Inner product spaces:

As compare to the Euclidean spaces, there are infinite dimensional spaces, where we can make sense of angle between two vectors. And hence allowing to draw unique normal to a subspace (or hyperplane).

Let X be a vector space on $F = (\mathbb{R}$ or $\mathbb{C})$. A bilinear form $\langle \cdot, \cdot \rangle : X \times X \longrightarrow F$ is called inner product if

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff. $x = 0$.

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

for all $\alpha, \beta \in F$ and $x, y, z \in X$.

Note that $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$ (by (iii) & (ii)).

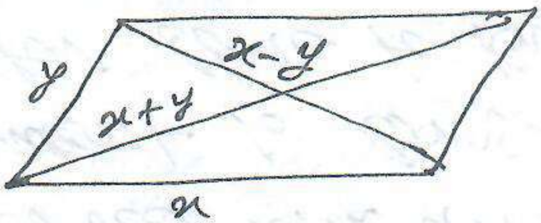
The space $(X, \langle \cdot, \cdot \rangle)$ is known as inner product space (IPS).

Note that if we write $\|x\| = \langle x, x \rangle^{1/2}$ then we later see that $\|\cdot\|$ is a norm on X. This will be followed by the inequality.

i.e. $\|x+y\| \le \|x\|+\|y\|, \ \forall x,y \in X$.
Thus $\langle \cdot,\cdot \rangle$ induces a norm on X. (156)
However, all norm on X need not produce inner product on X, which we see later, unless it satisfies the parallelogram law.

Parallelogram law:

For $x,y \in X$,



$$\|x+y\|^2+\|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right). \quad -(3)$$

Polarization identity:

Let X be an IPS. Then for $x,y \in X$, the following identity holds.

$$4\langle x,y\rangle = \|x+y\|^2-\|x-y\|^2 + i\left(\|x+iy\|^2-\|x-iy\|^2\right). \quad -(4)$$

Proof: Since

$$\|x+y\|^2-\|x-y\|^2 = 2\langle x,y\rangle+2\langle y,x\rangle$$

and $\|x+iy\|^2-\|x-iy\|^2 = 2\langle x,iy\rangle+2\langle iy,x\rangle$, the identity (4) follows.

Lemma: Let X be an inner product space, and $x_n, x \in X$. If $\|x_n - x\| \to 0$, then $\langle x_n - x, y\rangle \to 0$, $\forall\, y \in X$.

Converse need not be true. For $e_n \in l^2$, $\|e_n\|_2 = 1$, but $\langle e_n, y\rangle = y_n \to 0$, $\forall\, y \in l^2$.

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Proof: Suppose $\|x_n - x\| \to 0$. Note that w.l.g. we can assume $x = 0$. Thus, $x_n \to 0$.

By polarization identity,

$$4\langle x_n, y\rangle = \|x_n + y\|^2 - \|x_n - y\|^2 + i(\|x_n + iy\|^2 - \|x_n - iy\|^2)$$

$\Rightarrow \langle x_n, y\rangle \to 0$, $\forall\, y \in V$

Thus, while X is endowed with an inner product, norm convergence implies inner product wise convergent. By Riesz representation theorem, we come to know that, ~~norm~~ inner product wise convergence is same as weak convergence.

For $x, y \in X$, let $\|(x, y)\|_0 = \|x\| + \|y\|$.

Then $\langle \cdot, \cdot\rangle$ is a conti map on $(X \times Y, \|\cdot\|_0)$.

Suppose $\|(x_n, y_n) - (x, y)\|_0 \to 0$. Then

$\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$.

Now,

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle|$$

$$\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\|.$$

Since (y_n) is conv, (y_n) is bdd and $\|y_n\| \leq c, \forall n \geq 1$.

Hence, $|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$.

Note that $\langle \cdot, \cdot\rangle$ is uniformly conti.

<u>Theorem</u>: Let $(X, \|\cdot\|)$ be a n.l.s. Then $\|\cdot\|$ is induced by an inner product on X iff $\|\cdot\|$ satisfies parallelogram law.

Proof: Suppose $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$. Then for $\|x\| = \langle x, x\rangle^{1/2}$, it is easily followed that

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{--- (1)}$$

Conversely, suppose $\|\cdot\|$ satisfies (1).

