \sum_{n=1}^{\infty} \|\tilde{x}_n\| + \sum \frac{1}{2^n} < \infty.$$

Since X is complete, $\sum_{n=1}^{k} (x_n + m_n) \to y \in X$. ---(*)

Hence,

$$\left\|\sum_{n=1}^{k} \tilde{x}_n - \tilde{y}\right\| = \inf_{m \in M} \left\|\sum_{n=1}^{k} x_n - y + m\right\|$$

$$\le \left\|\sum_{n=1}^{k} x_n - y + \sum_{n=1}^{k} m_n\right\| \to 0$$

(by *)

Thus, $\sum \tilde{x}_n$ is convergent in X/M.

Ex. Let M be a complete subspace of a n.l.s X. If X/M is complete, then X complete.

Proof: Let $(x_n) \in X$ be such that

$\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then, we have

$$\sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum \|x_n\| < \infty.$$

Given, X/M is complete, and $\sum \|\tilde{x}_n\|$ is convergent, $\sum_{n=1}^{K} \tilde{x}_n \longrightarrow \tilde{y} \in X/M$.

i.e. for $\epsilon > 0$,

$$(1) \qquad \left\| \sum_{n=1}^{K} \tilde{x}_n - \tilde{y} \right\| < \epsilon, \text{ for } K \geq K_0.$$

Now, $\left\| \sum_{n=1}^{K} \tilde{x}_n - \tilde{y} \right\| = \inf_{m \in M} \left\| \sum_{n=1}^{K} x_n - y + m \right\|$.

for $\frac{1}{2^l} > 0$, $\exists\, m_l \in M$ such that

$$(2) \qquad \left\| \sum_{n=1}^{K} x_n - y + m_l \right\| < \left\| \sum_{n=1}^{K} \tilde{x}_n - \tilde{y} \right\| + \frac{1}{2^l}$$
$$< \epsilon + \frac{1}{2^l} \quad \text{for } K \geq K_0.$$

From (2), it follows that $\{m_l\}$ is a c.s. in M (which is complete). Hence, $m_l \to m \in M$. Letting $l \to \infty$ in (2), we get $\left\| \sum_{n=1}^{K} x_n - y + m \right\| \leq \epsilon$, $\forall\, K \geq K_0$

$\Rightarrow \sum_{n=1}^{K} x_n \to y - m \in X$. Thus, X is complete. [Note that $\|m_l - m_r\| < 2\epsilon + \frac{1}{2^l} + \frac{1}{2^r}$ $\forall\, \epsilon > 0 \Rightarrow \|m_l - m_r\| \leq \frac{1}{2^l} + \frac{1}{2^r}$, $\forall\, m, r$]

Ex. Since $C/C_0 \cong \mathbb{C}$ & C_0 is complete, it follows that C is complete.

Proposition: Let M be a closed subspace of a n.l.s. X. Define $\pi : X \longrightarrow X/M$ by $\pi(x) = \tilde{x}$. Then π is open, conti, and surjective.

Proof: Since

(i) $$\|\pi(x) - \pi(y)\| = \inf_{x \in M} \|x - y + m\| \leq \|x - y\|,$$

π is uniformly conti on X.

(ii) To show that π is open it is enough to show that

$$\pi(B_r(0)) = \tilde{B}_r(\tilde{0}).$$

where $\tilde{B}_r(\tilde{0}) = \{x + M \in X/M : \|x+M\| < r\}$.

Let $x + M \in \pi(B_r(0))$. Then $\exists\ y \in B_r(0)$ such that $x + M = y + M$, $\|y\| < r$

Hence, $\|x+M\| = \|y+M\| \leq \|y\| < r$.

$\Rightarrow x + M \in \tilde{B}_r(\tilde{0})$.

On the other hand, let $x + M \in \tilde{B}_r(\tilde{0})$.

Then $\|x + M\| < r \Rightarrow \inf_{m \in M} \|x + m\| < r$

$\Rightarrow \exists\ m_0 \in M$ s.t. $\|x + m_0\| < r$.

$\Rightarrow x+M = x+m_0+M = \pi(x+m_0)$,
where $\|x+m_0\| < r$.

Ex. Show that $\pi(B_r(x)) = \tilde{B}_r(\tilde{x})$.

Ex. Let X & Y be two n.l.s and T is a conti map on X onto Y. Show that $X/\ker T \cong Y$.

