Baire Category theorem:

This result, in a sense, dealt with indecomposibility of a complete metric spaces into its small constituent spaces. For instance, the plane cannot be written as countable union of lines. In general, a complete metric space cannot be written as countable union of nowhere dense sets. This is known as Baire Category theorem. However, we discuss this result for Banach spaces.

Def^n: A set $A \subset (X, \|\cdot\|)$ is said to be nowhere dense if $(\bar{A})^{\circ} = \phi$.

For example, $\mathbb{Z} \subset \mathbb{R}$ (with usual top) is nowhere dense in $\mathbb{R}$.

Also, $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \frac{1}{n}\mathbb{Z}$, where $\frac{1}{n}\mathbb{Z}$ is nowhere dense.

The Cantor set is nowhere dense in $[0,1]$ although it is uncountable

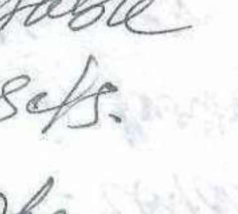
A normed linear space $(X, \|\cdot\|)$ is called of 1st category (meager, thin) if $X = \bigcup_{n=1}^{\infty} A_n$, where $(\overline{A_n})^{\circ} = \Phi$. 70

* That is, X is the countable union of nowhere dense sets.
* A normed linear space which is not of 1st category is known as of 2nd category.

<u>Theorem (BCT1)</u>: A complete n.l.s X cannot be written as countable union of nowhere dense sets.

Proof: Suppose $X = \bigcup_{n=1}^{\infty} A_n$, where $(\overline{A_n})^{\circ} = \Phi$. Then $\exists\, x_1 \in X \setminus \overline{A_1}$.

Since $\overline{A_1}$ is closed, $\exists$ a ball say $B_1 = B_{r_1}(x_1)$, $0 < r_1 < 1$ s.t.



$$\overline{B_1} \cap A_1 = \Phi$$

Since $(\overline{A_2})^{\circ} = \Phi \Rightarrow B_1 \not\subseteq \overline{A_2}$. Hence

$\exists\, x_2 \in B_1$ s.t. $x_2 \notin \overline{A_2}$.

Since A_2 is closed, $\overline{B_2} \cap A_2 = \Phi$, (71)
where $B_2 = B_{r_2}(x_2)$, $0 < r_2 < 1/2$.
Thus, $B_k \subset B_{k-1}$, $0 < r_k < \frac{1}{k}$ and
$\overline{B_k} \cap A_k = \Phi$.
Hence, for $j > k$, $x_j \in B_k$. This implies
$\|x_j - x_k\| < r_k < \frac{1}{k} \Rightarrow x_j$ is c.s. in X.
and $x_j \to x \in X$. Hence $\|x - x_k\| \leq r_k < \frac{1}{k}$
$\Rightarrow x \in \overline{B_k}$, $\forall k \geq 1$.
$\Rightarrow x \notin A_k \Rightarrow x \notin \bigcup_{k=1}^{\infty} A_k$.
Notice that we can always choose B_k s.t. $B_k \subset B_{k-1}$. A_k is nowhere dense.

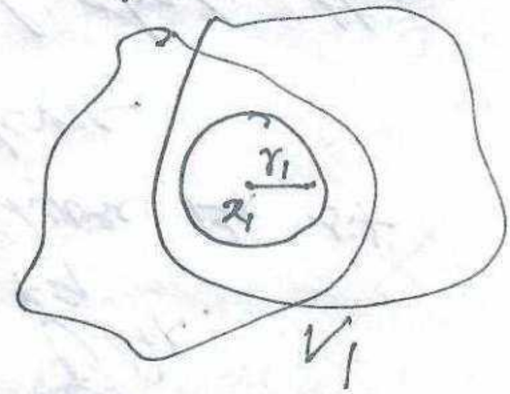
<u>Corollary</u>: If X is a Banach space, then for $X = \bigcup_{n=1}^{\infty} A_n$, $\exists\, n_0 \in \mathbb{N}$ s.t. $(\overline{A_n})^\circ \neq \Phi$.

The following version of the BCT is quite useful.

<u>Theorem</u> (BCT2): If X is a complete n.l.s. Then the countable intersection of open dense sets is also dense.

Proof: Let $\{V_n\}$ be seq^n of dense open sets in X. Then we claim $\overline{\bigcap V_n} = X$. For this, it is enough to show that every open set $W \subset X$ intersects $\bigcap_{n=1}^{\infty} V_n$. (72)

Since, $\overline{V_1} = X$, $W \cap V_1 \neq \Phi$, $\exists$ ball $B_1 = B_{r_1}(x_1)$ s.t. $\overline{B_1} \subset W \cap V_1$, with $r_1 < 1$. Again,



$\overline{V_2} = X$, $W \cap V_2 \neq \Phi$, and $\exists x_2 \in B_1$ s.t. $x_2 \subset W \cap V_2$.

Hence, $\exists$ $0 < r_2 < \frac{1}{2}$ s.t.

$\overline{B_2} = \overline{B_{r_2}(x_2)} \subset W \cap V_2$, where $B_2 \subset B_1$. By continuing this process, $\exists$ $B_K \subset B_{K-1}$, $K = 2, 3, \ldots$

and $\| x_j - x_K \| < r_K$ for $j > K$, $r_K < \frac{1}{K}$.

$\Rightarrow \{x_j\}$ is a C.C. in X and hence $x_j \to x \in X$. Thus, $\| x - x_K \| \le r_K < \frac{1}{K}$

$x \in \overline{B_K} \subset V_K \cap W$, $\forall K \ge 1$.

$\Rightarrow x \in (\bigcap_{K=1}^{\infty} V_K) \cap W$. $\Rightarrow \overline{\bigcap V_K} = X$.

Result: An infinite dim. Banach space X cannot have a countable Hamel basis. (73)

Proof: Suppose $E = \{e_1, e_2, \cdots\}$ is a countable Hamel basis for X. Consider

$$F_n = \text{span}\{e_1, \cdots, e_n\}.$$

Then $X = \bigcup_{n=1}^{\infty} F_n$. Hence, by BCT, $\exists\, n_0 \in \mathbb{N}$ s.t. $(\overline{F_{n_0}})^{\circ} \neq \Phi$. That is $\exists\, B_r(x)$ s.t. $B_r(x) \subset F_{n_0}$. Thus, $x + r.B_1(0) \in F_{n_0}$
$\Rightarrow B_1(0) \in F_{n_0} \Rightarrow k B_1(0) \subset F_{n_0}$, $\forall k \geq 1$
Hence $X = F_{n_0}$, which is a contradiction.

Ex. For $1 \leq p < \infty$, show that cardinality of Hamel basis for ℓ^p is c.

Write $B = \{(1, \alpha, \alpha^2, \cdots) : 0 < \alpha < 1\}$.
Then ~~each~~ B is a L.I. set in ℓ^p.
Hence $\#(B) \leq c \Rightarrow \#(B) = c$.
But B can be extended to a Hamel basis of X. Hence, $\#(\text{Hamel Basis}) = c$.

Remark: Later, we also show that $\#(H.B) \geq 2^{\aleph_0} = c$ (using Hahn-Banach thm).

Ex. Let $A \subset (X, \|\cdot\|)$. Show that

(i) $\overline{X \setminus A} = X \setminus A^{\circ}$

(ii) $(X \setminus A)^{\circ} = X \setminus \overline{A}$.

Ex. With the help of the above exercise, deduce BCT2 using BCT1.

Suppose $X = \overline{V_n}$, V_n - open in X.

Write $A_n = X \setminus V_n$, then $(\overline{A_n})^{\circ} = (X \setminus \overline{V_n}) = \Phi$.

If $\overline{\cap V_n} \subsetneq X$. Then $\exists\, B_r(x) \subset X$

s.t. $B_r(x) \cap (\cap V_n) = \Phi \Rightarrow B_r(x) \subset (\cap V_n)^c$

$\Rightarrow B_r(x) \subset \cup A_n$

$\Rightarrow \overline{B_r(x)} = \cup (A_n \cap \overline{B_r(x)}) = \cup (\tilde{A}_n)$

where $(\overline{\tilde{A}_n})^{\circ} = \Phi$, is a contradiction.

Ex. Let $f \in C^{\infty}(\mathbb{R})$ be such that for each $t \in \mathbb{R}$, $\exists\, n_t \in \mathbb{N}$ s.t. $f^{(n_t)}(t) = 0$.

Prove that $\exists$ an open interval $(a, b) \subset \mathbb{R}$ s.t.

$$f(t) = p(t), \ \forall\, t \in (a, b),$$

where p is a poly. on $\mathbb{R}$.

Let $E_m = \{ x \in \mathbb{R} : f^{(m)}(x) = 0 \}$. Then for $t \in \mathbb{R}$, $\exists\, n_t \in \mathbb{N}$ s.t. $f^{(n_t)}(t) = 0$. Hence, (75)

(1) — $\mathbb{R} = \bigcup_{m=1}^{\infty} E_m$. By BCT, $\exists\, m_0 \in \mathbb{N}$ s.t. $(\overline{E_{m_0}})^{o} = E_{m_0}^{o} \neq \emptyset$. $\Rightarrow \exists\, I_{m_0} \subset E_{m_0}$.

Notice that this will happen for almost all m except finitely many m. Otherwise (1) will not hold. Thus,

$$f^{(m_0)}(t) = 0, \quad \forall\, t \in (a,b).$$

$$\Rightarrow f(t) = p(t), \quad t \in (a,b).$$

<u>BCT2 fails:</u> $\mathbb{Q} \neq \bigcap (\mathbb{Q} \setminus \{q_n\}) = \emptyset$, where $\mathbb{Q} = \{q_1, q_2, \ldots, q_n, \ldots\}$, the set of rationals, because $(\mathbb{Q}, |\cdot|)$ is not a Banach space.

Note that if a n.l.s X is the union of countably many nowhere dense set, it is called of 1st category, else of 2nd category.

Ex. Show that $C_c(\mathbb{R})$ is a 1st category subspace of $C_0(\mathbb{R})$.

Continuous linear transformation:

Let X & Y be two n.l.s. A linear map $T: X \longrightarrow Y$ is said to be cont if for each $x \in X$, $\forall$ seqn $x_n \to x$, $Tx_n \longrightarrow Tx$.

Since T is linear, $T(x_n - x) \to 0$.

Thus, T is cont on X if $\forall\ y_n \to 0$, $Ty_n \to 0$.

That is, continuity of T on X is equivalent to cont. of T at "0."

Defn: A linear map $T: X \to Y$ is said to be bounded if $\exists\ M > 0$ s.t.

$$\|Tx\|_Y \leq M\|x\|_X . \quad \text{---} (*)$$

Note that norm in the LHS of $(*)$ is of the space Y whereas of X in the RHS.

Result: If $T: X \to Y$ is a linear map. Then FAE:

(i) T is conti on X (ii) T is cont at "0". (iii) T is bounded.

Proof: (i) $\Leftrightarrow$ (ii) is done. Now, consider (ii) $\Rightarrow$ (iii): Since T is cont. at "0", for $\epsilon > 0$, $\exists$ $\delta > 0$ s.t

$$\|x\| < \delta \Rightarrow \|Tx\| < \epsilon.$$ (77)

Let $y \in X$. Then $x = \frac{\delta}{2} \frac{y}{\|y\|} \in B_\delta(0)$.

Hence, $\|T(\frac{\delta}{2} \frac{y}{\|y\|})\| < \epsilon$

$$\Rightarrow \|Ty\| \leq \frac{2\epsilon}{\delta} \|y\|, \ \forall y \in X.$$

(iii) $\Rightarrow$ (ii) is obvious.

Norm of a linear transformation:

We know that T is bounded if

$$\|Tx\| \leq M\|x\| \text{ for } \forall x \in X.$$

The number

$$\|T\| := \inf\{M > 0 : \|Tx\| \leq M\|x\| \ \forall x \in X\}$$

is known as norm of T.

Note that for $\epsilon > 0$, $\exists M > 0$ s.t

$$M < \|T\| + \epsilon, \text{ where}$$

$$\|Tx\| \leq M\|x\|, \ \forall x \in X.$$

Hence, $\|Tx\| \leq (\|T\| + \epsilon)\|x\|, \forall x \in X.$

The above inequality is free of choice of $\epsilon>0$. Hence

$$\|Tx\| \leq \|T\| \|x\|.$$ (78)

<u>Lemma</u>: Let $T \in \mathcal{B}(X,Y)$, the space of all bdd linear maps from X to Y. Then

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

Proof: Let $\alpha = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$. Then for $x \neq 0$,

$$\frac{\|Tx\|}{\|x\|} \leq \alpha \Rightarrow \|Tx\| \leq \alpha \|x\|, \ \forall x \in X.$$

Since $\|T\|$ is infimum of all such α

$$\|T\| \leq \alpha.$$

We know that $\|Tx\| \leq \|T\| \|x\|$

$$\Rightarrow \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \leq \|T\| \Rightarrow \alpha = \|T\|.$$

If $\beta = \sup_{\|x\|=1} \|Tx\|$; then $\alpha \geq \beta$.

Let $x \neq 0$, then $\left\| \frac{x}{\|x\|} \right\| = 1$. Hence

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \beta \Rightarrow \alpha \leq \beta.$$

Thus, $\alpha = \beta$.

Write $\gamma = \sup_{\|x\|\le 1} \|Tx\|$. Then $\beta \le \gamma$. (79)

Let $x \neq 0$, and $\|x\| \le 1$. Write $y = \frac{x}{\|x\|}$.
Then $\|y\| = 1$, and $\|Ty\| \le \beta$.

or $\|Tx\| \le \beta \|x\| \le \beta \quad \forall \|x\| \le 1$.

Hence $\gamma \le \beta$.
Thus $\|T\| = \alpha = \beta = \gamma$.

Note that we mostly use $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

The space $B(X, Y) = \{ T : X \xrightarrow{bdd} Y \}$
is a n.l.s. under the norm $\|\cdot\|$ of map T.
This follows because $\|T\| = \sup_{\|x\|=1} \|Tx\| = 0$
$\Rightarrow \|Tx\| = 0, \ \forall \|x\| = 1 \Rightarrow \|Tx\| = 0$
$\forall x \in X$. Hence $Tx = 0 \ \forall x \in X \Rightarrow T = 0$.