Write

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2) \quad \text{--- (2)}$$

Then we claim that $\langle \cdot,\cdot \rangle$ stands for inner product on X. Notice that

(i) $\langle x,x\rangle = \|x\|^2 > 0$ & $\langle x,x\rangle = 0$ iff $x=0$.

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (from (2))

(iii) (a) $\langle ix,y\rangle = i\langle x,y\rangle$

(b) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

(Easily followed from parallelogram law)

Now, if $\alpha \in \mathbb{N}$, then from (iii)(b),

$$\langle \alpha x,y\rangle = \alpha\langle x,y\rangle.$$

For $\alpha \in \mathbb{Q}$, $\alpha = m/n \Rightarrow m = \alpha n$.

$$\langle \alpha n x,y\rangle = n\langle \alpha x,y\rangle$$

$$\Rightarrow \langle \alpha x,y\rangle = \alpha\langle x,y\rangle \quad \text{—(3)}$$

By continuity of inner product, for $\alpha \in \mathbb{R}$, $\alpha_n \in \mathbb{Q}$, $\alpha_n \to \alpha$, (3) holds for $\alpha \in \mathbb{R}$.

If $\alpha = a+ib$, then from (iii)(a), it follows that (3) holds for $\forall\, \alpha \in \mathbb{C}$.

Thus, $\langle \cdot,\cdot\rangle$ is an inner product, which is induced by $\|\cdot\|$.

Defn: An IPS $(X,\langle \cdot,\cdot\rangle)$ is said to be

Hilbert space if X is complete w.r.t the norm induced by $\langle\cdot,\cdot\rangle$. (160)

Ex. Let $f, g \in C[0,1]$, and write $\langle f, g\rangle = \int fg$. Then $\langle\cdot,\cdot\rangle$ is an IP on $C[0,1]$, but $(C[0,1], \|\cdot\|_2)$ is not complete.

Ex. For $1 \le p < \infty$, $(\ell^p, \|\cdot\|_p)$ becomes a Hilbert space iff $p=2$.

Proof: Since $(\ell^p, \|\cdot\|_p)$ is complete, it is enough to show that $\|\cdot\|_p$ produces an inner product iff $p=2$. But $\|\cdot\|_p$ produces IP iff

(1) $\|x+y\|_p^2 + \|x-y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2)$, $\forall x, y \in \ell^p$.

Let $x = (1,0,0,\ldots)$ & $y = (0,1,0,\ldots)$. Then it follows from (1) that $p=2$.

Ex. For $1 \le p < \infty$, $L^p(\mathbb{R})$ is a Hilbert space iff $p=2$.

For $f = \chi_{[0,1]}$ & $g = \chi_{[1,2]}$, by $\|f+g\|_p + \|f-g\|_p = 2(\|f\|_p + \|g\|_p)$, it follows that $p=2$. Notice that

Ex. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then $\langle x,y\rangle = \frac{1}{n}\sum_{p=0}^{n-1} w^p \|x+w^p y\|^2$, where w is an nth root of unity.

<u>Theorem</u>: Let M be a non-empty closed convex subset of a Hilbert space H. Then M has a unique element of the smallest norm.

Proof: If $0\in M$, then result holds trivially. Let $0\notin M$. Then write

$$\delta = \inf\{\|x\| : x\in M\}.$$

If $x,y\in M$, then $\frac{1}{2}(x+y)\in M$ and by parallelogram law,

$$\|x-y\|^2 = 2(\|x\|^2+\|y\|^2) - 4\left\|\frac{x+y}{2}\right\|^2$$

$$\leq 2(\|x\|^2+\|y\|^2) - 4\delta^2 \quad \text{---(1)}$$

Since δ is infimum, $\exists\ x_n\in M$ s.t. $\|x_n\|\to\delta$.

For $m,n\in\mathbb{N}$, from (1)

$$\|x_n - x_m\| \to 2\cdot 2\delta^2 - 4\delta^2 = 0.$$

Hence (x_n) is a C.S. in H and $x_n \to x \in M$ ($\because M$ closed). (162)

Thus $\delta = \inf_{y \in M} \|y\| = \lim \|x_n\| = \|x\|$.

Suppose $\exists$ $x_1, x_2 \in M$ s.t. $\|x_1\| = \|x_2\| = \delta$.