Note that in the Euclidean space $\mathbb{R}^n$, we can always draw a unique normal from a point onto a given hyperplane in $\mathbb{R}^n$. However, this may not be possible to do so for the infinite dim spaces. Riesz-lemma helps resolving this problem upto certain extent.

Riesz lemma: Let M be a proper closed subspace of a n.l.s X. Then for $0 < t < 1$, $\exists$ an unit vector $x_t \in X$ such that $dist(x_t, M) \geq t$.

Proof: Let $u \in X \setminus M$, and write

$$\delta = \inf_{m \in M} \|u - m\|. \text{ Then}$$

$\delta > 0$, because, if $\delta = 0$, then $\exists$ $m_n \in M$ s.t. $\lim \|u - m_n\| = 0 \Rightarrow u \in M$, which absurd.

For $0 < t < 1$, $\delta < \frac{\delta}{t}$. Hence by the defn of infimum $\exists$ $m_0 \in M$ such that

$$\delta \leq \|u - m_0\| \leq \frac{\delta}{t}. \quad \text{---(1)}$$

Set $x_t = \dfrac{u - m_0}{\|u - m_0\|}$. Then for $m \in M$,

$$\|m - x_t\| = \frac{\|m_1 - u\|}{\|u - m_0\|}, \text{ where}$$

$$m_1 = \|u - m_0\| m + m_0.$$

Thus, $\|m - x_t\| \geq \dfrac{\delta}{\|u - m_0\|} \geq t$.

<u>Remark</u>: (i) In general, $0 < t < 1$ is only allowed in Riesz lemma. However, if $\dim X < \infty$, there always exists x_t (unit vector) s.t. $\|x_t - M\| = 1$.

(ii) $X = \{ f \in C[0,1] : f(0) = 0 \}$ (52)
and $M = \{ f \in X : \int_0^1 f(t)\,dt = 0 \}$ then
M is a closed subspace of $(X, \|\cdot\|_\infty)$.
However, $\nexists$ $f \in X$ with $\|f\|_\infty = 1$ such that
$\|f - g\|_\infty \geq 1$, $\forall\, g \in M$
& $\inf_{g \in M} \|f - g\|_\infty = 1$.

For this, $\operatorname{dist}(f, M) = \inf_{g \in M} \|f - g\|_\infty = \inf_{h - f \in M} \|h\|_\infty$

Let $h(t) = \int f(t)$, then $\int h(t) = \int f(t)$.

Hence
$\operatorname{dist}(f, M) \leq \left| \int f(t)\,dt \right| \leq 1$, because
$\|f\|_\infty = 1$. By replacing f with $-f$,
we can assume that $\int f(t)\,dt \leq 1$.
But since $f(0) = 0$, $\int f(t)\,dt < 1$.
If $\int_0^1 f(t)\,dt = 1$, then $\int (1 - f(t))\,dt = 0$
Since $1 - f(t) \geq 0$, $\Rightarrow$ $f(t) = 1$ X
Hence $\operatorname{dist}(f, M) < 1$.

Theorem: The unit ball in a normed linear space X is compact iff X is a finite dim space.

Proof: Suppose $\overline{B_1(0)} = \{x \in X : \|x\| \leq 1\}$ is compact.

On contrary, suppose X is not a finite dim. space. Then for a unit vector $x_1 \in X$, $M_1 = \text{span}\{x_1\}$ is a closed proper subspace of X ($\because \dim X = \infty$).

Hence, by Riesz lemma, $\exists$ unit vector $x_2 \in X$ such that

$$\text{dist}(x_2, M_1) > 1/2 \cdot \quad (t = 1/2).$$

$$\Rightarrow \quad \|x_2 - x_1\| > 1/2$$

Write $M_2 = \text{span}\{x_1, x_2\}$. Then $\exists$ unit vector $x_3 \in X$ s.t. $\text{dist}(x_3, M) > 1/2$.

$\Rightarrow \|x_3 - x_i\| > 1/2, \ i = 1, 2.$

$\Rightarrow \{x_i\} \subset \overline{B_1(0)}$ such that $\|x_i - x_j\| > 1/2$. Hence $\overline{B_1(0)}$ is not compact, because $\{x_i\}$ has no conv. subseq^n. Thus, $\dim X < \infty$.