Ex. Let X be a n.l.s. and Y be a complete n.l.s., then $B(X, Y)$ is a Banach space.
proof: Let $\{T_n\}$ be a C.S. in $B(X, Y)$.
Then for $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$\|T_n - T_m\| < \epsilon \quad \forall\ n, m \geq N.$

$\Rightarrow \quad \|T_n x - T_m x\| \leq \epsilon \|x\|, \ \forall\ n, m \geq N.$

That is, $\{T_n x\}$ is a C.S. in Y, and Y is complete. Hence $T_n x \to y \in Y$.

Let $y = Tx$. Then T is linear.

$T(\alpha x + \beta y) = \lim T_n(\alpha x + \beta y)$

$= \alpha Tx + \beta Ty.$

<u>Claim</u>: T is bdd & $\|T_n - T\| \to 0$.

Note that $\{T_n\}$ is a C.S. in $B(X, Y)$, therefore, $\{T_n\}$ is a bdd seq.

That is, $\|T_n\| \leq M$ for some $M > 0$.

$$\sup_{x \neq 0} \frac{\|T_n x\|}{\|x\|} \leq M$$

$\Rightarrow \quad \|T_n x\| \leq M \|x\|.$

$\Rightarrow \quad \lim \|T_n x\| \leq M\|x\|$

i.e. $\|Tx\| \leq M\|x\|, \ \forall\ x \in X.$

Since, $\|T_n x - T_m x\| < \epsilon, \ \forall\ n, m \geq N$ and $\|x\| \leq 1$. Letting $m \to \infty$, we get

$\|T_n x - Tx\| \leq \epsilon, \quad \forall\ n \geq N, \ \|x\| \leq 1.$

$$\Rightarrow \sup_{\|x\|\le 1} \|(T_n - T)x\| \le \epsilon, \ \forall n \geq N.$$ (81)

Hence, $\|T_n - T\| < \epsilon, \ \forall n \geq N.$

Ex. For $1 \le p \le \infty$, let $x = (x_1, \cdots x_n, \cdots) \in l^p$, and

$$T(x_1, x_2, \cdots, x_n, \cdots) = (x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots).$$

Then $\|T\| = 1$.

For $1 \le p < \infty$,

$$\|Tx\|_p^p = \sum_{i=1}^{\infty} |\frac{x_i}{i}|^p \le \|x\|_p^p.$$

$$\Rightarrow \|T\| \le 1.$$

Take $x_0 = (1, 0, 0, \cdots)$. Then $\|Tx_0\| = 1 \geq \|T\|$

$$\Rightarrow \|T\| = 1.$$

If $T : l^\infty \to l^\infty$.

$$\|Tx\|_\infty = \sup_n |\frac{x_n}{n}| \le \|x\|_\infty.$$

$\Rightarrow \|T\| \le 1$, and $\|Tx_0\|_\infty = 1$.

Hence $\|T\| = 1$ for $1 \le p \le \infty$.

Ex. Let $T : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ be defined by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Then $|(Tf)(x)| \leq \int_0^1 |f(t)|\,dt \leq \|f\|_\infty, \ \forall x \in [0,1]$.

Hence $\|Tf\|_\infty \leq \|f\|_\infty \Rightarrow \|T\| \leq 1$. (82)

For $g(t)=1$, $\|Tg\|_\infty = 1 \geq \|T\| \Rightarrow \|T\|=1$.

Ex. let $\varphi : [0,1]\times[0,1] \xrightarrow{\text{cont}} \mathbb{C}$. Define

$T : C[0,1] \longrightarrow C[0,1]$ by

$$(Tf)(t) = \int_0^1 \varphi(s,t) f(s)\,ds$$

$|(Tf)(t)| \leq \|\varphi\|_\infty \|f\|_\infty, \ \forall t \in [0,1]$

$\Rightarrow \|Tf\|_\infty \leq \|\varphi\|_\infty \|f\|_\infty, \ \forall f \in C[0,1]$.

$\Rightarrow \|T\| \leq \|\varphi\|_\infty = \sup\limits_{s,t\in[0,1]} |\varphi(s,t)|$.

Note that if $f \in L^2[0,1]$, then

$Tf(t) = \int_0^1 \varphi(s,t) f(s)\,ds$ defines

a ~~bdd~~ linear map on $L^2[0,1]$ to $L^2[0,1]$.

For this

$$|(Tf)(t)| \leq \int_0^1 |\varphi(s,t)|\,|f(s)|\,ds$$
$$\leq \|\varphi(\cdot,t)\|_2 \|f\|_2$$

(by Cauchy Schwarz).

$$\Rightarrow \int_0^1 |Tf(t)|^2 dt \leq \int\int |\varphi(s,t)|^2 ds\,dt\ \|f\|_2^2$$

$\Rightarrow \|Tf\|_2 \le \|\varphi\|_2 \|f\|_2$, where (83)

$$\|\varphi\|_2 = \left(\iint |\varphi(s,t)|^2 ds\,dt\right)^{1/2}.$$

Thus, $\|T\| \le \|\varphi\|_2$.

Ex. Let $X = BC([0,\infty))$, the space of all bounded cont functions on $[0,\infty)$.

Define $$(Tf)(t) = \begin{cases} \frac{1}{t}\int_0^t f(s)ds, & t>0 \\ 0, & t=0 \end{cases}$$

Then T is linear.

$$|Tf(t)| \le \frac{1}{t}\int_0^t |f(s)|ds \le \frac{1}{t}\int_0^t \|f\|_\infty ds$$

ie $|Tf(t)| \le \|f\|_\infty$.

Hence Tf is bounded on $[0,\infty)$.

$$\lim_{t\to 0} Tf(t) = \lim_{t\to 0} \frac{1}{t}\int_0^t f(s)ds$$

$$= \lim_{t\to 0} \int_0^1 f(tz)\,dz \quad (\text{if } s = tz)$$

$$= \int_0^1 f(0)\,dz = f(0).$$

Hence, $Tf \in BC[0,\infty)$. For $g(t)=1$,

$$\|T\| = \sup_{\|f\|_\infty=1} \|Tf\|_\infty \ge \|Tg\|_\infty = 1 \Rightarrow \|T\|=1.$$

Ex. For $f \in C^1[0,1]$, define $Tf = f'$. Then for $f_n(t) = t^n$, $T(f_n)(t) = nt^{n-1}$ (84)

$\Rightarrow \|T(f_n)\|_\infty = n \to \infty$. Hence,

$T : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$

is not bounded.

However, for $f \in C^1[0,1]$, we define

$\|f\| := \|f\|_\infty + \|f'\|_\infty$. Then

$T : (C^1[0,1], \|\cdot\|) \longrightarrow (C[0,1], \|\cdot\|_\infty)$

is bounded.

$\|Tf\|_\infty = \|f'\|_\infty \le \|f\| \Rightarrow \|T\| \le 1$.

For $g_n(t) = \frac{t^n}{n+1}$, $\|g_n\| = \frac{1}{n+1} + \frac{n}{n+1} = 1$

$\|T\| = \sup_{\|f\|=1} \|Tf\|_\infty \ge \|Tg_n\|_\infty = 1 - \frac{1}{n+1} \to 1$.

Hence $\|T\| = 1$.

Ex. For $f \in L^2[0,1]$, define

$(Tf)(t) = \int_0^t f(s)\,ds$.

Show that $\|T\| \le 1$. Is $\|T\| = 1$?

Extension of Cont linear transformation:

Suppose f is a unif. cont function on $A \subseteq \mathbb{R}$. Then f can be extended uniformly to $\bar{A}$. (85)

For $x \in \bar{A}$, $\exists\, x_n \in A$ s.t. $x_n \to x$.

If $m > n$, then $x_n - x_m \to 0$ as $m \to \infty$ and $f(x_n) - f(x_m) \to 0$ as $m \to \infty$,
(since f is unif cont. on A).

Hence $f(x_n)$ is a C.S. in $\mathbb{R}$ and
let $\check{f}(x) = \lim f(x_n) \in \mathbb{R}$. Note that $\check{f}$ is well-defined.

(If $x_n, y_n \to x$, then $f(x_n) - f(y_n) \to 0$)

Now, for $x, y \in \bar{A}$, $\exists\, x_n, y_n \in A$ s.t. $x_n \to x$ & $y_n \to y$. Let $|x-y| < \delta$. for $\epsilon > 0$. Then

$$|\check{f}(x) - \check{f}(y)| \le |\check{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \check{f}(y)|$$
$$< 3\epsilon \quad \text{for } n \geq N_0.$$

Because, $\lim |x_n - y_n| < \delta \Rightarrow |x_n - y_n| < \delta$ for $n \geq N_0$.

That is, $|\tilde{f}(x)-\tilde{f}(y)| \leq 3\epsilon$ if $|x-y|<\delta$.
Thus $\tilde{f}$ is unif. cont. on $\bar{A}$. This ext. is unique. (86)

If $g : \bar{A} \xrightarrow[\text{cont}]{\text{unif}} \mathbb{R}$ s.t. $g = f$ on A.
Then for $x \in \bar{A}$, $\exists$ $x_n \in A$ s.t. $x_n \to x$.
$\tilde{f}(x) = \lim f(x_n) = \lim g(x_n) = g(x)$.
$\Rightarrow \tilde{f} = g$. ($\because$ g is unif cont on x)

Extension theorem:

Let M be a dense subspace of a n.l.s X and Y be a Banach space. Suppose $T : M \to Y$ is cont. Then $\exists !$ extension $\tilde{T}$ of T to X, with $\|\tilde{T}\| = \|T\|$

Proof: Let $x \in X$, then $\exists$ $x_n \in M$ s.t. $x_n \to x$. Since

$\|Tx_n - Tx_m\| \leq \|T\| \|x_n - x_m\| \to 0$,

$\{Tx_n\}$ is a C.S in Y, and hence convergent.

write $\tilde{T}(x) = \lim T(x_n)$.

Then $\tilde{T}$ is linear and bounded.

Note that $\|Tx_n\| \le \|T\| \|x_n\|$

$$\Rightarrow \lim \|Tx_n\| \le \|T\| \lim \|x_n\|$$

$$\Rightarrow \|\tilde{T}x\| \le \|T\| \|x\|, \ \forall \ x \in X$$

$(\because \left| \|Tx_n\| - \|\tilde{T}x\| \right| \le \|Tx_n - \tilde{T}x\| \to 0)$.

That is, $\|\tilde{T}\| \le \|T\|$.

But $\|\tilde{T}\| = \sup\limits_{\substack{\|x\|=1 \\ x\in X}} \|\tilde{T}x\| \ge \sup\limits_{\substack{\|x\|=1 \\ x\in M}} \|\tilde{T}x\| = \|T\|$.

Further, if S is another ext of T s.t $S = T$ on M. Then for $x \in X$,

$$\tilde{T}(x) = \lim T(x_n) = \lim S(x_n) = S(x),$$

because S is cont. on X.

Ex. Show that every linear map on a finite dim. n.l.s. X to a n.l.s. Y is bounded.

Let $X = \text{span}\{e_1, \ldots, e_n\}$. Then $x \in X$ is given by $x = \sum\limits_{i=1}^{n} x_i e_i$.

$$\|Tx\| = \|\sum x_i Te_i\|$$
$$\le \sum |x_i| \|Te_i\|$$
$$\le M \|x\|,$$

where $M = \max \|Te_i\| < \infty$.

Open mapping theorem:

This is a fundamental result of functional analysis, which tells conti. linear map having complete range is an open map. However, if we drop the linearity this need not be the case. For example, let $I = [0,1]$, $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$. Then the map $\varphi : S^1 \times I \longrightarrow B_2 = \{(x,y) : x^2+y^2 \leq 1\}$ is a continuous surjection but not a open map, where

$$\varphi(x,t) = (1-t)x \quad , \; x = (x_1, x_2)$$

φ does not send open set

$A = \{(x_1,x_2) \in S^1 : x_2 > 0\} \times I$ to open set in $\mathbb{B}^2$ as $\varphi(A) = \{(x_1,x_2) \in \mathbb{B}_2 : x_2 > 0\} \cup \{(0,0)\}$.

Thus, continuous surjection need not be an open map.

Theorem (OMT):

Let X and Y be two Banach spaces. Suppose $T : X \longrightarrow Y$ is continuous and onto. Then T is an open map.

Proof: If $G = \Phi$, then $T(G) = \Phi$ is open. (89)
Let $G \neq \Phi$, and $y \in T(G)$. Then
$\exists\, x \in G$ s.t. $y = Tx$. Given G is open,
$\exists\, r > 0$ s.t

$$B_X(x, r) \subset G$$

$$\Rightarrow T(B_X(x,r)) \subset T(G).$$

To show that $T(G)$ is open, it is enough
to show that $T(B_X(x,r))$ contains a
ball of Y, say $B_Y(y, r') \subset T(B_X(x,r))$
That is,

$$y + r' B_Y(y, 1) \subset Tx + r\, T(B_X(0,1))$$

$$\Rightarrow B_Y\left(0, \frac{r'}{r}\right) \subset T(B_X(0,1)) \quad -(*)$$

$$(\because\ y = Tx \ \&\ T \text{ linear})$$

Hence, to show $T(G)$ is open, it is
enough to show that $T(B_X(0,1))$ contains
a ball of Y of type $B_Y(0, \epsilon)$.

Note that $X = \bigcup_{n=1}^{\infty} B_X(0, n)$. Since
$Y = TX$, $\quad Y = \bigcup_{n=1}^{\infty} T(B_X(0,n))$.
But Y is complete, by BCT, $\exists\, n_0 \in \mathbb{N}$

such that $\left(\overline{T(B_X(0,n_0))}\right)^0 \neq \Phi$.

$\Rightarrow$ we can choose $\epsilon>0$ such that

$$B_Y(y_0, 4\epsilon n_0) \subset n_0 \overline{T(B_X(0,1))}$$

$$\Rightarrow B_Y(y_0, 4\epsilon) \subset \overline{T(B_X(0,1))}.$$

Since $y_0 \in \overline{T(B_X(0,1))}$, we get

$$y_0 = \lim T x_n, \text{ where } x_n \in B_X(0,1).$$

Note that $-x_n \in B_X(0,1)$, $-y_0 = -\lim T x_n$ belongs to $\overline{T(B_X(0,1))} = \overline{E}$ (say)

Thus, $B_Y(0, 4\epsilon) = B_Y(y_0, 4\epsilon) - y_0 \subset \overline{E} + \overline{E}$

That is, $2B_Y(0, 2\epsilon) \subset 2\overline{E}$

$$\Rightarrow B_Y(0, 2\epsilon) \subset \overline{E} \quad \text{--- (1)}$$

We claim that

$$B_Y(0, \epsilon) \subset T(B_X(0,1)) = E.$$

Let $y \in B_Y(0, 2\epsilon)$, then $y \in \overline{E}$, and hence $\exists\ y_1 \in E$ s.t. $\|y - y_1\| < \epsilon/2$.

$$\Rightarrow y - y_1 \in B_Y(0, \epsilon/2) \subset \tfrac{1}{4}\overline{E} \quad \text{--- (2)}$$

Similarly, $\exists\ y_2 \in \frac{1}{4}E$, s.t.

$$\|(y - y_1) - y_2\| < \epsilon/4 \quad \text{--- (3)}$$

By induction, $\exists\, y_n \in \frac{1}{2^n} E = \frac{1}{2^n} T(B(0,1))$

$\Rightarrow y_n = Tz_n$, where $\|z_n\| < \frac{1}{2^n}$

From (3), it follows that (92)

$$\|y - (y_1 + \cdots + y_n)\| < \frac{\epsilon}{2^n}$$

Let $z = \sum z_n$, then $\|z\| \le 1 < 2$.