Then $\frac{1}{2}(x_1 + x_2) \in M$ and

$$\delta \leq \| \tfrac{1}{2}(x_1 + x_2)\| \leq \tfrac{1}{2}\|x_1\| + \tfrac{1}{2}\|x_2\| = \delta$$

$$\Rightarrow \|x_1 + x_2\| = 2\delta.$$

By parallelogram law,

$$\|x_1 - x_2\|^2 = 2(\|x_1\|^2 + \|x_2\|^2) - (2\delta)^2$$
$$= 0.$$

Hence $x_1 = x_2$.

Corollary: Let M be a closed convex subset of a Hilbert space H. Then for each $x \in H$, there exists unique $x_0 \in M$ s.t.

$$\operatorname{dist}(x, M) = \|x - x_0\|.$$

Proof: Note that

$$\inf_{y \in M} \|x - y\| = \inf_{z \in x - M} \|z\|.$$

Since $x - M$ is closed & convex, $\exists ! x_0$ in $x - M$ s.t

$$d(x, M) = \|x - x_0\|.$$

Notice that, from above, it follows that for each $x \in H$, $\exists !$ closest element of M.

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Ex. Let S be a subset of an IPS X. Write $S^{\perp} = \{x \in X : \langle x, y \rangle = 0, \forall y \in S\}$.

Then (i) (a) $S \cap S^{\perp} \subseteq \{0\}$

(b) $S \cap S^{\perp} = \{0\}$ if S is a subspace of X.

(ii) $\{0\}^{\perp} = X$ & $X^{\perp} = \{0\}$

(iii) $S^{\perp}$ is a closed subspace of X.

(iv) $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$.

(v) $S \subset S^{\perp\perp}$.

(Hint: $S \perp S^{\perp} \Rightarrow \langle S, S^{\perp} \rangle = 0 \Rightarrow S \subset (S^{\perp})^{\perp} = S^{\perp\perp}$)

<u>Theorem</u>: Let M be a closed subspace of a Hilbert space H. Then

(a) for $x \in H$, $\exists !$ $u \in M$ and $v \in M^{\perp}$ s.t. $x = u + v$. That is,

$$H = M \oplus M^{\perp}$$

(b) $\exists$ $P : H \to M$ with $Px = u$, and $Q : H \to M^{\perp}$ with $Qx = v$, such that

(i) $P(M) = M$, $Q(M^\perp) = M^\perp$
$P(M^\perp) = \{0\}$ and $Q(M) = \{0\}$.

(ii) $P^2 = P$ and $Q^2 = Q$

(iii) $P, Q \in B(H)$ and $\|P\| = 1 = \|Q\|$

(iv) $P(x)$ is a unique closest element of M to x, whereas $Q(x)$ is a unique closest element of $M^\perp$ to x.

Proof: Let $x \in H$. Then $x + M$ is a closed convex subset of H.

By previous result, $\exists$ unique smallest norm element of $x+M$, say $Qx \in x+M$

Let $P(x) = x - Q(x)$.

Claim: Q maps H onto $M^\perp$. For this, let $z = Qx$ and $y \in M$ with $\|y\| = 1$.

Write $\alpha = \langle y, z \rangle$. Then

$z - \alpha y = Qx - \alpha y \in x + M$.

Since $\|z\| = \inf\limits_{y \in x+M} \|y\|$, it follows that $\|z\|^2 \leq \|z - \alpha y\|^2 = \|z\|^2 - |\alpha|^2$,

which is possible iff $\alpha = 0$.

Thus, $\langle Qx, y \rangle = 0$, $\forall y \in M$.

That is, $Qx \in M^\perp$, $\forall x \in H$. Thus, Q maps H into $M^\perp$ & P maps H into M.

Note that if $x \in H$, then

$$x = Px + Qx = u + v \in M + M^\perp$$

$$\Rightarrow H = M \oplus M^\perp \quad (\because M \cap M^\perp = \{0\})$$

(of) (i): Let $x \in M^\perp$, then $Px = x - Qx \in M^\perp \cap M = \{0\}$.

$$\Rightarrow Q(M^\perp) = M^\perp \quad (\because P(M^\perp) = \{0\})$$

Similarly, if $x \in M$, then $Q(M) = \{0\}$

$$\Rightarrow P(M) = M.$$

Thus, P & Q both are onto map.

(ii) For $x \in H$, $x = Px + Qx$. Then

$$Px = P^2x + P(Qx) = P^2x$$

and $Qx = Q(Px) + Q^2x = Q^2x$.