Conversely, suppose $\dim X < \infty$. (54)

Claim: $\overline{B_1(0)}$ is compact. Let

$X = \operatorname{span}\{e_1, \dots, e_n\}$. Then for $x \in X$,

$x = x_1 e_1 + \dots + x_n e_n$.

Define $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. Then $\|\cdot\|_1$ is a norm on X. Let $T : X \longrightarrow \mathbb{R}^n$ be defined by $T(x) = \sum_{i=1}^{n} x_i \cdot Te_i$.

Then T is linear bijection.

Further, $\|T(x)\|_1 \leq \sum \|Te_i\|_1 |x_i|$

$$\leq \max_i \|Te_i\| \sum |x_i|$$

so $\|T(x)\|_1 \leq K \|x\|_1$, where $K = \max_i \|Te_i\|_1$.

For $x, y \in X$,

$\|T(x) - T(y)\|_1 \leq K \|x - y\|_1$.

Hence, T is continuous w.r.t. $\|\cdot\|_1$ on X. But all norms on X are equivalent, hence T is cont. w.r.t. its given norm $\|\cdot\|$.

Similarly, $T^{-1} : \mathbb{R}^n \longrightarrow X$ is continuous, Let $\mathbb{R}^n = \text{span}\{f_1, f_2, \dots, f_n\}$, $f_i = (0, 0, \underset{i^{th}}{1}, 0, \dots 0)$.

Then for $y \in \mathbb{R}^n$, $y = \sum_{i=1}^{n} y_i f_i$. Thus, (55)

$T^{-1}(y) = \sum y_i T^{-1} f_i = \sum y_i e_i$. $(\because T^{-1} f_i = e_i)$

Hence, as similar to the above, T^{-1} is continuous w.r.t $\|\cdot\|$, and ~~hence~~ it implies that T^{-1} is conti on $\mathbb{R}^n$ w.r.t $\|x\|_2$. Thus X is top. homeomorphic to $\mathbb{R}^n$.

Notice that $T(\overline{B_1(0)})$ is a closed & bdd set in $\mathbb{R}^n$. Hence $T^{-1}(T(\overline{B_1(0)})) = \overline{B_1(0)}$ is a compact set in X.

Separable Banach Spaces:

Eventually, separability helps determing the size of a space. If the space admit a countable dense set, we say the space is seperable.

Defn: A n.l.s. space $(X, \|\cdot\|)$ is said to be separable if $\exists$ a countable dense set $A \subset X$. That is, $\overline{A} = X$.

For example, $\mathbb{Q}$ (the set of rationals) is a countable dense subset of $\mathbb{R}$.

Like-wise, $\mathbb{Q}^n$ and $\mathbb{Q}^n + i\mathbb{Q}^n$ are countable dense subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively. It can be easily seen that $(\mathbb{R}^n, \|\cdot\|_p)$ is separable ~~seper~~ separable for $1 \le p \le \infty$. However, $(\ell^p, \|\cdot\|_p)$ is separable for $1 \le p < \infty$, and not separable for $p = \infty$. (56)

We know that $\overline{C_{00}} = \ell^p$. For $x \in \ell^p$,
$x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$, let
$X_n = (x_1, \ldots x_n, 0, 0, \ldots)$. Then

$$\|X_n - x\|_p \to 0 \quad \text{as } n \to \infty. \quad -(1)$$

Since $x_i \in \mathbb{C}$, $\exists\ x_i^k \in \mathbb{Q} + i\mathbb{Q}$ s.t

$$|x_i^k - x_i|^p \to 0 \quad i = 1, 2, \ldots, n.$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i^k - x_i|^p\right)^{1/p} \to 0$$

$$\|X_n^k - X_n\|_p \to 0. \quad -(2)$$

where, $X_n^k = (x_1^k, \ldots, x_n^k)$.

From (1) & (2),

$$\|X_n^k - x\|_p \le \|X_n^k - X_n\|_p + \|X_n - x\|_p \to 0.$$

That is, $\overline{C_{00}(\mathbb{N}, \mathbb{Q} + i\mathbb{Q})} = \ell^p(\mathbb{N}, \mathbb{C})$.