Since X is complete, $z \in X$. Given T is cont, $y = \sum y_n = \sum Tz_n = T(\sum z_n)$

i.e. $y = Tz$.

That is, $B_Y(0, 2\epsilon) \subset T(B_X(0,2))$

$$\Rightarrow \quad B_Y(0,\epsilon) \subset T(B_X(0,1)) \quad \text{---(4)}$$

Note that the inclusion (4) is very fundamental, and often we use this as open mapping theorem.

Ex. Let $T \in B(X,Y)$, X & Y both are Banach spaces. If T is onto, then

$$\tilde{T} : X/\ker T \to Y$$

is one-one, onto. Thus $X/\ker T \overset{\sim}{=} Y$.

Inverse mapping theorem (IMT):

Let X & Y be two Banach spaces. If $T: X \to Y$ is a continuous linear bijection, then T^{-1} is continuous.

Proof: Since T is cont. & onto, by OMT, $\exists\, \epsilon > 0$ such that

$$B_Y(0,\epsilon) \subset T(B_X(0,1)).$$

Let $y \in B_Y(0,\epsilon)$, then $\|y\| < \epsilon$ and $\exists\, x \in B_X(0,1)$ s.t. $y = Tx$

$\Rightarrow \|T^{-1}y\| = \|x\| < 1$ if $\|y\| < \epsilon$

$\Rightarrow \|T^{-1}y\| < 1$ if $\|y\| \leq \epsilon/2$

That is, $\|T^{-1}(\frac{2y}{\epsilon})\| < \frac{2}{\epsilon}$ if $\|\frac{2y}{\epsilon}\| \leq 1$.

Hence, $\|T^{-1}(z)\| < 2/\epsilon$ if $\|z\| \leq 1$.

Thus, T^{-1} is bounded.

Remark: If T is 1-1, onto cont, then T is open, by OMT. It follows that T is a closed map.

For F to be closed set in X, (97)

$T(F) = \{Tx : x \in F\}$. Let $Tx_n \to y \in Y$.

Then Tx_n is a c.s. in Y and

$$\|x_n - x_m\| = \|T^{-1}(Tx_n - Tx_m)\|$$

$$\leq \|T^{-1}\| \cdot \|Tx_n - Tx_m\| \to 0.$$

But X is complete, hence $x_n \to x \in X$.

$\Rightarrow \quad Tx_n \to Tx = y, \quad \therefore x \in F.$

Thus, $T(F)$ is closed.

Closed graph theorem (CGT):

It is easy to see that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its graph $G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is a closed set in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

However, converse of the above result need not be true, unless the function is at least linear. For example, the graph of the function $f(x) = \frac{1}{x}$, $x \neq 0$, is closed but f is not continuous.

The closed graph theorem plays a vital role ... while deciding the boundedness of a linear map. (9)

<u>Theorem (CGT):</u>

Let X & Y be two Banach spaces, & $T: X \to Y$ be linear. Then T is continuous iff G_T is closed in $X \times Y$.

<u>Proof:</u> Let $G_T = \{(x, Tx) : x \in X\}$, and

$$\|(x,y)\|_G = \|x\|_X + \|y\|_Y .$$

Then $(X \times Y, \|\cdot\|_G)$ is a Banach space.

Suppose T is continuous. Consider $(x_n, Tx_n) \to (x, y)$. Then

$$\|(x_n, Tx_n) - (x, y)\|_G \to 0$$

$$\Rightarrow \|x_n - x\| + \|Tx_n - y\| \to 0$$

That is, $x_n \to x$, $Tx_n \to y$. But T is cont, $\Rightarrow y = \lim Tx_n = Tx$.

Thus, G_T is closed in $X \times Y$.

Conversely, suppose G_T is closed. Then G_T is a Banach space in $X \times Y$.

Consider: $\pi_1 : G_T \longrightarrow X$, $\pi_1(x, Tx) = x$

and $\pi_2 : G_T \rightarrow Y$, $\pi_2(x, Tx) = Tx$.

Then π_1 is a 1-1, onto conti map

$$\|\pi_1(x,Tx)\|_X = \|x\|_X \le \|(x,Tx)\|_{G_T}.$$

Hence, $\pi_1^{-1} : X \longrightarrow G_T$ is cont. (by IMT).

Thus, $T = \pi_2 \circ \pi_1^{-1}$ is continuous.

$$\begin{tikzcd} x \arrow[r, mapsto, "\pi_1^{-1}"] \arrow[rr, bend right, "\pi_2 \circ \pi_1^{-1} = T"'] & (x, Tx) \arrow[r, "\pi_2"] & Tx \end{tikzcd}$$

Remark: Completeness of the spaces in OMT and CGT is essential. For example $I : (l^1, \|\cdot\|_1) \longrightarrow (l^1, \|\cdot\|_\infty)$ is continuous, but its converse is not continuous. Since $\|I(x)\|_\infty = \|x\|_\infty \le \|x\|_1$.
However, $\|I^{-1}(x)\|_1 = \|x\|_1 \not\le \alpha \|x\|_\infty$ for any $\alpha > 0$. Note that $(l^1, \|\cdot\|_\infty)$ is not complete.

Similarly, graph of $I : (\ell^1, \|\cdot\|_\infty) \to (\ell^1, \|\cdot\|_1)$ is closed but I is not continuous. (98)

Also, $T : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$, given by $T(f)(t) = f'(t)$, is not continuous, however its graph G_T is closed. If $(f_n, Tf_n) \to (f, g)$.

Then $\|f_n - f\|_\infty \to 0$ & $\|f_n' - g\|_\infty \to 0$

Thus, $\int_0^s g(t)dt = \int_0^s \lim f_n'(t)dt = \lim \int_0^s f_n'(t)dt$

That is, $\int_0^s g(t)dt = f(s) - f(0)$

$\Rightarrow \quad g = f' = T(f)$.

Ex. Let $1 \leq p \leq \infty$, and $a = (a_1, \cdots a_n, \cdots)$ is a seq^n in $\mathbb{C}$ s.t. for each $x = (x_1, \cdots x_n, \cdots) \in \ell^p$, $(a_1 x_1, a_2 x_2, \cdots) \in \ell^p$. Show that $Tx = (a_1 x_1, a_2 x_2)$ is continuous.

Let $a \cdot x = (a_1 x_1, a_2 x_2 \cdots \cdots)$.

By closed graph thm, to show T is conti, it is enough to show that for $x^k \to 0$ in ℓ^p and $Tx^k \to y$ in $\ell^p \Rightarrow y = T0 = 0$.

$x \Rightarrow y = 0 = 0.$

Note that $\|x^k\|_p \to 0$ & $\|a_n x_n^k - y\|_p \to 0$

$\Rightarrow x_n^k \to 0, \forall n.$ & $a_n x_n^k \to y_n \ \forall n.$

Hence, $a_n x_n^k \to 0 \Rightarrow y_n = 0.$ (97)

Ex. Let $\varphi \in L^\infty(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, define $T(f) = \varphi f$. Then T is bounded on $L^1(\mathbb{R})$, and $\|T\| = \|\varphi\|_\infty$.

$\|T(f)\|_1 \leq \|\varphi\|_\infty \cdot \|f\|_1 \Rightarrow \|T\| \leq \|\varphi\|_\infty.$

Notice that $E = \{x \in \mathbb{R} : |\varphi(x)| > \|\varphi\|_\infty - \epsilon\}$ is a set of positive Lebesgue measure. Let $F \subset E$ be such that $0 < m(F) < \infty$. Write $f_0 = \frac{1}{m(F)} \chi_F \operatorname{sign}(\varphi)$. Then

$$T(f_0) = \int_F |\varphi| \cdot \frac{1}{m(F)} > \frac{m(F)}{m(F)} (\|\varphi\|_\infty - \epsilon).$$

That is, $T(f_0) > \|\varphi\|_\infty - \epsilon, \ \forall \epsilon > 0$

Hence $T(f_0) \geq \|\varphi\|_\infty \geq \|T\|$

$\Rightarrow \|T\| = \|\varphi\|_\infty.$

Ex. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a measurable function s.t. $\varphi f \in L^1(\mathbb{R}), \ \forall f \in L^1(\mathbb{R}).$

Show that T defined by $Tf = \varphi f$ is a bounded linear transformation on $L^1(\mathbb{R})$. (98)

By closed graph thm, it is enough to show that if $f_n \xrightarrow{L^1} 0$ & $Tf_n \xrightarrow{L^1} g$. Then $g = T0 = 0$.

Recall that every Cauchy seq^n in L^1 has conv. subsequence, which converges point wise a.e.

$$\left.\begin{array}{l} Tf_{n_k}(x) \xrightarrow{a.e.} g \Rightarrow \varphi(x) f_{n_k}(x) \xrightarrow[a.e]{p.w.} g(x) \\ \text{since } f_{n_k} \xrightarrow{L^1} 0 \Rightarrow f_{n_{k,m}}(x) \xrightarrow[a.e]{p.w} 0 \end{array}\right\}$$

But then $\varphi(x) f_{n_{k,m}}(x) \xrightarrow[a.e]{p.w.} 0$. Thus $g(x) = 0$ a.e.

Ex. If $1 \le p \le \infty$, and $\varphi: \mathbb{R} \to \mathbb{C}$ is a msble function s.t. $f\varphi \in L^p(\mathbb{R})$, $\forall f \in L^p(\mathbb{R})$. then $T(f) = f\varphi$ is continuous.

If $\varphi f \in L^1(\mathbb{R})$, $\forall f \in L^1(\mathbb{R})$, then it can be shown that $\varphi \in L^\infty(\mathbb{R})$.

If $\varphi \notin L^\infty(\mathbb{R})$, then $\forall n \in \mathbb{N}$

$\exists$ a set E_n of +ve measure such that

$$\chi_{E_n}(x)\,|\varphi(x)| > n.$$ (99)

Let F_n be a set of finite measure in E_n.
That is, $0 < m(F_n) < \infty$.

Then $$\int_{F_n} \chi_{F_n} |\varphi| > n \int_{F_n} \chi_{F_n} = n\, m(F_n)$$

$$\Rightarrow \sum \frac{1}{n^2} \int_{F_n} \frac{\chi_{F_n}(x)\,|\varphi(x)|}{m(F_n)}\, dx > \sum n \cdot \frac{1}{n^2} = \infty.$$

$$\Rightarrow \int \left(\sum \frac{1}{n^2} \frac{\chi_{F_n}}{m(F_n)}\right) |\varphi| = \infty.$$

That is, $\int f\,|\varphi| = \infty$, where

$f = \sum \frac{1}{n^2} \frac{\chi_{F_n}}{m(F_n)} \in L^1(\mathbb{R})$, is a contradiction. Hence $\varphi \in L^\infty(\mathbb{R})$ and as earlier, $\|T\| = \|\varphi\|_\infty$.

Banach Steinhaus thm (or Uniform boundedness principle)

We know that a seq^n of continuous functions on $\mathbb{R}$ which is point-wise bounded need not be uniformly bounded.

However, Osgood (1897) had shown that if $f_n : [0,1] \to \mathbb{R}$ be a seqn of cont. functions which is point-wise bounded, then $\exists$ an interval $[a,b] \subset [0,1]$ such that f_n is uniformly bounded. (100)

Uniform boundedness principle (UBP) ensure that a seqn of pointwise bounded operators on a Banach space is uniformly bounded.

Theorem (Banach Steinhaus thm):

Let X be a Banach space and Y be a n.l.s. Suppose $\{T_i\}_{i\in I} \subset B(X,Y)$. Then either $\exists\, M > 0$ s.t

$$\left.\begin{array}{l} \|T_i\| \le M, \ \forall i \in I \\ \text{or} \quad \varphi(x) = \sup\limits_{i\in I} \|T_i(x)\| = \infty \end{array}\right\} (*)$$

for all x belonging to a dense G_δ set in X.

Proof: Consider $X_n = \{x \in X : |\varphi(x)| > n\}$.

Then $X_n^c = \bigcap_{i\in I}\{x\in X : \|T_i(x)\| \le n\}$
is a closed set because each T_i is continuous on X. Hence, X_n is an open set in X.

Note that if all of X_n are dense in X. Then by BCT, $\overline{\bigcap X_n} = X$. Hence,

$$\phi(x) = \infty \quad \text{on } \bigcap X_n = G \quad (G_\delta\text{-set}).$$

If some of $\overline{X_n} \neq X$. Then $\exists\, B(x_0, r)$ in X s.t.

$$B(x_0, r) \cap X_n = \Phi$$

$$\Rightarrow \phi(x) \le n \quad \forall\, x\in B(x_0, r)$$

That is, $\phi(x + x_0) \le n, \ \forall\ \|x\| < r$.

$$\Rightarrow \|T_i(x)\| \le n, \text{ if } \|x\| < r, \ \forall\, i\in I.$$

If $\|x\| \le r/2$, then

$$\|T_i(x)\| \le \|T_i(x + x_0)\| + \|T_i(x_0)\|$$
$$\le 2n \quad \forall\, i\in I.$$

$$\Rightarrow \|T_i(x)\| \le \frac{4n}{r} \quad \text{if } \|x\| \le 1.$$

Thus, $\|T_i\| \le \frac{4n}{r} \quad \forall\, i\in I$.

Corollary 1: If $\{T_i\}_{i\in I} \in B(X, Y)$ is such that for each $x\in X$, $\exists\ M_x > 0$ with

$\sup_{i\in I}\|T_i(x)\| \le M_x < \infty.$

Then $\exists\, M > 0$ such that $\sup_{i\in I}\|T_i\| \le M$.