(iii) Note that P & Q are linear because

$$x = Px + Qx, \ \forall x \in H.$$

Here,

$$P(\alpha x + \beta y) - \alpha Px - \beta Py$$

$$= \alpha x + \beta y - Q(\alpha x + \beta y) - \alpha(x - Qx) - \beta(y - Qy)$$

$$= \alpha Qx + \beta Qy - Q(\alpha x + \beta y) \in M^\perp \cap M = \{0\}.$$

$\Rightarrow P$ & Q are linear.

Also, $x = Px + Qx$, $(\because Px \perp Qx)$

$\Rightarrow \|x\|^2 = \|Px\|^2 + \|Qx\|^2$

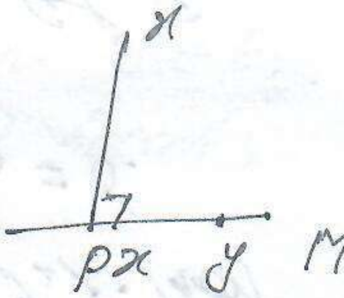
$\Rightarrow \|Px\| \leq \|x\| \Rightarrow \|P\| \leq 1$, & $\|Q\| \leq 1$.

Let $x_0 \in M$ & $\|x_0\| = 1$. Then

$$\|P\| = \sup_{\|x\|=1} \|Px\| \geq \|Px_0\| = \|x_0\| = 1.$$

Hence $\|P\| = 1 = \|Q\|$.

(iv) Let $y \in M$. Then



$$\|x-y\|^2 = \|Px - y + Qx\|^2$$
$$= \|Px - y\|^2 + \|Qx\|^2 \quad \text{——} (*)$$

In $(*)$ minimum will attain iff $y = Px$.

$$\Rightarrow \inf_{y \in M} \|x-y\| = \|x - Px\|$$

and similarly, $\inf_{y \in M^\perp} \|x-y\| = \|x - Qx\|$.

Riesz - Representation theorem.

If H is a Hilbert space then $H^* \cong H$, where H^* is the Conti. dual of H.

Proof: By HBT, we are knowing that $H \subseteq H^*$. For $y \in H$, define

$$f_y(x) = \langle x, y \rangle \quad \text{——} (*)$$

Then $|f_y(x)| \le \|x\| \|y\| \Rightarrow \|f_y\| \le \|y\|$.

Also, $\|y\|^2 = \langle y, y \rangle = f_y(y) \le \|f_y\| \|y\|$

$\Rightarrow \|f_y\| = \|y\|$. (167)

Thus $f_y \in H^*$, $\forall\, y \in H$.

Conversely, suppose $f \in H^*$. Write $M = \operatorname{Ker} f$. Then for $z \in M^\perp$ with $\|z\| = 1$

$$x - \frac{f(x)}{f(z)} z \in \operatorname{Ker} f$$

$$\Rightarrow \left\langle x - \frac{f(x)}{f(z)} z, z \right\rangle = 0$$

$$\Rightarrow f(x) = \langle x, \overline{f(z)} z \rangle = \langle x, y \rangle = f_y(x),$$

where $y = \overline{f(z)} z$. Hence $f = f_y$.

Notice that representation in (*) is unique. If $f(x) = \langle x, y_1 \rangle = \langle x, y_2 \rangle$. Then

$y_1 - y_2 \perp x \quad \forall\, x \in H \Rightarrow y_1 - y_2 = 0$.

Orthonormal Set:

A subset E of a Hilbert space H is said to be orthonormal if each $e \in E$, $\|e\|=1$, and for each pair of points $e_i, e_j \in E$, $e_i \perp e_j$ if $i \neq j$.

Note that the family $\mathcal{F}$ of all orthonormal sets in H is partially ordered under set inclusion. If $\{E_i\}_{i\in I}$ is a chain in $\mathcal{F}$, then $\bigcup_{i\in I} E_i$ is an upper bound for $\{E_i\}_{i\in I}$.

Hence, by Zorn's lemma $\mathcal{F}$ has a maximal element, say E_∞. E_∞ is known as ortho-normal basis of H.

That is, every Hilbert space has an orthonormal basis (ONB).