We shall show that $l^\infty(\mathbb{N},\mathbb{C})$ is not seperable by proving that l^∞ cannot be the union of countabely many balls of arbitrarily small radius. (57)

Let $A=\{\tilde{x}_1, \tilde{x}_2, \cdots\}$ be a countable set in l^∞. Consider

$$S=\{x=(x_1,x_2,\cdots)\in l^\infty : x_i\in\{0,1\}\}.$$

Then S is an uncountable set. For this,

$$S\ni x \to y = \frac{x_1}{2}+\frac{x_2}{2^2}+\cdots, \quad x_i\in\{0,1\}.$$

Then the map is bijection from S to $[0,1]$. Let $x,y\in S$ be such that $x\neq y$. Then $\|x-y\|_\infty=1$ (Check!).

Hence, $\{B(x,\frac{1}{2}) : x\in S\}$ is an uncountable disjoint collection of open balls in l^∞. Since A is countable, $\exists\, x_0\in S$ s.t.

$$B(x_0,\tfrac{1}{2})\cap A=\phi.$$

Thus, A (an arbitrary countable set) in l^∞ cannot be dense.

Ex. Show that $\overline{c_{00}} = c_0$, and hence deduce that c_0 is seperable. (58)

Ex. $(C[0,1], \|\cdot\|_\infty)$ is a seperable Banach space.

By Weierstrass ~~theorem~~, approximation theorem, for $\epsilon>0$, $\exists$ a polynomial p on $[0,1]$ such that $\|P-f\|_\infty < \epsilon$, if f was in $C[0,1]$. Note that

$$P(x) = \sum_{i=0}^{k} a_i x^i, \quad a_i \in \mathbb{C}.$$

Since $\overline{\mathbb{Q}+i\mathbb{Q}} = \mathbb{C}$, $\exists\ b_i^n \in \mathbb{Q}+i\mathbb{Q}$ such that $|b_i^n - a_i| < \frac{\epsilon}{k}$ for $n \geq n_0$.

$$\Rightarrow \left\| \sum_{i=1}^{k} b_i^n x^i - \sum_{i=1}^{k} a_i x^i \right\| \leq \sum |b_i^n - a_i| \, \|x^i\|_\infty < \epsilon.$$

Write $\mathbb{P}(\mathbb{Q}) = \{ Q(x) = \sum_{i=1}^{k} b_i x^i : b_i \in \mathbb{Q} \}$

Then $\|Q-P\|_\infty < \epsilon$. Hence,

$$\|Q-f\|_\infty < 2\epsilon.$$

Thus $\mathbb{P}(\mathbb{Q})$ is dense in $C[0,1]$, where $\mathbb{P}(\mathbb{Q})$ is countable.

Ex. For $1 \le p < \infty$, $L^p([0,1])$ is separable, however, $L^\infty([0,1])$ is not seperable.

The separability of $L^p([0,1])$ $(1 \le p < \infty)$ is followed due to the result that

$$\overline{C[0,1]} = L^p[0,1], \quad 1 \le p < \infty.$$ (59)

(This, we shall prove later.)

Since $\overline{\mathbb{P}(\mathbb{Q}+i\mathbb{Q})} = C[0,1]$, and $\overline{C[0,1]} = L^p([0,1])$, for $f \in L^p[0,1]$, $\exists\, g \in C[0,1]$ s.t. $\|g-f\|_p < \epsilon$ and for $g \in C[0,1]$, $\exists\, P_n \in \mathbb{P}(\mathbb{Q}+i\mathbb{Q})$ such that

$$\|g-P\|_p \le \|g-P\|_\infty \quad \text{(notice this)}$$

Hence,

$$\|P-f\|_p \le \|g-f\|_p + \|g-P\|_\infty < 2\epsilon.$$

However, $L^\infty[0,1]$ is not separable.

For $t \in (0,1)$, write $f_t = \chi_{[0,t)}$.

Then for $s \ne t$, $s,t \in (0,1)$,

$$\|f_s - f_t\|_\infty = 1.$$

Thus $\mathcal{S} = \{B_{1/2}(f_t) : t \in (0,1)\}$ is an uncountable collection of disjoint open balls in $L^\infty[0,1]$.

If $A = \{g_1, g_2, \cdots\} \in L^\infty[0,1]$, then $\exists\, t_0 \in (0,1)$ such that $B_{1/2}(f_{t_0}) \cap A = \Phi$.

Ex. Let M be a closed subspace of a n.l.s X. Then X is separable iff M and X/M both are separable.