(this is known as UBP).

<u>Alternative proof of Corollary I:</u>

For $n \in \mathbb{N}$, let

$S_n = \{x \in X : \|T_i x\| \le n, \forall i \in I\}$.

By hypothesis, for each $x \in X$, $\exists\, n \in \mathbb{N}$ s.t. $x \in S_n \Rightarrow X = \bigcup S_n$.

Since each S_n is closed, by BCT, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$B(x_0, r) \subset S_{n_0}.$$

Let $x \in X$ & $\|x\| \le r/2$. Then $x + x_0 \in B(x_0, r)$ and

$$\|T_i x\| \le \|T_i(x + x_0)\| + \|T_i(x_0)\|$$
$$\le 2n_0.$$
$$\Rightarrow \|T_i\| \le \frac{4n_0}{r}, \quad \forall i \in I.$$

Remark: Completeness of X is essential. Let $P(\mathbb{R})$ be the space all poly-s on $\mathbb{R}$ of the form $p(x) = a_0 + a_1 x + \cdots + a_d x^d$.

Define $\|P\| = \sup_{0\le j\le d} |a_j|$. Then $(P(\mathbb{R}), \|\cdot\|)$ (103)

is an incomplete n.l.s, since by BCT, a complete n.l.s. cannot has countable Hamel basis as $P(\mathbb{R})$ has $\{1, x^2, \cdots\}$ a Hamel Basis.

Define $T_n : P(\mathbb{R}) \longrightarrow \mathbb{R}$ by

$T_n(P) = a_0 + a_1 + \cdots + a_{n-1}$. Then

$|T_n(P)| \le |a_0| + \cdots + |a_d|$

$\Rightarrow |T_n(P)| \le (d+1)\|P\|, \ \forall n \geq 1$.

$\Rightarrow \{T_n\}$ is a point-wise bounded seq^n of bounded operators on $(P(\mathbb{R}), \|\cdot\|)$, but $\{T_n\}$ is not uniformly bounded.

$\|T_n\| = \sup_{\|P\|=1} |T_n(P)| \geq \|T_n(P_n)\| = n$,

where $P_n(x) = 1 + x + \cdots + x^{n-1}$.

$\Rightarrow \|T_n\| \geq n, \ \forall n$.

Corollary 2 of UBP:

Let $\{T_n\} \in B(X,Y)$ and for each $x \in X$, $\{T_n x\}$ has limit in Y. Define

$T(x) = \lim T_n(x)$. Then $T \in B(X,Y)$

and $\|T\| \le \underline{\lim} \|T_n\|$.

Proof: By UBP, it follows that $\{\|T_n\|\}$ is a bounded seqn. Thus, (104)

$$\|T_n\| \leq M \quad \text{for some } M>0.$$

$$\Rightarrow \|T_n x\| \leq M\|x\|$$

$$\Rightarrow \lim \|T_n x\| \leq M\|x\|$$

$$\Rightarrow \|Tx\| \leq M\|x\|$$

$$\Rightarrow T \in B(X,Y).$$

Further, $\|T_n x\| \leq \|T_n\|\,\|x\|$

$$\Rightarrow \underline{\lim} \|T_n x\| \leq \underline{\lim} \|T_n\|\,\|x\|$$

$$\Rightarrow \lim \|T_n x\| \leq \underline{\lim} \|T_n\|\,\|x\|$$

$$\Rightarrow \|Tx\| \leq \underline{\lim} \|T_n\|\,\|x\|, \ \forall x \in X$$

$$\Rightarrow \|T\| \leq \underline{\lim} \|T_n\|.$$

Application of UBP:

Let $f: [-\pi, \pi] \longrightarrow \mathbb{C}$ be an integrable function. Then its Fourier series can be expressed as

$$f(t) \sim \sum \hat{f}(n) e^{int}$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) e^{-ins}\, ds$,

known as Fourier coeff. of f.

Let $S_m(f)(t) = \sum_{n=-m}^{m} \hat{f}(n) e^{int}$. Then for 105
$f \in C^1[-\pi,\pi]$, $S_m(f) \to f$ uniformly.
However, F.S. of continuous function f need not converge even point-wise.
Note that
$$S_m(f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_m(t-s)\,ds,$$
where $D_m(t) = \sum_{m=-n}^{n} e^{imt}$, known as Dirichlet kernel. Also,
$$D_m(t) = \begin{cases} \dfrac{\sin(m+\frac{1}{2})t}{\sin\frac{1}{2}t} & \text{if } t \neq 2k\pi \\ 2m+1 & \text{if } t = 2k\pi, \end{cases}$$
where $k \in \mathbb{Z}$.

<u>Lemma I</u>: $\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(t)|\,dt = \infty$.

proof: For $t \in \mathbb{R}$, $|\sin t| < |t|$. Hence,
$$\int_{-\pi}^{\pi} |D_n(t)|\,dt \geq 4 \int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{t}\right| dt$$
$$= 4 \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin t|}{t}\,dt$$
$$> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\,dt$$

$$> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{k\pi}\, dt$$ 106

$$= \frac{8}{\pi} \sum_{k=1}^{n} \frac{1}{k} \to \infty \text{ as } n \to \infty.$$

Lemma 2: Let $X = C[-\pi, \pi]$ be equipped with $\|\cdot\|_\infty$. Define

$$T_n(f) = S_n(f)(0), \text{ for } f \in X.$$

Then $\{T_n\} \in \mathcal{B}(X, \mathbb{C})$, and

$$\|T_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt.$$

Proof: Since $T_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(t)\, dt$,

it follows that

$$|T_n(f)| \le \|f\|_\infty \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt$$

& $\|T_n\| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt.$

Now, let $E_n = \{t \in [-\pi, \pi] : D_n(t) \ge 0\}$.

Define $f_m(t) = \dfrac{1 - m\, d(t, E_n)}{1 + m\, d(t, E_n)}$

Then $f_m \in C[-\pi, \pi]$, $\|f_m\|_\infty \le 1$ and $f_m(t) \to 1$ if $t \in E_n$ & $f_m(t) \to -1$ if $t \in E_n^c$.

By DCT, it follows that

$$T_n(f_m) = \frac{1}{2\pi}\int_{E_n} f_m(t)\, D_n(-t) + \frac{1}{2\pi}\int_{E_n^c} f_m(t)\, D_n(t)$$

$$\Rightarrow T_n(f_m) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt \quad \text{as } m \to \infty.$$

Thus, $\|T_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt \to \infty$.

Hence, by Banach Steinhaus thm (BST), $\exists$ a dense G_δ-set G in X s.t.

$$|T_n(f)| = |S_n(f)(0)| \to \infty$$

for each $f \in G$. That is, F.S. of f in G diverge at $t = 0$.

<u>Hahn-Banach Theorem (HBT):</u>

Hahn-Banach thm is a very much fundamental result of functional analysis, which dealt with extension of bounded linear functional to higher dim. spaces. And thereby ensuring existence of enough bounded linear functional on any n.l.s, making an interesting dual of the space

However, the proof of HBT is carried out for more general class of linear (108) functional which are dominated by sub-linear functional instead of $|f(x)| \le M\|x\|$.

Let $(X, \|\cdot\|)$ be a normed linear space. A map $p : X \to \mathbb{R}$ is said to be sub-linear if

(i) $p(x+y) \le p(x) + p(y)$, $\forall\, x, y \in X$

and (ii) $p(\alpha x) = \alpha\, p(x)$, $\forall\, \alpha \ge 0$.

Notice that

$$p(\lambda x + (1-\lambda) y) \le \lambda p(x) + (1-\lambda) p(y).$$

Hence, p is a convex function but need not be non-negative.

Ex. $p : \mathbb{R}^2 \to \mathbb{R}$ by

$$p(x_1, x_2) = x_1 + |x_2|$$

is a sub-linear functional.

The proof of the HBT shall be carried out in two steps, namely, real and complex versions. However, complex version will be followed by real version.

HBT (Real case):

Let X_0 be a subspace of a n.l.s. X, and $f_0 : X_0 \longrightarrow \mathbb{R}$ be a linear functional which satisfies $f_0(x) \le p(x), \forall x \in X_0$, for some sub-linear functional p on X. Then f_0 can be extended to X as $f : X \longrightarrow \mathbb{R}$ with

$$f(x) \le p(x), \forall x \in X \ \& \ f|_{X_0} = f_0.$$

The proof of this result requires the following Zorn's lemma (or maximal principle).

Zorn's lemma:

Every partial ordered set having each chain has an upper bound contains a maximal element.

Proof of HBT (Real Case):

If $X_0 = X$, then trivial. Suppose $x_1 \in X \setminus X_0$, and write

$$X_1 = \{x + \alpha x_1 : x \in X_0 \ \& \ \alpha \in \mathbb{R}\}.$$

Then X_1 is a subspace X. For $x, y \in X_0$,

$$f_0(x) + f_0(y) = f_0(x+y) \le p(x+y)$$
$$\le p(x+x_1) + p(y-x_1).$$

That is,

$$f_0(y) - p(y - x_1) \le p(x + x_1) - f_0(x) \quad \text{—(1)}$$

Let $a = \sup_{y \in X_0} \{ f_0(y) - p(y - x_1) \}$ and

$b = \inf_{x \in X_0} \{ p(x + x_1) - f_0(x) \}$.

Then $a \le b$. If $a = b$, it will be clear from further calculation that $\exists$ only one extension of f_0 to X_1.

Suppose $a < c < b$. Then

$$f_0(x) - p(x - x_1) \le a < c,$$

i.e. $\quad f_0(x) - c < p(x - x_1) \quad$ —(2)

and $\quad p(x + x_1) - f_0(x) \ge b > c$

$\Rightarrow \quad f_0(x) + c \le p(x + x_1) \quad$ —(3)

By multiplying (2) & (3) with $\alpha > 0$ and replacing $\alpha x \to x$, we get

$$\left.\begin{aligned} f_0(x) - \alpha c &\le p(x - \alpha x_1) \\ f_0(x) + \alpha c &\le p(x + \alpha x_1) \end{aligned}\right\} \text{—(4)}$$

For $\alpha \in \mathbb{R}$, write

$$f_1(x + \alpha x_1) = f_0(x) + \alpha c, \quad x \in X_0.$$

Then $f_1 : X_1 \longrightarrow \mathbb{R}$ is a linear map and $f_1(y) \le \beta(y)$, $\forall\, y \in X_1$; (by (4)). (111)

If $X_1 = X$, then f_1 is a desire ext. of f_0. Otherwise, Consider the following family of earlier extensions.

$$\mathcal{F} = \{(Y, f) : X_0 \subset Y \subset X,\ f/X_0 = f_0\}.$$

Then $\mathcal{F} \ne \emptyset$, as $(X_1, f_1) \in \mathcal{F}$.

Write $(Y_1, f_1) \le (Y_2, f_2)$ iff

$Y_1 \le Y_2$ & $f_2/Y_1 = f_1$.

Then "$\le$" is a partial order relation.

Let $\mathcal{G} = \{(Y_\alpha, f_\alpha) : \alpha \in I\}$ be a totally ordered (chain) in $\mathcal{F}$. Write

$$Y = \bigcup_{\alpha \in I} Y_\alpha \text{ and } g : Y \longrightarrow \mathbb{R}$$

such that $g/Y_\alpha = f_\alpha$, $\forall\, \alpha \in I$.

Then (Y, g) is an upper bound for $\mathcal{G}$.

Thus, by Zorn's lemma, $\exists$ a maximal element, say (Y_∞, f_∞) in $\mathcal{F}$.

If $Y_\infty \subsetneq X$, then (Y_∞, f_∞) can be added one more dimension that gives (Y'_∞, f'_∞), which shall Contradict that (Y_∞, f_0) was maximal. Hence $Y_\infty = X$ and we write $f_\infty = f$. Then f is a desire linear functional on X. (112)

Cor: If $f_0 : X_0 \subset X \longrightarrow \mathbb{R}$ is a conti. linear map, then $f_0(x) \leq \|f_0\| \|x\|, \forall x \in X_0$.
Write $p(x) = \|f_0\| \|x\|$. Then HBT (Real Case), $\exists f : X \xrightarrow{\text{linear}} \mathbb{R}$ s.t

$$f(x) \leq \|f_0\| \|x\|, \forall x \in X$$

$$\Rightarrow \|f\| \leq \|f_0\|.$$

On the other hand, we have

$$\|f\| = \sup_{\|x\|=1} |f(x)| \geq \sup_{\|x\|=1, x \in X_0} |f_0(x)| = \|f_0\|.$$

Thus, $\|f\| = \|f_0\|$.

HBT (Complex Case):

Let X_0 be a subspace of a n.l.s X, and $f_0 : X_0 \rightarrow \mathbb{C}$ be a linear map, which satisfies $\operatorname{Re} f_0(x) \leq p(x), \forall x \in X_0$, for some sub-linear functional on X. Then $\exists$ a linear map $f : X \rightarrow \mathbb{C}$ s.t. $\operatorname{Re} f(x) \leq p(x), \forall x \in X$ and $f|_{X_0} = f_0$.

Proof: Let $g_0(x) = \mathrm{Re}\, f_0(x)$, for $x \in X_0$. Then $g_0(x) \le p(x)$, $\forall\, x \in X_0$, and by (113) real version of HBT, $\exists\ g : X \to \mathbb{R}$ s.t. $g(x) \le p(x)$, $\forall\, x \in X$ & $g|_{X_0} = g_0$.

Since, $g_0(ix) = \mathrm{Re}\, f_0(ix)$
$$= \mathrm{Re}\{i f_0(x)\}$$
$$= -\mathrm{Im}\, f_0(x),$$
it follows that
$$f_0(x) = \mathrm{Re}\, f_0(x) + \mathrm{Im}\, f_0(x)$$
$$= g_0(x) - i g_0(ix).$$
Define $f : X \to \mathbb{C}$ by
$$f(x) = g(x) - i g(ix) \quad \text{—— } (*)$$
Then $\mathrm{Re}\, f(x) = g(x) \le p(x)$, $\forall\, x \in X$.

Note that f is complex linear.

From $(*)$, $f(ix) = i f(x)$, and
$$f((a+ib)x) = f(ax) + f(ibx)$$
$$= a f(x) + ib f(x)$$
$$= (a+ib) f(x).$$

Cor: If $f_0 : X_0 \subset X \to \mathbb{C}$ is conti linear functional, then $\exists$ an extension f of f_0 to X such that $\|f\| = \|f_0\|$ & $f|_{X_0} = f_0$.