Lemma: Let $\{e_1, \ldots e_n\}$ be an orthonormal set in Hilbert space H. Then for each $x \in H$,

(i) $$x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \perp e_k, \ \forall\ k = 1, 2, \ldots, n. \quad \text{--- (1)}$$

(ii) $\left\| \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\|^2 = \sum_{j=1}^{n} |\langle x, e_j \rangle|^2$ —(2)

<u>Proof</u>: $\langle x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0.$

$\Rightarrow \langle x - S_n, e_k \rangle = 0$, where $S_n = \sum_{j=1}^{n} \langle x, e_j \rangle e_j$.

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$$\|S_n\|^2 = \langle S_n, S_n \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle x, e_j \rangle \overline{\langle x, e_k \rangle}$$

i.e. $\left\| \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\|^2 = \sum_{j=1}^{n} |\langle x, e_j \rangle|^2.$

Note that

$$0 \le \|x - S_n\|^2 = \langle x - S_n, x \rangle - \langle x - S_n, S_n \rangle$$
$$= \|x\|^2 - \langle S_n, x \rangle - 0 \ \text{(by (1))}.$$
$$= \|x\|^2 - \sum |\langle x, e_j \rangle|^2.$$

Hence $\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \le \|x\|^2, \ \forall x \in H.$ —(3)

This is known as finite Bessel inequality.

<u>Lemma</u>: Let $\{e_\alpha\}_{\alpha \in I}$ be an orthonormal set (ONS), then for each $x \in H$, the set $E = \{\alpha \in I : \langle x, e_\alpha \rangle \neq 0\}$ is countable.

Proof: Let $E_n = \{ e_\alpha : |\langle x, e_\alpha \rangle|^2 > \frac{\|x\|^2}{n} \}$.

Then by finite Bessel inequality, E_n contains at most $n-1$ elements.

Also, if $e_\alpha \in E$, then $\exists$ $n \in \mathbb{N}$ s.t

$|\langle x, e_\alpha\rangle|^2 > \frac{\|x\|^2}{n}$, else $\langle x, e_\alpha\rangle = 0$.

Hence, $E = \bigcup E_n$ is countable. (17D)

Bessel inequality:

Let $\{e_\alpha\}_{\alpha \in I}$ be an ONS in a Hilbert space H. Then for each $x \in H$,

$$\sum_{\alpha \in I} |\langle x, e_\alpha\rangle|^2 \leq \|x\|^2 < \infty. \quad \text{—(1)}$$

Note that sum in (1) is on countable set, since E is countable.

proof: For $x \in H$, $\exists$ only countably many e_α s.t $\langle x, e_\alpha\rangle \neq 0$. Let us write them as $\{\langle x, e_n\rangle \neq 0 : n \in \mathbb{N}\}$. Then from finite Bessel inequality,

$$\sum_{n=1}^{k} |\langle x, e_n\rangle|^2 \leq \|x\|^2 < \infty.$$

Letting $k \to \infty$, we get the required.

Now, we are going to see that every vector in a Hilbert space has series expansion. This is known as Parseval formula. This will be followed by the fact that every Hilbert space has an ONB.

Parseval identity:

Let $\{e_\alpha\}_{\alpha\in I}$ be an ONB of a Hilbert space H. Then for $x\in H$, (17)

$$x = \sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha$$

and $$\|x\|^2 = \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2$$ both the sums are countable.

Proof: Let $x\in H$, then $\langle x, e_\alpha\rangle \neq 0$ only for countably many $\alpha\in I$. Let

$$E = \{ e_n : \langle x, e_n\rangle \neq 0,\ n\in\mathbb{N}\}.$$

Write $y_n = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$. Then by Bessel inequality, for $m > n$, it follows that

$$\|y_m - y_n\|^2 = \sum_{i=n+1}^{m} |\langle x, e_i\rangle|^2 \to 0 \text{ as } n\to\infty.$$

Hence (y_n) is a C.S. in H, and $y_n \to y\in H$.