Proof: suppose X is separable and $E=\{e_1, e_2, \ldots\}$ be a countable dense set in X. That is $\overline{E}=X$.

Then M is separable, since $\overline{E\cap M}=M$.

Let $x+M \in X/M$. Then $x \in X$, and for $\epsilon>0$, $\exists$ $e_i \in E$ s.t. $\|x-e_i\|<\epsilon$.

Hence,

$$\|x+M-(e_i+M)\| \leq \|x-e_i\|<\epsilon.$$

Thus, X/M is separable, and $E+M=\{e_i+M : e_i \in E\}$ is a countable dense set in X/M.

Conversely, suppose M & X/M both are separable. Let $E_1=\{f_1, f_2, \ldots\}$ and $E_2=\{g_1+M, g_2+M, \ldots\}$ be countable dense sets in X & X/M

respectively. Let $E = \{ f_i + g_j : f_i \in E_1,\ g_j + M \in E_2 \}$.

Claim: E is a dense subset of X. Let $x \in X$. Then for $\epsilon > 0$, $\exists\ g_j + M \in E_2$ s.t.

$$\| x + M - (g_j + M) \| < \epsilon$$

$$\Rightarrow \inf_{m \in M} \| x - g_j - m \| < \epsilon.$$

Hence $\exists\ m_0 \in M$ such that

$$\| x - g_j - m_0 \| < \epsilon \quad \text{---(1)}$$

Again, $\overline{E_1} = M$ and $m_0 \in M$, hence for $\epsilon > 0$, $\exists\ f_i \in E_1$ s.t. $\| m_0 - f_i \| < \epsilon$ ---(2)

From (1) and (2), we get

$$\| x - (f_i + g_j) \| < 2\epsilon$$

Thus, E is a dense set in X and X separable.

Dense subspaces of $L^p(\mathbb{R})$:

Lemma: The space of simple integrable functions are dense in $L^p(\mathbb{R})$, $1 \le p < \infty$.

Proof: Let $S_p = \{ \varphi : \mathbb{R} \xrightarrow[\text{m'ble}]{\text{simple}} \mathbb{R}\ \&\ \varphi \in L^p(\mathbb{R}) \}$.

If $f \in L^p(\mathbb{R})$, then f is measurable, and hence, there exists a seq^n of simple m'ble functions φ_n s.t

$\varphi_n \xrightarrow{p.w.} f$ & $|\varphi_n| \uparrow |f|$ p.w. This gives $|\varphi_n|^p \le |f|^p \in L^1(\mathbb{R})$. & $\varphi_n \in S_p$. (62)

Now, $|f-\varphi_n|^p \le 2^{p+1} |f|^p \in L^1(\mathbb{R})$. By DCT, $\lim \int |f-\varphi_n|^p = \int \lim |f-\varphi_n|^p = 0$

i.e. $\lim \|f-\varphi_n\|_p = 0$. Hence $\overline{S_p} = L^p(\mathbb{R})$, for $1 \le p < \infty$.

Note that the above result followed by the following

<u>Theorem</u>: Let $f: \mathbb{R} \to [-\infty, \infty]$ be a m'ble function. Then $\exists$ a seq^n of simple functions φ_n s.t. $\varphi_n \xrightarrow{p.w.} f$ on $\mathbb{R}$, $|\varphi_n| \uparrow |f|$ p.w. and $\varphi_n \longrightarrow f$ uniformly on any set $A \subset \mathbb{R}$ on which f is bdd.

There are more classes of functions which are dense in $L^p(\mathbb{R})$, $1 \le p < \infty$.

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$). Then

$\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$ is

is known as support of function f.

If $\mathrm{supp}(f) \subset K$, and K is compact, then we say f is compactly supported.

ex. $f(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x|<1 \\ 0 & \text{o.w.} \end{cases}$

is a compactly supported function on $\mathbb{R}$ with $\mathrm{supp}(f) = \{x : |x| \le 1\}$.

In fact, given any compact set in $\mathbb{R}$, we can always construct a compactly supported continuous function on $\mathbb{R}$.

<u>Urysohn's lemma</u>: Let K & O be compact and open sets in $\mathbb{R}$. If $K \subset O$, then $\exists$ a continuous function f on $\mathbb{R}$ s.t.
$f = 1$ on K, $f = 0$ on O^c and
$0 \le f(x) \le 1$, $\forall x \in \mathbb{R}$.