Proof: Let $p(x) = \|f_0\|\,\|x\|$, where
$|f_0(x)| \le \|f_0\|\,\|x\|$, $\forall x \in X_0$.

$\Rightarrow \operatorname{Re} f_0(x) \le \|f_0\|\,\|x\|$, $\forall x \in X$.

By complex version of the HBT,
$\exists\, f : X \xrightarrow{\text{linear}} \mathbb{C}$ s.t. $\operatorname{Re} f(x) \le \|f_0\|\,\|x\|$, $\forall x \in X$. Claim $|f(x)| \le \|f_0\|\,\|x\|$.

Note that $|f(x)| = e^{-i\theta} f(x)$, $\theta \in [0, 2\pi]$.

But then
$$|f(x)| = \operatorname{Re}\{f(e^{-i\theta}x)\}$$
$$\le \|f_0\|\,\|e^{-i\theta}x\|.$$

Hence, $|f(x)| \le \|f_0\|\,\|x\|$
& $\|f\| \le \|f_0\|$.

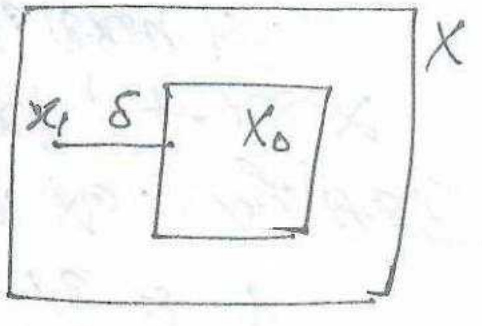
But $\|f\| = \sup\limits_{\|x\|=1} |f(x)| \ge \sup\limits_{\substack{\|x\|=1\\ x\in X_0}} |f_0(x)| = \|f_0\|$.

Thus $\|f\| = \|f_0\|$.

The next result will tell, how HBT helps producing enough bdd linear functionals on a n.l.s X. In fact, each point of X gives a functional (continuous) on X. If X^* denotes the space of all cont linear functional, then X can be embedded into X^*.

Theorem: Let X_0 be a proper subspace of a n.l.s X. Suppose for $x_1 \in X \setminus X_0$, $\delta = \text{dist}(x_1, X_0) > 0$. Then $\exists f \in X^*$ s.t. $\|f\| = 1$, $f(x_1) = \delta$ and $f(X_0) = \{0\}$.

Proof: Let $X_1 = \{x + \lambda x_1 : \lambda \in \mathbb{C}, x \in X_0\}$.



Write $f_1(x + \lambda x_1) = \lambda \delta$.

Then f_1 is linear on X_1,

$f_1(x_1) = \delta$ and $f_1(X_0) = \{0\}$.

Claim: $\|f_1\| = 1$.

Let $x \in X_0$, & $\lambda (\neq 0) \in \mathbb{C}$. Then

$|f_1(x - \lambda x_1)| = |\lambda| \delta \le |\lambda| \, \| \frac{x}{\lambda} - x_1 \|$

& $|f_1(x - \lambda x_1)| \le \| x - \lambda x_1 \|$

$\Rightarrow \|f_1\| \le 1$.

Since $|f_1(x - x_1)| = \delta = \inf_{x \in X_0} \| x - x_1 \|$,

$\exists$ seq^n $x_n' \in X_0$ such that $\|x_n' - x_1\| \to \delta$

$$\Rightarrow \frac{|f_1(x_n' - x_1)|}{\|x_n' - x_1\|} \to 1.$$

Hence $\|f_1\| = \sup \dfrac{|f(x + \lambda x_1)|}{\|x + \lambda x_1\|} \ge \dfrac{|f_1(x_n' - x_1)|}{\|x_n' - x_1\|} \to 1$.

Thus, $\|f_1\| = 1$. By HBT, f_1 can be

extended to X as f with $\|f\| = \|f_1\| = 1$.
This completes the proof of theorem. (116)

Cor: Let M be a closed proper subspace of X. Then $\exists$ $f \in X^*$ s.t. $\|f\| = 1$ and $f(M) = \{0\}$.

The next result is known as Hahn-Banach separation theorem, which means that two points can be separated by a hyperplane.

<u>Theorem</u>: Let X be a n.l.s, and $0 \neq x_0 \in X$. Then $\exists$ $f \in X^*$ s.t. $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

<u>Proof</u>: Write $X_0 = \{\alpha x_0 : \alpha \in \mathbb{C}\}$. Then X_0 is a closed subspace. Define
$f_0(\alpha x_0) = \alpha \|x_0\|$. Then f_0 is a bounded linear functional on X_0. By HBT, $\exists$ $f \in X^*$ s.t. $|f(x)| \leq \|x\|$, $\forall x \in X$ and $f|_{X_0} = f_0$. ($\because p(x) = \alpha_0 \|x_0\|$).

$\Rightarrow$ $\|f\| \leq 1$. But,
$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_0)|}{\|x_0\|} = 1. \quad \#$$

Note that if $x_1 \neq x_2$, $x_1, x_2 \in X$. Then $\exists$ $f \in X^*$ such that $f(x_1) \neq f(x_2)$.

Ex. If M be a closed proper subspace of a n.l.s X. Then for each $x_1 \in X \setminus M$, $\exists f \in X^*$ s.t. $f(M) = \{0\}$ & $f(x_1) = 1$. (117)

For $x_1 \in X \setminus M$, $\exists f \in X^*$ s.t
$f(M) = \{0\}$ & $f(x_1) = \delta = \text{dist}(x_1, M)$
with $\|f\| = 1$.
Write $f'(x) = \frac{1}{\delta} f(x)$. Then $f'(x_1) = 1$.

Notice that any n.l.s X can be embedded into X^* (the dual of X), it is expected that if X^* is separable then X is so. However, converse is not true. We see later that $(l^1)^* \cong l^\infty$ (not separable).

<u>Theorem</u>: Let X be a Banach space. If X^* is separable, then X is separable.

Proof: Note that X^* is separable iff $S_{X^*} = \{f \in X^* : \|f\| = 1\}$ is separable.
Let $A = \{f_n \in X^* : \|f_n\| = 1\}$ and $\bar{A} = S_{X^*}$.
Since $\|f_n\| = 1$, $\exists$ unit vector $x_n \in S_X$ s.t. $|f_n(x_n)| > 1/2$.
Let $D = \{x_k \in X : x_k = \sum_{n=1}^{k} \alpha_n x_n, \alpha \in \mathbb{Q}, k \in \mathbb{N}\}$.

Then D is countable and $\overline{D}$ is a closed subspace of X. If $\overline{D} \neq X$. Then $\exists f \in X^*$ s.t. $f(\overline{D}) = \{0\}$, & $\|f\| = 1$.
Since $f \in S_{X^*} = \overline{A}$, $\exists f_n \in A$ s.t. $\|f_n - f\| \to 0$. But, then (118)

$$\frac{1}{2} \leq |f_n(x_n)| = |f_n(x_n) - f(x_n)|$$
$$\leq \|f_n - f\| \|x_n\|$$
$$= \|f_n - f\| \to 0,$$

which is a contradiction. Thus $\overline{D} = X$.

Dual space of normed linear space:

Dual space of a n.l.s space play vital role for understanding the space itself. The space of all bounded linear functional on X is known as dual of X and is denoted by X^*. We know that if X^* is separable then X is also separable.

By Hahn-Banach separation theorem, we know that for any $0 \neq x \in X$, $\exists$ $f_x \in X^*$ s.t. $\|f_x\| = 1$ & $f_x(x) = \|x\|$.

Thus, X is embedded into the Banach space $X^{**} = \mathcal{B}(X^*, F)$. (119)

We say X is isomorphic to Y if $\exists$ $T : X \xrightarrow{\text{linear}} Y$ which one-one, onto and $\|Tx\| = \|x\|$.

If $X^* \cong Y$, it means that $\exists$ $T : Y \xrightarrow{\text{linear}} X^*$ one-one, onto s.t. $Ty \in X^*$ and $\|Ty\| = \|y\|$.

<u>Dual of $\ell^p(\mathbb{N})$:</u>

For $1 < p < \infty$, let $\ell_n^p = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{C}\}$ and $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ then $(\ell_n^p)^* \cong \ell_n^q$, where, $\frac{1}{p} + \frac{1}{q} = 1$.

For this, consider $T_y(x) = x \cdot y$ for $x, y \in \mathbb{C}^n$. Then $T_y \in (\ell_n^p)^*$ and $T : \ell_n^q \to (\ell_n^p)^*$ is one-one. T is onto.

Let $f \in (\ell_n^p)^*$, and $\{e_i\}$ is the S.B. of $\mathbb{C}^n$. Write $y_i = f(e_i)$. Then for $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, then

$f(x) = \sum x_i f(e_i) = x \cdot y = T_y(x)$. Thus, (120)

T is onto & $T_y = f$. Thus,

$$|T_y(x)| = |x \cdot y| \le \|x\|_p \|y\|_q$$

$$\Rightarrow \quad \|T_y\| \le \|y\|_q .$$

To show other inequality (or equality),

$$\text{let } x_i = \begin{cases} \dfrac{|y_i|^q}{y_i} & \text{if } y_i \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

Then $x_0 = (x_1, x_2, \ldots, x_n)$ satisfies

$$\|x_0\|_p^p = \sum \left| \frac{|y_i|^q}{y_i} \right|^p = \sum |y_i|^{(q-1)p} = \sum |y_i|^q .$$

Hence, $\|x_0\|_p = \|y\|_q^{q/p}$.

Now, $\|T_y\| = \sup\limits_{x \neq 0} \dfrac{|T_y(x)|}{\|x\|_p} \ge \dfrac{|x_0 \cdot y|}{\|x_0\|} = \dfrac{\sum |y_i|^q}{\|y\|_q^{q/p}}$

That is, $\|T_y\| \ge \|y\|_q^{q(1-\frac{1}{p})} = \|y\|_q$.

$$\Rightarrow \|T_y\| = \|y\|_q .$$

Thus, $T : l_n^q \longrightarrow (l_n^p)^*$ is an isometric isomorphism.

When $p=1$, let $f \in (l_n^1)^*$. Then for $y_i = f(e_i)$

and $y = (y_1, \ldots, y_n)$,

$$T_y(x) = \sum x_i y_i = \sum x_i f(e_i) = f(x).$$

Thus, T is onto, and (121)

$$|T_y(x)| \le \|x\|_1 \|y\|_\infty$$
$$\Rightarrow \|T_y\| \le \|y\|_\infty$$

On the other hand, Consider

$$x_i = \begin{cases} \dfrac{|y_i|}{y_i} & \text{if } |y_i| = \|y\|_\infty \\ 0 & \text{o.w.} \end{cases}$$

Let $x_0 = (x_1, x_2, \ldots, x_n)$. Then

$$\|T_y\| = \sup_{x \ne 0} \frac{|x \cdot y|}{\|x\|_1} \ge \frac{\left|\sum \frac{|y_i|}{y_i} \cdot y_i\right|}{\sum \left|\frac{|y_i|}{y_i}\right|}$$

That is, $\|T_y\| \ge \dfrac{\sum |y_i|}{\sum 1} = \dfrac{m\|y\|_\infty}{m} = \|y\|_\infty$.

(Since, may be only m many y_i's are non-zero)

Hence, for $1 \le p < \infty$, $(l_n^p)^* \cong l_n^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

When $p = \infty$, $(l_n^\infty)^* \cong l_n^1$. Likewise we can define $T : l_n^1 \longrightarrow (l_n^\infty)^*$ by $T_y(x) = x \cdot y$. Then T is one-one.

For $f \in (l_n^\infty)^*$, let $y_i = f(e_i)$ and $y = (y_1, \ldots, y_n)$. Then for $x = (x_1, \ldots, x_n) \in l_n^\infty$,

$T_y(x) = x \cdot y = \sum x_i f(e_i) = f(x)$. 122
$\Rightarrow$ $T_y = f$. Hence, T is onto.

Also, $\|T_y\| \leq \|y\|_1$.

Consider, $x_i = \begin{cases} \frac{|y_i|}{y_i} & \text{if } y_i \neq 0 \\ 0 & \text{o.w.} \end{cases}$

and let $x_0 = (x_1, \cdots x_n)$. Then

$$\|T_y\| = \sup_{x \neq 0} \frac{|x \cdot y|}{\|x\|_\infty} \geq \frac{\left|\sum \frac{|y_i|}{y_i} y_i\right|}{\|x_0\|_\infty} = \|y\|_1$$

$\Rightarrow$ $\|T_y\| = \|y\|_1$.

Let $f \in (\ell^p)^*$, and $\{e_i\}_{i \in \mathbb{N}}$ be a Schauder basis of ℓ^p, for $1 < p < \infty$.

Write $y = (y_1, \cdots y_n, \cdots) = (f(e_1), \cdots, f(e_n), \cdots)$.

Then $f(x) = f\left(\lim \sum_{i=1}^{n} x_i e_i\right)$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(e_i) x_i = x \cdot y =: T_y(x).$$

Then $T_y = f$. Hence $T : \ell^q \longrightarrow (\ell^p)^*$ is onto, and one-one by its def^n.

Since, $x \in \ell^p$, $\|T_y\| \leq \|y\|_q$ if we can show that $y \in \ell^q$.

For this, let $Y_n = (y_1, \dots, y_n) = (f(e_1), \dots, f(e_n))$.

and define $f_n : l_n^p \to \mathbb{C}$ by (123)

$$f_n(X_n) = f_n(x_1, \dots x_n) = \sum_{i=1}^{n} x_i y_i, \quad X_n = (x_1, \dots, x_n).$$

Then $\|Y_n\|_q = \|f_n\| = \sup_{\|X_n\|_p = 1} |f_n(X_n)|$

That is, $\|Y_n\|_q \leq \sup_{\|x\|_p=1} |f(x)| = \|f\| < \infty$. (Notice this)

Letting $n \to \infty$, $\|y\|_q \leq \|f\| < \infty$.

Hence, $y \in l^q$. But $\|f\| = \|Ty\|$

$\Rightarrow \|Ty\| = \|y\|_q$.

Thus, $1 < p < \infty$, $(l^p)^* \cong l^q$.

When $p = 1$, $(l^1)^* \cong l^\infty$.

Define $T : l^\infty \to (l^1)^*$ by

$T_y : l^1 \to \mathbb{C}$ with $T_y(x) = x \cdot y$.