Now, we need to show that $x = y$.

$$\langle x-y, e_j\rangle = \langle x, e_j\rangle - \langle y, e_j\rangle$$

$$= \langle x, e_j\rangle - \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} \langle x, e_i\rangle e_i, e_j\right\rangle$$

$(\because \|y_n - y\| \to 0$ & $\langle\cdot,\cdot\rangle$ is conti$)$

$$= \langle x, e_j\rangle - \langle x, e_j\rangle = 0.$$

Further, if $e_\beta \in \{e_\alpha\}_{\alpha\in I}$ and $e_\beta \notin E$, then then $\langle x, e_\beta\rangle = 0$. Hence

$$\langle x-y, e_\beta\rangle = \langle x, e_\beta\rangle - \lim_{n\to\infty}\sum_{i=1}^{n}\langle x, e_i\rangle\langle e_i, e_\beta\rangle$$

$$= 0 - \lim_{n\to\infty}\sum_{i=1}^{n}\langle x, e_i\rangle\, 0 = 0$$ 179

Thus, $\langle x-y, e_\alpha\rangle = 0, \ \forall \alpha \in I.$

$\Rightarrow \quad x - y = 0.$

$\Rightarrow \quad x = \sum_{\alpha\in I}\langle x, e_\alpha\rangle e_\alpha.$

Now,

$$\|x\|^2 - \sum_{\alpha\in I}|\langle x, e_\alpha\rangle|^2 = \|x\|^2 - \sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2$$

$$= \lim_{n\to\infty}\left(\|x\|^2 - \sum_{i=1}^{n}|\langle x, e_i\rangle|^2\right)$$

$$= \lim_{n\to\infty}\left\| x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i\right\|^2$$

$= 0$ (by first part)

<u>Gram - Schmidt process:</u>

Let $\{x_1, x_2, \ldots\}$ be a L.I. set in a Hilbert Space H. Write $e_1 = \frac{x_1}{\|x_1\|}$ and

$$e_n = x_n - \sum_{j=1}^{n-1}\langle x_j, e_j\rangle e_j, \quad n \geq 2.$$

Then $\{e_1, e_2, \ldots\}$ is an orthonormal set in H.

EX. It is easy to see that $\{e^{-t^2/2}, te^{-t^2/2}, \ldots\}$ is a L.I. set in $L^2(\mathbb{R})$.

Write $f_n(t) = (-1)^n e^{-t^2/2}\frac{d^n}{dt^n}e^{-t^2}$; $n = 0, 1, 2, \ldots$

Using Gram-Schmidt process and

integration by parts. Then

$f_n(t) = H_n(t)e^{-t^2/2}$, where H_n is Hermite polynomial. (173)

The set $\{f_n : n \in \mathbb{N}\}$ is an ONB for $L^2(\mathbb{R})$.
To show completeness of this set, suppose $\exists\, g \in L^2(\mathbb{R})$ s.t. $\int_{\mathbb{R}} g(t)\, e^{-t^2/2}\, t^n\, dt = 0$. —(1)

For $G(z) = \int_{-\infty}^{\infty} g(t) e^{-t^2/2} e^{tz}\, dt$, G is an entire function on $\mathbb{C}$, and its all derivatives is 0 at 0. Hence, $G(z) \equiv 0$.
In particular, $\int_{-\infty}^{\infty} g(t) e^{-t^2/2} e^{itx}\, dt = 0$

By Fourier Inversion, $g = 0$.

<u>Theorem</u>: A Hilbert space is separable iff it has countable orthonormal basis (ONB).

Proof: Suppose H is separable and $A = \{x_1, x_2, \ldots\}$ be a dense set in H. By Q2, Assignment 2, H must have an infinite L.I dense set say $B = \{u_n\}_{n=1}^{\infty} \subset H$.
By Gram-Schmidt process, we can assume that $\{u_n\}_{n=1}^{\infty}$ is an ONS. Since

$\overline{\operatorname{span} B} = H$, it follows that $\{\psi_n\}_{n=1}^{\infty}$ is an ONB for H.

Conversely, if H has a countable ONB, then it is also a Schauder basis, and hence separable.

Theorem: Every infinite dimensional separable Hilbert space is isomorphic to ℓ^2.

Proof: Let $A = \{e_1, e_2, \dots\}$ be an ONB for H.

Define $F : H \longrightarrow \ell^2$

$$F(x) = (\langle x, e_1\rangle, \langle x, e_2\rangle, \dots)$$

Then $\|F(x)\|^2 = \sum |\langle x, e_n\rangle|^2 = \|x\|^2 < \infty$.

$\Rightarrow \|F(x)\| = \|x\|$.

Hence, F is one-one isometry.

Let $y \in \ell^2$ and $y = (y_1, y_2, \dots)$. Write

$x = \sum y_n e_n$. Then $\|x\|^2 = \sum |\langle y_n, e_n\rangle|^2 < \infty$,

and $F(x) = y$. Hence F is onto.

This proves the result.

— x —