Proof: Let $f(x) = \dfrac{d(x, O^c)}{d(x, O^c) + d(x, K)}$, where

$d(x, A) = \inf_{y \in A} |x-y|$. Then f will satisfies all conclusion of the result.

Let $C_c(\mathbb{R}) = \{ f : \mathbb{R} \xrightarrow{\text{cont}} \mathbb{R},\ \mathrm{supp} f \subset K,\ K \text{ cpt} \}$.
Then $(C_c(\mathbb{R}), \|\cdot\|_\infty)$ is a normed linear space

Note that for $f \in C_c(\mathbb{R})$, $\mathrm{supp}(f) \subset K$, and $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| < \infty$.

Theorem: $C_c(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$, for $1 \le p < \infty$.

proof: Let $f \in L^p(\mathbb{R})$. Then $\exists$ a seq^n φ_n of simple m'ble functions such that $|\varphi_n| \uparrow |f|$ p.w. In fact, $\forall\ \epsilon > 0$,

$\exists\ \varphi \in S_p$ s.t. $\|f - \varphi\|_p < \epsilon/2$ — (1)

Since $\varphi \in S_p \subset L^p$, we can write

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \quad m(E_i) < \infty, \ \forall\ \alpha_i \neq 0.$$

Since $m(E_i) < \infty$, for each $\epsilon > 0$, $\exists$

$K_i \subset E_i \subset O_i$ (K_i cpt & O_i open)

such that $m(O_i \setminus K_i) < \left(\frac{\epsilon}{2|\alpha_i| n}\right)^p$ — (2)

By Urysohn's lemma, $\exists$ a function $g_i \in C_c(\mathbb{R})$ such that $g_i|_{K_i} = 1$ & $g_i|_{O_i^c} = 0$, with $0 \le g_i(x) \le 1$, $\forall\ x \in \mathbb{R}$.

Hence, $\int_{\mathbb{R}} |\chi_{E_i} - g_i|^p = \int_{O_i} |\chi_{E_i} - g_i|^p = \int_{O_i \setminus K_i} |\chi_{E_i} - g_i|^p$

$$\leq m(O_i \setminus K_i) < \left(\frac{\epsilon}{2|\alpha_i|}\right)^p \frac{1}{n^p}.$$

That is, $\|\chi_{E_i} - g_i\|_p < \frac{\epsilon}{2|\alpha_i| n}$. Let us denote $g = \sum_{i=1}^{n} \alpha_i g_i$. Then $\varphi - g = \sum_{i=1}^{n} \alpha_i(\chi_{E_i} - g_i)$.

Hence,

$$\|\varphi - g\|_p \leq \sum |\alpha_i| \, \|\chi_{E_i} - g_i\|_p < \epsilon/2 \quad \text{—③}$$

From (1) & (3), we get

$\|g - f\|_p \leq \|g - \varphi\|_p + \|\varphi - f\|_p < \epsilon$, where $g \in C_c(\mathbb{R})$. Hence, $\overline{C_c(\mathbb{R})} = L^p(\mathbb{R})$, $1 \leq p < \infty$.

Notice that if $m(E) < \infty$, then $\exists$ $K \subset E \subset O$ s.t. $m(O \setminus E) \leq m(O \setminus K) < \epsilon$, for $\epsilon > 0$.

Then $\|\chi_O - \chi_E\|_p < \epsilon^{1/p}$. But $O = \dot{\bigcup}_{n=1}^{\infty} I_n$, and $m(O \setminus \dot{\bigcup}_{n=1}^{k} I_n) < \epsilon$ for $k \geq k_0$ (for some $k_0 \in \mathbb{N}$).

Let $\psi_k = \sum_{n=1}^{k} \chi_{I_n}$. Then $\|\chi_O - \psi_k\|_p < \epsilon^{1/p}$

This shows that $L^p(\mathbb{R})$ can be constructed over $\{\chi_{I_n} : I_n \text{- open \& bound intervals}\}$.