Then $\|Ty\| \leq \|y\|_\infty$. Now, claim T is onto. Let $f \in (l^1)^*$, and write $y_i = f(e_i)$, where $\{e_i\}$ is a Schauder basis of l^1. Then we need to show that $y = (y_1, y_2, \dots) \in l^\infty$.

Define $f_n : l_n^1 \longrightarrow \mathbb{C}$ by

$$f_n(X_n) = X_n \cdot Y_n = \sum_{i=1}^{n} x_i y_i = T_{Y_n}(X_n).$$

Then $\|Y_n\|_\infty = \|T_{Y_n}\| = \|f_n\|$ $(\because f_n = T_{Y_n})$.

Now, $\|Y_n\|_\infty = \sup_{\|X_n\|_1 = 1} |f_n(X_n)| \leq \sup_{\|x\|_1 = 1} |f(x)| = \|f\|$.

Letting $n \to \infty$, $\|y\|_\infty \leq \|f\| < \infty$.

Further,

$$f(x) = f\left(\lim \sum_{i=1}^{n} x_i e_i\right) = \lim \sum_{i=1}^{n} f(e_i) x_i = x \cdot y.$$

i.e. $f(x) = T_y(x) \Rightarrow f = T_y$.

Hence, T is onto, and

$$\|y\|_\infty \leq \|f\| = \|T_y\| \leq \|y\|_\infty.$$

(by Holder's inequality)

Note that $(l^\infty)^*$ is not isomorphic to l^1, else l^∞ will be separable, being l^1 is separable. However, l^1 is embedded in $(l^\infty)^*$ via the map

$$T : l^1 \longrightarrow (l^\infty)^*, \quad T_y(x) = x \cdot y.$$

$\|T_y\| \leq \|y\|_1$ (by Holder's inequality)

If $x_i = \begin{cases} \frac{|y_i|}{y_i}, & y_i \neq 0 \\ 0 & \text{o.w.} \end{cases}$, and

Let $x_0 = (x_1, x_2, \dots)$. Then $\|x_0\|_\infty = 1$,

and $|T_y(x_0)| = \left| \sum \frac{|y_i|}{y_i} y_i \right| = \|y\|_1$. (125)

Thus, $\|T_y\| \geq \|y\|_1 \geq \|T_y\|$.

That is, $\|T_y\| = \|y\|_1$.

Remark: One the reason, we cannot show T is onto, lies with the fact that ℓ^∞ has no Schauder basis.

Lemma: Let M be a dense subspace of a n.l.s. X. Then $M^* = X^*$.

proof: Let $f \in M^*$, then $f : M \xrightarrow{\text{cont}} \mathbb{C}$ can be extended uniquely to X as $\tilde{f} \in X^*$.

Conversely, if $g \in X^*$, then $g|_M \in M^*$.

It follows that $(c_{00}, \|\cdot\|_p)^* \cong \ell^q$, $1 \leq p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Ex. Show that $(c_0, \|\cdot\|_\infty)^* \cong \ell^1$.

(Hint: Since $\{e_i\}_{i \in \mathbb{N}}$ is a Schauder basis for c_0, the proof follows as earlier).

Next, we shall discuss the dual of $C[0,1]$ and $L^p(\mathbb{R})$. The dual of $C[0,1]$ is the space of all functions of bounded variation, whereas dual of $L^p(\mathbb{R})$, $(1 \le p < \infty)$ is $L^q(\mathbb{R})$. (126)

Functions of BV:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function and
$P = \{ a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots \le x_n = b \}$
be a partition of $[a,b]$.

Let $V_a^b(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

If $P_0 = \{a,b\}$. Then $f(b) - f(a) \le V_a^b(f,P)$.

Now, $\sup_P V_a^b(f,P) =: V_a^b(f)$

= total variation of f on $[a,b]$.

If $V_a^b(f) < \infty$, we say f is of bounded variation (BV) on $[a,b]$.

Note that for the partition $P_1 = \{ a < x < b \}$

$$|f(x) - f(a)| + |f(b) - f(x)| \le V_a^b(f).$$

$$\Rightarrow |f(x)| \le |f(a)| + V_a^b(f).$$

Hence, if $f \in BV[a,b]$. Then f is bdd.

For $x \in [a,b]$, let $V(x) = V_a^x(f)$. Then (127)

for $x < y$, $V_a^y(f) - V_a^x(f) = V_x^y(f)$

$\geq |f(y) - f(x)|$.

$\Rightarrow V$ is an $\uparrow$ function of x.

Further, $f = v - (v - f)$ & $v - f \uparrow$.

Thus, f is a difference of two $\uparrow$ functions.

Ex. $f(x) = \begin{cases} x \cos \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{o.w.} \end{cases}$

is a continuous function on $[0,1]$ but not of BV. For $P_n = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots, \frac{1}{3}, \frac{1}{2}, 1\}$,

$V_0^1(f, P_n) = 2\left(\frac{1}{2n} + \frac{1}{2n-1} + \dots + \frac{1}{3} + \frac{1}{2}\right) + 1 \to \infty$.

Remark: Since $f \in BV[a,b]$ is difference of monotone functions & monotone function can have at most countable point of discontinuity, it follows that $f \in R[a,b]$.

Ex. For $f \in BV[0,1]$, write $\|f\| = |f(0)| + V_a^b(f)$.
Then $(BV[0,1], \|\cdot\|)$ is a Banach space

Lemma: Let $f_n, f \in [0,1] \longrightarrow \mathbb{R}$ be s.t. $f_n \to f$ point-wise. Then
$$V(f_n, P) \to V(f, P).$$

Proof: $|V(f_n,P) - V(f,P)|$
$$\leq \sum_{i=1}^{k} \big| |f_n(x_i) - f_n(x_{i-1})| - |f(x_i) - f(x_{i-1})| \big|$$
$$\longrightarrow 0.$$

Note that $\|f\| = 0 \Rightarrow |f(0)| = 0$ & $V_a^b(f) = 0$.

Let $P = \{0, x, 1\}$. Then
$$|f(x) - f(0)| + |f(1) - f(x)| \leq V_a^b(f) = 0.$$
$\Rightarrow f = 0.$

Suppose $\{f_n\}$ be a C.C. in $(BV[0,1], \|\cdot\|)$. Then for $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.
$$\|f_n - f_m\|_{BV} < \epsilon \quad (\because \|\cdot\| = \|\cdot\|_{BV})$$
$\Rightarrow |(f_n - f_m)(x)| \leq |(f_n - f_m)(0)| + V_a^b(f_n - f_m).$

$\Rightarrow |f_n(x) - f_m(x)| \leq 2\epsilon$ for large n, m. & $\forall\, x \in [0,1]$. Thus, $\|f_n - f_m\|_\infty \leq 2\epsilon$.

Hence, $f_n \xrightarrow{unif} f$.

Let P be any partition of $[0,1]$. Then
$$|f(0) - f_n(0)| + V(f - f_n, P)$$

$$= \lim_{m\to\infty} \left[|f_m(0) - f_n(0)| + V(f_m - f_n, P) \right]$$

$$\le \sup_{m \geq N} \left[|f_m(0) - f_n(0)| + V(f_m - f_n, P) \right]$$

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$$\le \sup_{m \geq N} \|f_m - f_n\|_{BV} \le \epsilon, \quad \forall P$$

$\Rightarrow \|f - f_n\|_{BV} \le \epsilon, \quad \forall n \geq N.$

Take $\epsilon = 1$, then $f - f_N \in BV[0,1]$.

Hence, $f = f - f_N + f_N \in BV[0,1]$.

Ex. Suppose f is diff on $[a,b]$ and $f' \in R[a,b]$. Then $V_a^b(f) = \int_a^b |f'(x)|dx$.

Note that

$$V(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum |f'(t_i)| \Delta x_i,$$

for some $t_i \in (x_{i-1}, x_i)$ (by MVT).

$\Rightarrow L(P,|f'|) \le V(P,f) \le U(P,|f'|)$.

Since $|f'| \in R[a,b]$, $\exists$ an $\uparrow$ seq^n of partitions P_n s.t.

$$\lim L(P_n, f) = \lim U(P_n, f) = \int_a^b |f'|$$

Hence $V_a^b(f) = \int_a^b |f'|$.

Note that $\sup_P V(P,f) = \lim V(P_n, f)$.

Riemann Stieltjes integration:

Let f be a bounded function on $[a,b]$ and α be an increasing (non-constant) function on $[a,b]$. For partition P of $[a,b]$, let $L(P,f,\alpha)=\sum_{i=1}^{n} m_i \Delta\alpha_i$, where

$\Delta\alpha_i = \alpha(x_i)-\alpha(x_{i-1})$, $m_i=\inf\limits_{[x_{i-1},x_i]} f(x)$.

and $U(P,f,\alpha)=\sum_{i=1}^{n} M_i \Delta\alpha_i$, $M_i=\sup\limits_{[x_{i-1},x_i]} f(x)$.

If $\sup\limits_{P} L(P,f,\alpha)=\inf\limits_{P} U(P,f,\alpha)$, we say f is R-S integrable and denote by $\int_a^b f(x)\,d\alpha(x)$.

Note that α can be replaced by a function $g\in BV[a,b]$, since $g=\alpha_1-\alpha_2$ where α_i's are ↗ functions on $[a,b]$.

Theorem: $(C[0,1])^* \cong BV[0,1]$.

Proof: Let $g\in BV[0,1]$, then for $f\in C[0,1]$

(*) $\quad \varphi_g(f)=\int_0^1 f\,dg$, where

$g = g_1 - g_2$, g_1 & g_2 ↑ and non-negative functions. The integral in (*) is in Riemann-Stieltje's sense. Hence, (131)

$$|\varphi_g(f)| \leq \int_0^1 |f|\,|dg| \leq \|f\|_\infty \int_0^1 |dg|.$$

Now, $\int_0^1 |dg| = \lim_{n\to\infty} \sum_{i=1}^{n} |\Delta g_i| = \lim_{n\to\infty} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$,

where $P_n = \{0, x_1, \ldots x_{i-1}, x_i, x_n = 1\}$ ↑ seqⁿ of partitions of $[0,1]$. Thus,

$$\int_0^1 |dg| = V_0^1(g) \leq \|g\|_{BV}.$$

i.e. $|\varphi_g(f)| \leq \|f\|_\infty \|g\|_{BV}$.

$\Rightarrow \varphi_g \in (C[0,1])^*$.

Conversely, we shall show that every $\varphi \in (C[0,1])^*$, $\exists\ g \in BV[0,1]$ s.t. (*) holds. For this, let $\chi_0 = 0$, and $0 < t \leq 1$, $\chi_t = \chi_{[0,t)}$. Write $g(t) = \varphi(\chi_t)$.

We show that $g \in BV[0,1]$:

Suppose $g \in BV[0,1]$. Then,

$$\varphi(\chi_t) = g(t) - g(0) = \int_0^t 1 \cdot dg = \int_0^1 \chi_t\, dg.$$

This implies, (*) holds for every step-function.

If $f \in C[0,1]$, then $\exists$ a seq^n ψ_n of simple step functions s.t $\psi_n \to f$ uniformly. (132)

$$|\varphi(\psi_n) - \int f dg| = |\int (\psi_n - f) dg| \le \|\psi_n - f\|_\infty \int |dg|$$

i.e. $|\varphi(\psi_n) - \int f dg| \le \|\psi_n - f\|_\infty \cdot \|g\|_{BV} \to 0$.

$\lim \varphi(\psi_n) = \int f dg$. since φ is cont linear functional & $\psi_n \to f$ unif,

$$\varphi(f) = \int f dg.$$

Now, for $P = \{0, t_1, \cdots t_{i-1}, t_i, \cdots, t_n = 1\}$,

$$\sum_{i=1}^n |g(t_i) - g(t_{i-1})| = \sum_{i=1}^n [g(t_i) - g(t_{i-1})] \operatorname{sign}[g(t_i) - g(t_{i-1})]$$

$$= \sum_{i=1}^n [\varphi(\chi_{t_i}) - \varphi(\chi_{t_{i-1}})] \operatorname{sign}[g(t_i) - g(t_{i-1})]$$

$$= \varphi\left(\sum (\chi_{t_i} - \chi_{t_{i-1}}) \operatorname{sign}(g(t_i) - g(t_{i-1}))\right)$$

$$= \varphi\Big(\underbrace{\sum_{i=1}^n \chi_{[t_{i-1}, t_i)} \operatorname{sign}(g(t_i) - g(t_{i-1}))}_{f_n}\Big)$$

Notice that $\|f\|_\infty = 1$. Since φ can be extended to $L^1[0,1]$ without changing its norm,

$$\sum |g(t_i) - g(t_{i-1})| \le \|\varphi\|, \quad \forall P.$$

Since $g(0) = 0$, $\quad V_0^1(g) \le \|\varphi\|$.

But then $\|g\|_{BV} \le \|\varphi\|$.

Also, $\varphi(f) = \int_0^1 f dg$, implies $|\varphi(f)| \le \|f\|_\infty \|g\|_{BV}$.

Thus, $\|g\|_{BV} = \|\varphi\|$. This completes the proof.

Next, we discuss that if $1 \le p < \infty$, then $(L^p(\mathbb{R}))^* \cong L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Define a map $T : L^q(\mathbb{R}) \longrightarrow (L^p(\mathbb{R}))^*$ by $T_g(f) = \int_{\mathbb{R}} fg$. Then by Holder's Inequality. $|T_g(f)| \le \|f\|_p \|g\|_q$.

$$\Rightarrow \|T_g\| \le \|g\|_q \qquad \text{———} \quad (*)$$

This shows that $T_g \in (L^p(\mathbb{R}))^*$.

Next, we show that equality holds in $(*)$.

When $p=1$, $q=\infty$, this case has been discussed on page 97. Consider $1 \le p < \infty$.

Let $f_0 = \dfrac{|g|^{q-1}}{\|g\|_q^{q-1}} \operatorname{sign} g$. Then $\|f_0\|_p = 1$.

Now, $T_g(f_0) = \int \dfrac{|g|^q}{\|g\|_q^{q-1}} = \|g\|_q$.

Thus, $\|T_g\| = \|g\|_q$, $\forall\, p,\ 1 \le p < \infty$.

This implies that T given by $(*)$ is an one-one continuous map, which is isometry. The map T is also onto, whose proof requires Radon-Nikodym theorem. Hence skip the proof over here.

This concludes that any $\varphi \in (L^p(\mathbb{R}))^*$ is given by $(*)$. That is, $\exists\, g \in L^q(\mathbb{R})$ s.t. $\varphi(f) = \int_{\mathbb{R}} fg$.

When $p=\infty$, $q=1$. $L^1(\mathbb{R}) \not\cong (L^\infty(\mathbb{R}))^*$, however $L^1(\mathbb{R})$ is embedded in $(L^\infty(\mathbb{R}))^*$, since (134)

$$\varphi_g(f) = \int fg, \quad g \in L^1(\mathbb{R}), \quad \varphi_g \in (L^\infty(\mathbb{R}))^*.$$

Next, we show that $L^1(\mathbb{R}) \not\cong (L^\infty(\mathbb{R}))^*$. If $L^1(\mathbb{R}) \cong (L^\infty(\mathbb{R}))^*$. Then consider $S(\mathbb{R})$ as the space of all essentially bounded simple functions on $\mathbb{R}$. We know that $\overline{S(\mathbb{R})} = L^\infty(\mathbb{R})$. For $\varphi \in S(\mathbb{R})$, define

$T(\varphi) = \varphi(0)$. Then $\|T\| = 1$ (Check!) Hence, T can be extended to $L^\infty(\mathbb{R})$. But, then $\exists f \in L^1(\mathbb{R})$ s.t. $T = T_f$ $(\because (L^\infty(\mathbb{R}))^* \cong L^1(\mathbb{R}))$ For $I \subset \mathbb{R}\setminus\{0\}$, I a bounded interval,

$$0 = T(\chi_I) = \int f\chi_I = \int_I f, \ \forall I$$

$\Rightarrow f = 0$ a.e., which contradicts $\|T\| = 1$.

Let M be a subspace (or subset) of a n.l.s. X. Write $M^\perp = \{ f \in X^* : f(M) = \{0\} \}$. Then $M^\perp$ is a closed subspace of X^*. Since $f_n \to f \Rightarrow f(y) = \lim f_n(y) = 0$, $\forall y \in M$. That is, $f(M) = \{0\}$.

The following result is very very important and useful.

Theorem: Let M be a subspace of a n.l.s. X. Then $\overline{M} = X$ iff $M^{\perp} = \{0\}$.

Proof: Suppose $\overline{M} = X$, and $f \in M^{\perp}$. Then $f(M) = \{0\}$. Let $x \in X$, then $\exists$ $x_n \in M$ s.t. $x_n \to x$. Hence,

$$f(x) = \lim f(x_n) = 0, \ \forall \ x \in X.$$

That is, $M^{\perp} = \{0\}$.

Next, suppose $M^{\perp} = \{0\}$. On contrary, suppose $\overline{M} \neq X$. Then $\exists$ $x_0 \in X \setminus \overline{M}$ & By HBT, $\exists$ $f_0 \in X^*$ s.t. $f_0(\overline{M}) = \{0\}$ & $f_0(x_0) = 1$. This implies $M^{\perp} \neq \{0\}$, which is a contradiction. Hence $\overline{M} = X$.

Note that the subspace $M^{\perp}$ of X^* is known as annihilator space of M.

Ex. Let $X = C[0,1]$ with sup norm. Then $\{f \in X : f(0) = 0\}^{\perp} \neq \{0\}$. Also,

$$\{f \in X : \int_0^1 f(t)\,dt = 0\}^{\perp} \neq \{0\}.$$

Weak and Weak* topologies:

A weak top. eventually is a top. having fewer number of open sets. To make a given family of function to be continuous, we may not require full strength of the parent topology.

Def^n: Let X be a non-empty set, and $\mathcal{F} = \{f_i : X \to \mathbb{C}, i \in I\}$. A weak top. w.r.t. to $\mathcal{F}$ is the smallest top. $\mathcal{T}_{\mathcal{F}}$ on X that makes each f_i continuous on X.

$\mathcal{J}_{\mathcal{F}} = \{ \bigcap_{i=1}^{k} f_i^{-1}(O_i) : O_i \subset \mathbb{C},\ O \text{ open} \}$

is the base for $\mathcal{T}_{\mathcal{F}}$. That is, $O \in \mathcal{T}_{\mathcal{F}}$, can be expressed as $O = \bigcup_{i=1}^{\infty} F_i$, where $F_i \in \mathcal{J}_{\mathcal{F}}$.

That is, each open set in $\mathcal{T}_{\mathcal{F}}$ is the countable union of finite intersection of members of the form $f_i^{-1}(O)$, O open in $\mathbb{C}$.

Weak topology on $(X, \|\cdot\|)$.

The weak top. on $(X, \|\cdot\|)$ is the weakest top. on X that makes each $f \in X^*$ conti on X. It is easy to see that weak top.

on X is Hausdorff. If $x_1, x_2 \in X$, and $x_1 \neq x_2$, then by HBT, $\exists f \in X^*$ s.t.

$f(x_1 - x_2) = \|x_1 - x_2\| \neq 0$, with $\|f\| = 1$.

$\Rightarrow f(x_1) \neq f(x_2)$. Hence, $\exists$ open sets U & V in $\mathbb{C}$ s.t. $f(x_1) \in U$ & $f(x_2) \in V$.

It follows that $f^{-1}(U) \cap f^{-1}(V) = \phi$ (Exercise)

Since $f^{-1}(U)$ & $f^{-1}(V) \subset \tau_w$ (weak top. on X),

$\Rightarrow (X, \tau_w)$ is a Hausdorff space.

Suppose $x_n, x \in X$, and $x_n \to x$ in τ_w. Then for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if

$x \in f^{-1}(B_{\epsilon/2}(0))$, $\forall f \in X^*$,

then $x_n, x \in f^{-1}(B_{\epsilon/2}(0))$, $\forall n \geq N$, $\forall f \in X^*$.

That is, $|f(x_n) - f(x)| < \epsilon$ for all $n \geq N$, and whenever $f \in X^*$. Thus,

$x_n \xrightarrow{\tau_w} x$ or $x_n \xrightarrow{w} x$

iff $\forall f \in X^*$, $f(x_n) \to f(x)$.

<u>Weak* topology on X^*:</u>

Let $X^{**} = (X^*)^* = \{ g : X^* \xrightarrow{\text{linear}} \mathbb{C}, \text{cont} \}$.

This is known as the second dual of X.

Now, consider a sub collection of X^{**}.
For $x \in X$, define $F : X \longrightarrow (X^*)^*$ by

$F_x(f) = f(x)$, where $F_x : X^* \longrightarrow \mathbb{C}$.

Let $\mathcal{J} = \{ F_x : X^* \to \mathbb{C},\ x \in X,\ F_x(f) = f(x),\ f \in X^* \}$.

Weak* top. on X^* is the smallest top. on X^* that makes each F_x continuous.

Note that $|F_x(f)| = |f(x)| \leq \|f\|\,\|x\|$

$\Rightarrow \|F_x\| = \sup_{\|f\| \leq 1} |F_x(f)| \leq \|x\|$.

If $x \neq 0$, by HBT, $\exists$ $f \in X^*$ with $\|f\| = 1$ s.t. $f(x) = \|x\|$. Hence, $\|F_x\| = \|x\|$.

Thus, $F_x \in X^{**}$ and $F : X \longrightarrow X^{**}$ is a one-one isometry.

The collection $\mathcal{E} = \{ \bigcap_{i=1}^{k} F_{x_i}^{-1}(O) : O \subset \mathbb{C},\ O \text{ open},\ x_i \in X \}$

is a subbase for the weak* top. τ_{w^*}.

Thus, if $f_n, f \in X^*$, and $f_n \longrightarrow f$ in τ_{w^*}, then for $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

if $f \in F_x^{-1}(B_{\epsilon/2}(0))$, $\forall$ $x \in X$, then

$f_n, f \in F_x^{-1}(B_{\epsilon/2}(0))$, $\forall$ $n \geq N$, $\forall$ $x \in X$.

That is, $|F_x(f_n) - F_x(f)| < \epsilon$, $\forall$ $n \geq N$, $\forall$ $x \in X$.

Hence, $|f_n(x) - f(x)| < \epsilon$, $\forall\, n \geq N$, and whenever $x \in X$. Thus,

$f_n \xrightarrow{\tau_{w^*}} f$ or $f_n \xrightarrow{w^*} f$ iff

$$f_n(x) \longrightarrow f(x), \ \forall\, x \in X.$$

This means, $f_n \to f$ pointwise.

Note that $x_n, x \in X$ & $x_n \to x$ in $(X, \|\cdot\|)$, it follows that

$$|f(x_n) - f(x)| \leq \|f\| \, \|x_n - x\| \to 0.$$

Hence norm conv. (or strong) conv implies weak conv, but converse need not be true. Ex. If $e_n \in l^2$, $e_n = (0, \ldots, \underset{n\text{th}}{1}, 0, \ldots)$, Then for each $y \in l^2$,

$$f_y(e_n) = y_n \to 0.$$

Hence $e_n \to 0$ weakly, however, $\|e_n\| = 1 \Rightarrow e_n \not\to 0$ in norm.

Ex. We know that $(L^2[-\pi, \pi])^* = L^2[-\pi, \pi]$ and $T \in (L^2[-\pi, \pi])^*$, is given by

$T(f) = T_g(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}$, for some $g \in L^2[-\pi, \pi]$.

Let $v_n(t) = e^{-int}$, $n \in \mathbb{Z}$. Then

$$\|v_n\|_2 = 1 \Rightarrow v_n \not\to 0 \text{ in } L^2[-\pi, \pi].$$

(Note that $\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2\right)^{1/2}$)

Also, $T(\nu_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} g(t)\,dt = \hat{g}(n)$.

Let $\mathcal{P} = \{ P_K : P_K(t) = \sum_{n=-K}^{K} \alpha_n e^{int},\ K = 0,1,2,\dots \}$

Then $\overline{\mathcal{P}} = L^2[-\pi,\pi]$. That is, trigonometric polys are dense in $L^2[-\pi,\pi]$.

Hence, for each $\epsilon > 0$, $\exists$ P_K s.t.

$$\|P_K - g\|_2 < \epsilon.$$

Then $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} P_K(t)\,dt = 0, \quad \forall\ n > K.$

That is, $T_{P_K}(\nu_n) = 0, \quad \forall\ n > K.$

Hence,

$$\begin{aligned}|T_g(\nu_n)| &= |T_g(\nu_n) - T_{P_K}(\nu_n)| \\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}(g - P_K)(t)\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot|g - P_K| \le \|g - P_K\|_2 < \epsilon\end{aligned}$$

That is, $|T_g(\nu_n)| < \epsilon, \quad \forall\ n > K.$

Thus, $T_g(\nu_n) \to 0, \quad \forall\ g \in L^2[-\pi,\pi]$.

$\Rightarrow \nu_n \to 0$ weakly in $L^2[-\pi,\pi]$.

However, every weakly convergent seq^n is bounded.

Theorem: Let X be a n.l.s. Then every weakly Cauchy seq^n in X is bdd.

Proof: Let $x_n \in X$, and $f(x_n)$ is C.S for each $f \in X^*$. Then $f(x_n)$ is a C.S in $\mathbb{C}$, and hence bounded. 141

Therefore, $|f(x_n)| \leq M_f, \ \forall f \in X^*$.

i.e. $|F_{x_n}(f)| \leq M_f, \ \forall f \in X^*$.

Note that $F_{x_n} : X^* \longrightarrow \mathbb{C}$ is a seq^n of bdd linear functionals on X^*.

By UBP, F_{x_n} is uniformly bounded, and hence $\|F_{x_n}\| \leq M$.

That is, $\|x_n\| \leq M$ $(\because \|F_x\| = \|x\|)$.

Ex. If $f_n, f \in X^*$, and $f_n \to f$ in X^* then $f_n \to f$ in the weak* top. of X^*.

$|F_x(f_n) - F_x(f)| = |f_n(x) - f(x)| \leq \|f_n - f\| \|x\| \to 0$,

$(\because \|f_n - f\| \to 0$ is given$)$.

However, converse need not be true.

Let $x = (x_1, x_2, \ldots x_n, \ldots) \in l^2$, and $f_n \in (l^2)^*$

is defined by $f_n(x) = x \cdot e_n = x_n$.

Then $f_n(x) \to 0$, $\forall\, x \in l^2$, that is,

$f_n \xrightarrow{w^*} 0$, but $\|f_n\| = 1$ (Exercise),

$\Rightarrow f_n \not\to 0$ in $(X^*, \|\cdot\|)$, when $X = l^2$.

Note that $|f_n(x) - f_m(x)| = |x \cdot (e_n - e_m)|$

and $\|f_n - f_m\| = \|e_n - e_m\| = \sqrt{2}$ if $n \neq m$.

Hence, f_n is not even a c.s. in $((l^2)^*, \|\cdot\|)$.

Theorem: Let M be a proper dense subspace of X^*. If $x_n \in X$ is a uniformly bdd seq^n and $f(x_n) \to f(x)$, $\forall\, f \in M$. Then $f(x_n) \to f(x)$, $\forall\, f \in X^*$.

Proof: Let $f \in X^*$, then for $\epsilon > 0$, $\exists\, f_i \in M$ s.t. $\|f - f_i\| < \epsilon$, if $i \geq i_0$.

$$|f(x_n) - f(x)| \leq |f(x_n) - f_i(x_n)| + |f_i(x_n) - f_i(x)| + |f_i(x) - f(x)|$$

$$\leq \|f - f_i\|\,\|x_n\| + |f_i(x_n) - f_i(x)| + \|f_i - f\|\,\|x\| \quad \text{---(1)}$$

Since $(x_n) \in X$ is uniformly bounded, $\exists$ $C > 0$ s.t. $\|x_n\| \leq C$, $\forall\, n \geq 1$. ---(2)

For $\epsilon > 0$, $\exists\ n_0 \in N$ s.t

$$|f_i(x_n) - f_i(x)| < \epsilon, \quad \forall\ n \geq n_0 \quad -(3)$$

From (1), (2) & (3), it follows that (143)

$$|f(x_n) - f(x)| < (1 + c + \|x\|)\epsilon, \quad \forall\ n \geq n_0,$$

$\forall\ f \in X^*$. That is, $f(x_n) \to f(x)$, $\forall f \in X^*$.

<u>Reflexive spaces:</u>

A normed linear space X which is isometrically isomorphic to its 2nd dual X^{**} is known as reflexive space. Since, X^{**} is a Banach space, it follows that $\therefore$ each reflexive n.l.s must be a B.S.