That is, for $f \in L^p(\mathbb{R})$, and $\epsilon > 0$, $\exists$

$\psi = \sum \alpha_i \chi_{I_i}$, $|\alpha_i| < \infty$, $m(E_i) < \infty$

such that $\|f-\psi\|_p < \epsilon$. Note that the function ψ is known as step function.
Thus, $L^p(\mathbb{R})$, $1 \le p < \infty$ is a seperable Banach space. In fact,

$L^p(\mathbb{R}) = \overline{\text{span}\{\chi_I : I \subset \mathbb{R}, \text{ bdd open interval}\}}$.

Remark: If $f \in L^p[a,b]$, then $\|\psi - f\|_p < \epsilon$ and $\psi \in R[a,b]$. Hence $\overline{R[a,b]} = L^p[a,b]$, if $1 \le p < \infty$.

However, $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$ because $\overline{C_c(\mathbb{R})}^{\|\cdot\|_\infty} = C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Ex. Show that $\overline{C_c(\mathbb{R})} = C_0(\mathbb{R})$.

Let $f \in C_0(\mathbb{R})$. Then for $\epsilon > 0$, $\exists$ cpt set $K \subseteq \mathbb{R}$ s.t. $|f(x)| < \epsilon$, $\forall x \in K^c$.

By Urysohn's lemma, $\exists$ open set $O \supset K$ and $g \in C_c(\mathbb{R})$ s.t. $g = 1$ on K & $g = 0$ on O^c, with $0 \le g(x) \le 1$ $\forall x \in \mathbb{R}$.

Write $h = fg$. Then $h \in C_c(\mathbb{R})$

and $|f(x) - h(x)| = |f(x)(1-g(x))| \le |f(x)| < \epsilon$, $\forall x \in \mathbb{R}$

Hence, $\|f - h\|_\infty \le \epsilon$. Thus, $\overline{C_c(\mathbb{R})} = C_0(\mathbb{R})$.

Ex. From the above discussion about $L^p(\mathbb{R})$, deduce that $C_0(\mathbb{R})$ is separable.

We end this discussion by mentioning the following charaterization of separable Banach space.

Banach Mazur theorem:

Every separable Banach space X is linearly isometric to a subspace of $C[0,1]$.
(Ref. to Fabian, Page 240, Theorem 5.8).
Note that this we shall prove while discussing weak* topology.

Schauder basis: Let $(X, \|\cdot\|)$ be a n.l.s. A Schauder basis for X is a seq^n $\{e_n\}$ in X such that each $x \in X$ has unique ~~rep~~ representation

$$(*) \qquad x = \sum_{i=1}^{\infty} a_i e_i, \quad a_i \in \mathbb{R} \,(\text{or } \mathbb{C})$$

Where convergence in $(*)$ is in the sense

$$\left\| \sum_{i=1}^{k} a_i e_i - x \right\| \to 0 \text{ as } k \to \infty.$$

If $\dim(X) < \infty$, then Hamel basis and Schauder basis both coincide.

For $(c_0, \|\cdot\|_\infty)$, $\{e_n = (0, 0, \overset{n\text{th}}{1}, 0, \dots) : n \in \mathbb{N}\}$ is a Schauder basis, but not a Hamel basis. Hamel basis of c_0 is uncountable having cardinality of continuum. 68

<u>Theorem:</u> If a n.l.s. $(X, \|\cdot\|)$ has a Schauder basis, then $(X, \|\cdot\|)$ is separable.

<u>Proof:</u> Let $E = \{e_j : j = 1, 2, \dots\}$ be a Schauder basis for X. Then for $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$$\left\| x - \sum_{j=1}^{n} a_j e_j \right\| < \epsilon \quad \text{---(1)}$$

In particular, $\left\| x - \sum_{j=1}^{n_0} a_j e_j \right\| < \epsilon$.

Since $a_j \in \mathbb{C}$, $\exists\ a_j^k \in \mathbb{Q} + i\mathbb{Q}$ s.t.

$$|a_j^k - a_j| < \frac{\epsilon}{n_0}, \quad \forall\ k \geq k_0.$$

Hence

$$\left\| x - \sum_{j=1}^{n} a_j^k e_j \right\| < \epsilon + \frac{\epsilon}{n_0} \cdot n_0 = 2\epsilon$$

Let $E_{n_0}^k = \left\{ \sum_{i=1}^{n_0} a_i^k e_i : n_0 \in \mathbb{N} \right\}$. Then

$E = \cup E_{n_0}^k$ is a countable dense set in X. Hence, X is separable.