Note that, in general, any conti linear functional on X^* need not be of form F_x, for some $x \in X$. However, to each $x \in X$,

$$F_x : X^* \longrightarrow \mathbb{C}, \text{ defined by}$$

$$F_x(f) = f(x), \text{ implies that}$$

X is embedded into X^{**}. That is,

$$F : X \longrightarrow X^{**} \text{ is a one-one}$$

map which is isometry. but need not be onto. For example,

$$l^1 \longrightarrow (l^1)^* = l^\infty \longrightarrow (l^\infty)^* \supsetneq l^1.$$

Hence, $F: \ell^1 \longrightarrow (\ell^1)^{**}$ is not onto.

Def^n: A n.l.s. X is said to be reflexive if F is an onto map. (144)

Ex. Let $1<p<\infty$, then $(\ell^p)^{**} \cong \ell^p$, and $(L^p(\mathbb{R}))^{**} \cong L^p(\mathbb{R})$.

Ex. Any finite dim. space is reflexive.

Ex. $c_0 \longrightarrow c_0^* = \ell^1 \longrightarrow (\ell^1)^* = \ell^\infty$

But $c_0 \subsetneq \ell^\infty$ (proper closed subspace).

Hence, c_0 is not reflexive.

Next, we shall show that weak* top. of X^* is metrizable, and hence, compactness and sequential compactness on X^* are equivalent.

Let X be a separable Banach space and $\{x_k\}_{k=1}^\infty$ be a countable dense set in X.

For $f, g \in X^*$, define

$$d(f,g) = \sum \frac{1}{2^k} \frac{|f(x_k) - g(x_k)|}{\|x_k\|}. \quad \text{--- (1)}$$

Then d is a metric on X^*.

<u>Theorem</u>: Let $f_n, f \in X^*$. Then FAE

[I] $\exists\ C > 0$ s.t. $\|f_n\| \le C,\ \forall\ n \geq 1$

(fn uniformly bounded on X^*) (145)

and $d(f_n, f) \to 0$.

(II) $f_n(x) \longrightarrow f(x),\ \forall\ x \in X$

($f_n \to f$ in the weak* top. of X^*)

Proof: Since $d(f_n, f) \to 0$, by (1), it follows that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|f_n(x_k) - f(x_k)|}{\|x_k\|} = 0$$

$$\Rightarrow f_n(x_k) \to f(x_k),\ \forall\ k \geq 1 \quad -(2)$$

Let $x \in X$. Then $\exists\ x_{k_\ell} \in \{x_1, \dots x_n, \dots\}$ s.t. $x_{k_\ell} \to x$. Hence,

$$|f_n(x) - f(x)| \le |f_n(x) - f_n(x_{k_\ell})| + |f_n(x_{k_\ell}) - f(x_{k_\ell})| + |f(x_{k_\ell}) - f(x)|$$

$$\le \|f_n\| \|x_{k_\ell} - x\| + |f_n(x_{k_\ell}) - f(x_{k_\ell})| + \|f\| \|x_{k_\ell} - x\|.$$

Since, $\|f_n\| \le C$, it follows that $f_n(x) \to f(x),\ \forall\ x \in X$.

Suppose (II) is true. That is,

$$f_n(x) \longrightarrow f(x),\ \forall\ x \in X.$$

Then $f_n(x)$ is a bounded seq^n in $\mathbb{C}$.
Hence $|f_n(x)| \leq C_x$, $\forall n \geq 1$. (146)
By UBP, we get $\|f_n\| \leq C$.
Since, $f_n(x) \to f(x) \Rightarrow (f_n - f)(x) \to 0$,
from (1), it is clear that we can assume $f = 0$. Hence,

$$d(f_n, 0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|f_n(x_k)|}{\|x_k\|} < \infty \quad -(2)$$

From (2), for $\epsilon > 0$, $\exists$ $k_0 \in \mathbb{N}$ s.t.

$$\sum_{k=k_0+1}^{\infty} \frac{1}{2^k} \frac{|f_n(x_k)|}{\|x_k\|} < \epsilon/2$$

On the other hand, $f_n(x) \to 0$, $\forall x \in X$,
for $\epsilon > 0$, $\exists$ $N_0 \in \mathbb{N}$ s.t.

$$\sum_{k=1}^{k_0} \frac{1}{2^k} \frac{|f_n(x_k)|}{\|x_k\|} < \epsilon/2, \ \forall n \geq N_0.$$

Thus, $d(f_n, 0) < \epsilon$, $\forall n \geq N_0$.

Banach-Alaoglu theorem:

We know that, unit ball in an infinite dim. Banach space is not compact. This implies, $B^* = \{f \in X^* : \|f\| \leq 1\}$ will not compact in $(X^*, \|\cdot\|)$. However, the following weak result holds.

Theorem (Banach-Alaoglu):
Let X be a n.l.s. Then the closed unit ball $B^* = \{f \in X^* : \|f\| \leq 1\}$ is weak* compact in X^*.

Proof: For $x \in X$, let $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$.
Then D_x is compact in $\mathbb{C}$. Write
$$D = \prod_{x \in X} D_x .$$
Then, by Tychonoff theorem for product top, D is compact.
Define $\varphi : B^* \to D$ by
$$\varphi(f) = (f(x))_{x \in X} . \quad (\because |f(x)| \leq \|x\|)$$
Then φ is 1-1, continuous linear map.
If $\varphi(f) = 0$, then $(f(x))_{x \in X} = 0 \Rightarrow f(x) = 0, \forall x \in X$.
Thus $f = 0$.

<u>φ is continuous:</u> Let $f_\alpha \in B^*$ & $f_\alpha \xrightarrow{w^*} f$.
Then $f_\alpha(x) \to f(x), \forall x \in X$.
$\Rightarrow \varphi(f_\alpha) = (f_\alpha(x))_{x \in X} \to (f(x))_{x \in X}$.
(Note that conv. in product top. is co-ordinate wise)
Since D is compact, and top. on D is Hausdorff, to show $\varphi(B^*)$ is compact.

it is enough to show that $\varphi(B^*)$ is closed in D. Let $\xi \in \overline{\varphi(B^*)} \subset D$. Then (148)
$\xi = (\xi_x)_{x\in X}$; and $|\xi_x| \le \|x\|$. Also,
$\exists\ f_n \in B^*$ s.t. $\varphi(f_n) \to \xi = (\xi_x)_{x\in X}$

i.e. $(f_n(x))_{x\in X} \to (\xi_x)_{x\in X}$

$\Rightarrow f_n(x) \to \xi_x$ (Co-ordinate wise conv)

Let $\xi_x = f(x)$. Then f is linear, and
$|f(x)| \le \|x\|$. This implies: $f \in B^*$.
Thus, $\xi = (f(x))_{x\in X} \in \varphi(B^*)$.
$\Rightarrow \varphi(B^*)$ is closed in D, and hence compact in D.

<u>Corollary</u>: Every n.l.s. X is isometrically isomorphic to a subspace of $C(K)$, where K is some compact Hausdorff space.

Proof: Let $K = B^*$. Then by BAT, B^* is weak* compact. Define

$$\varphi : X \longrightarrow C(K) \text{ by}$$
$$\varphi_x(f) = f(x), \quad f \in K.$$

Then $\varphi_x : K \to \mathbb{C}$ is continuous, because
$f_n, f \in K$, & $f_n \xrightarrow{w^*} f \Rightarrow f_n(x) \to f(x)$,
$\forall x \in X$. Hence, $\varphi_x(f_n) \to \varphi_x(f)$.

Note that $|\varphi_x(f)| \le \|f\| \|x\|$. By H.B.T,
φ_x can be extended to X^*. Hence, (149)
$\|\varphi_x\| \le \|x\|$. (we identify $\hat{\varphi_x}$ as φ_x)
once again, by H.B.T, $\exists f \in X^*$ s.t
$f(x) = \|x\|$ & $\|f\| = 1$. Thus, $\|\varphi_x\| = \|x\|$. —(1)
Hence, X is isometrically isomorphic to $\varphi(X)$, a subspace of $C(K)$.

Remark 1. If X is a Banach space, then $\varphi(X)$ is a closed subspace of $C(K)$. For this, let $\varphi_{x_n} \to \varphi$. Then $\|\varphi_{x_n} - \varphi_{x_m}\| = \|x_n - x_m\|$, implies (x_n) is a C.S. in X, hence $x_n \to x$.
Thus, $\lim \varphi_{x_n}(f) = \lim f(x_n) = f(x) = \varphi(f)$.
$\Rightarrow \varphi = \varphi_x$. Thus, $\varphi_{x_n} \to \varphi_x \in \varphi(X)$.

Remark 2: Every Banach space X is isometrically isomorphic to a closed subspace of $C(B^*)$.

Adjoint of a linear transformation:

Let X & Y be two normed linear spaces, and $T: X \longrightarrow Y$ be a linear map. Define
$T^*: Y^* \longrightarrow X^*$ by $T^*(g) = g \circ T$.
Then T^* linear.

<u>Theorem</u>: Let $T \in B(X, Y)$. Then $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. (150)

Proof: By def^n, $T^*(g)(x) = g(Tx)$.

$$\Rightarrow |T^*(g)(x)| = |g(Tx)| \le \|g\| \, \|Tx\|$$

$$\Rightarrow \|T^*(g)\| \le \|g\| \, \|T\| \quad (\because \|Tx\| \le \|T\| \|x\|)$$

$$\|T^*\| = \sup_{\|g\|=1} \|T^*(g)\| \le \|T\|.$$

Notice that $T^*(g)(x) = g(Tx)$. For $Tx \neq 0$, $\exists\ g_0 \in Y^*$ s.t. $\|g_0\| = 1$ & $g_0(Tx) = \|Tx\|$.

Hence $\sup\limits_{\|x\| \le 1} \|Tx\| = \sup\limits_{\|x\| \le 1} |T^*(g_0)(x)| \le \|T^*(g_0)\|$.

$$\Rightarrow \|T\| \le \|T^*(g_0)\| \le \sup_{\|g\| \le 1} \|T^*(g)\| = \|T^*\|.$$

Ex. Suppose $T : X \longrightarrow Y$ is invertible. Then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$

Proof: If $T^*(g) = 0$, for some $g \in Y^*$. Then $g \circ T = 0 \Rightarrow g(Tx) = 0, \ \forall\ x$. But $Y = TX \Rightarrow g(y) = 0, \ \forall\ y \in Y \Rightarrow g = 0$.

<u>T^* is onto</u>: If $T^*(g) = f$. Then $g \circ T = f$ $\Rightarrow g = f \circ T^{-1} \in Y^*$.

By IMT, T^* is invertible, since T^* is cont. Also, $(T^*)^{-1} : X^* \longrightarrow Y^*$, and

$(T^*)^{-1}(f)(y) = g(y)$, where $T^* g = f$ &

$g \circ T = f$, since T^* is onto. (151)

Also, $y = Tx$ for some $x \in X$. Hence

$$(T^*)^{-1}(f)(y) = f(x) = (f \circ T^{-1})(y) = (T^{-1})^*(f)(y).$$

$$\Rightarrow (T^*)^{-1} = (T^{-1})^*.$$

Annihilator Subspaces:

Recall that if M is a subspace of X, then $M^{\perp} = \{f \in X^* : f(M) = \{0\}\}$ is known as annihilator subspace of M. Also, we know that $\overline{M} = X$ iff $M^{\perp} = \{0\}$.

__Lemma__: Let N be a subspace of X^*, which separates point on the n.l.s X. Then N is weak* dense in X^*.

That is $\overline{N}^{w^*} = X^*$.

Proof: By the previous result, it is enough to show that $N^{\perp} = \{0\}$ in the weak* top. of X^*.

$$N^{\perp} = \{F_x \in (X^*)^* : F_x(N) = \{0\}\}$$
$$= \{F_x \in X^{**} : F_x(f) = 0, \forall f \in N\}$$
$$= \{F_x \in X^{**} : f(x) = 0, \forall f \in N\}.$$

Since N separates points on X, for $x \neq 0$, $\exists\, f \in N$ s.t. $f(x) \neq 0$. Hence, $N^{\perp} = \{0\}$.

Further, we conclude from previous result that $N^{\perp} = \{0\}$ iff $\overline{N} \overset{w^*}{=\!=} X^*$. (152)

Now let M be subspace of X. Consider

$$\varphi : X^* \longrightarrow M^* \text{ by}$$

$$\varphi(f) = f|_M \text{. Then } \ker\varphi = M^{\perp},$$

and φ is an onto map by HBT.

If $g \in M^*$, then $\exists\, f \in X^*$ s.t. $f|_M = g$ and $\|f\| = \|g\|$. Thus,

$$\tilde{\varphi} : X^*/M^{\perp} \longrightarrow M^*$$

is an onto isometry, where

$$\tilde{\varphi}(f + M^{\perp}) = \varphi(f), \text{ and}$$

$$\|\tilde{\varphi}(f + M^{\perp})\| = \|\varphi(f)\| = \|f|_M\| = \|g\| = \|f\|.$$

Thus, $X^*/M^{\perp} \cong M^*$.

Ex. Let be a closed proper subspace of X. Then $(X/M)^* \cong M^{\perp}$.

__Proof:__ Define $\varphi : (X/M)^* \longrightarrow M^{\perp} \subset X^*$ by

$$\varphi(\tilde{f})(x) = \tilde{f}(\tilde{x}) = \tilde{f}(\pi(x)) = \tilde{f}(x + M).$$

Note that $\varphi(\tilde{f})(M) = \tilde{f}(\pi(M)) = \tilde{f}(\tilde{0}) = \{0\}$.

$\Rightarrow \varphi(\tilde{f}) \in M^{\perp}$. (153)

φ is onto: Let $g \in M^{\perp} \subset X^*$. Then

$\tilde{g}(x+M) = g(x)$. Hence, $\tilde{g}$ is well-defined

and $\tilde{g} \in (X/M)^*$. Now,

$\varphi(\tilde{g})(x) = \tilde{g}(\tilde{x}) = \tilde{g}(\pi(x)) = \tilde{g}(x+M) = g(x)$.

$\Rightarrow \varphi(\tilde{g}) = g \Rightarrow \varphi$ is onto.

φ is an isometry:

$$\begin{aligned}
\|\varphi(\tilde{f})\| &= \sup_{\|x\|\le 1} |\varphi(\tilde{f})(x)| \\
&= \sup_{\|x\|\le 1} |\tilde{f}(\tilde{x})| \\
&= \sup_{\|\tilde{x}\|\le 1} |\tilde{f}(\tilde{x})| \quad (\because \widetilde{B(0,1)} = B(\tilde{0},1)) \\
&= \|\tilde{f}\|.
\end{aligned}$$